Chapter 5

SIGNED DOMATRIC NUMBER OF A GRAPH.
5.1 Introduction:

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants. The idea of dominating function \( f : V \rightarrow \{0,1\} \) satisfying \( \sum_{u \in N(v)} f(u) \geq 1 \), for all vertices \( v \) in \( G \), is itself emerged from the definition of dominating sets by simply assigning 1 to the vertices of a dominating set \( D \) and 0 to the vertices of \( V - D \). The definition of domatic number \( d(G) \) is well known in the literature and defined as the maximum cardinality taken over all partitions in which each set is a dominating set. Now, the question is how this parameter can be looked in terms of dominating functions.

For example, consider the following graph \( G \) as shown in the Figure-5.1.

\[ \text{Figure-5.1} \]
The largest domatic partition

\[ P = \{ D_1 = \{ u_1, u_5 \}, D_2 = \{ u_2, u_6 \}, D_3 = \{ u_3, u_7 \}, D_4 = \{ u_4, u_8 \} \} \]

which gives us that \( d(G) = 4 \). If we look at these dominating sets \( D_i \) in terms of dominating functions, that is, \( f_i : V \rightarrow \{0, 1\} \) corresponds to the dominating set \( D_i \), for \( i = 1, 2, 3, 4 \) then for every vertex \( u_j \in D_i \), we have \( \sum_{i=1}^{4} f_i(u_j) = 1 \), since \( u_j \) is in exactly one of the sets \( D_i \), as \( D_i \) form a partition of the vertex set \( V(G) \).

In view of the above, we can redefine the concept of domatic number \( d(G) \) as the maximum number of dominating functions \( f_1, f_2, \ldots, f_d \) satisfying \( \sum_{i=1}^{d} f_i(u) = 1 \). How this idea can be extended to the set of signed dominating functions? As the signed dominating function \( f : V \rightarrow \{-1, 1\} \) involves the functional values \( +1 \) and \( -1 \), it may happen that \( \sum_{i=1}^{d} f_i(u) > 1 \), if we consider each \( f_i \) as a signed dominating function. For example, consider the graph \( G_2 \) of the Figure-5.2.

\[ \text{FIGURE-5.2} \]
By the property of signed dominating functions, the vertices $u_1$ and $u_2$ must receive +1 by any signed dominating function. The only vertices which receive $-1$ are $u_3$ and $u_4$. Thus, there are three signed dominating functions $f_1$, $f_2$ and $f_3$ which assigns +1 and $-1$ as in the following Figure-5.3 (a)-(c) in the order given below:

**FIGURE-5.3 (a), (b), (c)**
Now, can we conclude the Signed domatic number \( d_s \) (let us denote it!) is three? The answer is no, since \( \sum_{i=1}^{3} f_i(u_i) = 3 > 1 \). Then again, we cannot say it is two for the same reason; but definitely, we conclude that \( d_s(G) = 1 \). This motivates L. Volkman and B. Zelinka to introduce the concept of signed domatic number denoted by \( d_s(G) \) for a graph \( G \) in [11]. They defined the signed domatic number \( d_s(G) \) as the maximum number of signed dominating functions \( f_1, f_2, ..., f_d \) on a vertex set \( V(G) \) satisfying \( \sum_{i=1}^{d} f_i(u) \leq 1 \), for all vertices \( u \) of \( G \). This can be seen in the following example. Consider the graph \( G = K_8 \) - complete graph on eight vertices.

Define a function \( f_i : V \rightarrow \{-1, 1\} \) for \( i = 1, 2, 3 \) by

- \( f_1(u_1) = f_1(u_2) = f_1(u_3) = f_1(u_4) = f_1(u_5) = f_1(u_6) = f_1(u_7) = -1 \),
- \( f_2(u_2) = f_2(u_3) = f_2(u_4) = f_2(u_5) = f_2(u_6) = f_2(u_7) = 1 \),\( f_3(u_1) = f_3(u_2) = f_3(u_3) = f_3(u_4) = f_3(u_5) = f_3(u_6) = f_3(u_7) = -1 \),
- \( f_3(u_1) = f_3(u_2) = f_3(u_3) = f_3(u_4) = f_3(u_5) = f_3(u_6) = f_3(u_7) = 1 \),\( f_3(u_1) = f_3(u_2) = f_3(u_3) = f_3(u_4) = f_3(u_5) = f_3(u_6) = f_3(u_7) = -1 \).

The assignment of +1 and −1 by \( f_i \) can be seen in the following Figure 5.4 (a)-(c).

![Graph](image.png)
Also one can easily verify that $\sum_{j=1}^{3} f_j(u) \leq 1$. Now, the question is, can we get more than three dominating functions $f^*_i$ satisfying $\sum_{i=1}^{d \geq 4} f_i(u) \leq 1$ for every vertex $u$?
The answer is no! and this can be discussed later. Hence, the signed domatic number \( d_s(K_8) = 3 \). This beautiful idea of signed domatic number was introduced by L. Volkmann and B. Zelinka in [11], but no progress has been made by any researchers after this paper. This made us to consider this parameter for further investigation.

5.2 Some existing results:

L. Volkmann and B. Zelinka [11] obtained some properties of signed domatic number and also found exact value of signed domatic number \( d_s(G) \) for certain class of wellknown graphs namely, complete graphs, Cycles, Wheels and Fans. All these results, we report here.

**Proposition 5.2.1** [11]: The signed domatic number \( d_s(G) \) is well-defined for each graph \( G \).

**Proposition 5.2.2** [11]: Let \( G \) be a graph of order \( n \) with signed domination number \( \gamma_s(G) \) and signed domatic number \( d_s(G) \), \( \gamma_s(G)d_s(G) \leq n \).

**Proposition 5.2.3** [11]: If \( G \) is a graph with minimum degree \( \delta(G) \), then

\[
1 \leq d_s(G) \leq \delta(G) + 1.
\]
PROPOSITION 5.2.4 [11]: The signed domatic number is an odd integer.

COROLLARY 5.2.4.1 [11]: If $T$ is a tree, then $d_a(T) = 1$.

In fact, for any graph with pendant vertex has a signed domatic number one.

THEOREM 5.2.5 [11]: If $G = K_n$ is the complete graph of order $n$, then

- $d_s(G) = n$, if $n$ is odd
- $d_s(G) = p$, if $n = 2p$ and $p$ is odd
- $d_s(G) = p - 1$, if $n = 2p$ and $p$ is even.

THEOREM 5.2.6[11]: Let $C_n$ be a cycle of length $n \geq 3$. If $n$ is divisible by 3, then $d_s(C_n) = 3$ and $d_s(C_n) = 1$ in the remaining cases

THEOREM 5.2.7[11]: Let $G$ be a fan of order $n$. If $n = 3$, then $d_s(G) = 3$ and if $n \neq 3$, then $d_s(G) = 1$.

THEOREM 5.2.8[11]: If $G$ is a wheel of order $n$, then $d_s(G) = 1$. 

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5.3 New Results:

In this section, we are going to prove $\gamma_s(G) + d_s(G) \leq n + 1$ and also $d_s(G) + d_s(G) \leq n + 1$ and also we characterise the extremal class of graphs for which both the bounds attain. Further, We consider to find the exact value of signed domatic number of a circulant graphs.

THEOREM 5.3.1: If $G$ is a graph of order $n$, and $\gamma_s(G) \geq 0$ then $\gamma_s(G) + d_s(G) \leq n + 1$. Further, the equality holds if and only if $G$ is a complete graph of odd order or $G$ is a graph in which every vertex is a support or a pendent vertex.

PROOF: Let $G$ be a graph of order $n$. The inequality follows from the fact that $a + b \leq ab + 1$, for any two non-negative integer $a$ and $b$. By the Proposition 5.2.2, we have $\gamma_s(G) + d_s(G) < \gamma_s(G).d_s(G) + 1 < n + 1$.

The only thing remains to prove the equality. Suppose that,

$$\gamma_s(G) + d_s(G) = n + 1 \quad (5.1)$$

Then, $n + 1 = \gamma_s(G) + d_s(G) \leq \gamma_s(G).d_s(G) + 1 \leq n + 1$.

This implies that $\gamma_s(G) + d_s(G) = \gamma_s(G).d_s(G) + 1$.

This shows that

$$\gamma_s(G).d_s(G) = n \quad (5.2)$$
Solving (5.1) and (5.2) simultaneously, we have either
\[ \gamma_s(G) = 1, \quad d_a(G) = n \quad \text{or} \quad \gamma_a(G) = n, \quad d_s(G) = 1. \]

If \( \gamma_s(G) = 1, \quad d_a(G) = n, \)

then \( n = d_s(G) \leq \delta(G) + 1 \) by the proposition 5.2.5.

Therefore, \( \delta(G) \geq n - 1 \) implies that \( \delta(G) = n - 1 \) which in turn gives that \( G \) is a complete graph. But by the Theorem 5.2.5, the order of the complete graph must be odd and hence in this case \( \gamma_s(G) = 1 \) holds. If \( \gamma_s(G) = n \) and \( d_a(G) = 1 \), then by the Corollary 5.2.4.1, \( G \) must be a graph in which every vertex is either a support or a pendant vertex.

Converse is obvious.

**PROPOSITION 5.3.2:** If \( \gamma_s(G) > \frac{n}{2} \) then \( d_s(G) = 1 \), where \( n \) is the order of \( G \).

**PROOF:** Let \( G \) be a graph of order \( n \). Suppose that \( \gamma_s(G) > \frac{n}{2} \). Then by the Proposition 5.2.2, we have

\[ \frac{n}{2} d_s(G) < \gamma_s(G) d_a(G) \leq n. \]

This implies that \( d_s(G) < 2 \) which in turn implies that \( d_s(G) = 1 \).

**COROLLARY 5.3.2.1:** If \( G \) is a Petersen graph, then \( d_s(G) = 1 \).

**PROOF:** Suppose \( G \) is a Petersen graph, then by Proposition 2.4.1,

\[ \gamma_s(G) = 8 > \frac{10}{2}. \]

Thus, \( d_s(G) = 1 \) follows from the above Proposition 5.3.2.
Next Theorem deals with the domatic number of a graph and that of its complement. Such results in graph theory are popularly known as Nordhaus-Gaddum type results.

Before going to prove the Nordhaus-Gaddum type result, we prove the following Lemma.

**Lemma 5.3.3**: If $G$ is a non-complete regular graph of order $n$, then $d_s(G) < \frac{n}{2}$.

**Proof**: Let $G$ be a non-complete regular graph of order $n$ with regularity $r$.

We prove that $d_s(G) < \frac{n}{2}$. For if, $d_s(G) > \frac{n}{2}$, then, by the Proposition 5.2.2, we have $\frac{3}{2} \gamma_s(G) < \gamma_s(G) d_s(G) \leq n$. This implies $\gamma_s(G) < 2$ and hence $\gamma_s(G) \leq 1$.

But, we know that $\gamma_s(G) \geq \frac{n}{r+1}$, for any regular graph of degree $r$. Thus, $\frac{n}{r+1} \leq 1$ which gives us that $r \geq n - 1$ and in turn we conclude that $G$ is a complete graph, a contradiction to the hypothesis; which proves the result.

With the help of above Lemma 5.3.3 and Proposition 5.2.3, we prove the following Theorem.

**Theorem 5.3.4**: Let $G$ be a graph of order $n$, then $d_s(G) + d_s(\bar{G}) \leq n + 1$ and the equality holds if and only if $G$ is a complete graph of odd order or $G$ is its complement.

**Proof**: Let $G$ be a graph of order $n$. By the Proposition 5.2.3, we have $d_s(G) \leq \delta(G) + 1$ and $d_s(\bar{G}) \leq \delta(\bar{G}) + 1$.

Thus, we have
Thus, the inequality holds.

Next, let \( d_s(G) + d_s(\overline{G}) = n + 1 \).

By the proof of the above part, we conclude that \( G \) is a regular graph. Now, we claim that \( G \) is a complete graph. For, if \( G \) is not complete and regular graph, then by the Lemma 5.3.3,

\[
d_s(G) \leq \frac{n}{2} \quad \text{and} \quad d_s(\overline{G}) \leq \frac{n}{2},
\]

which implies that

\[
n + 1 = d_s(G) + d_s(\overline{G}) \leq \frac{n}{2} + \frac{n}{2} = n,
\]

a contradiction.

But, by the **Theorem 5.3.1**, \( G \) is a complete graph of odd order and converse is obvious.

### 5.4 Signed domatic number of circulant graphs:

The circulant graphs are emerged from circulant matrices. An \( n \times n \) matrix \( S = [s_{ij}] \) is said to be a **Circulant matrix**, if its entries satisfy \( s_{ij} = s_{i,j-i+1} \) where the subscripts are reduced modulo \( n \) and lie in the set \( \{1, 2, 3, \ldots, n\} \). In other words, row \( i \) of \( S \) is obtained from the first row of \( S \) by a cyclic shift of \( i-1 \) steps, and so any circulant matrix is determined by its first row. If the first row of \( S \) is \( (s_1, s_2, s_3, \ldots, s_{n-1}, s_n) \), then \( i^{th} \) row of \( S \) is \( (s_{n-i+2}, s_{n-i+3}, \ldots, s_{n-i}, s_n) \).

The circulant graph is a graph \( G \) whose vertices can be ordered so that the adja-
The adjacency matrix $A(G)$ is a circulant matrix. For example, the graph $G$ of the Figure 5.5, is a circulant graph.

\[
A(G) = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
5 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
6 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
7 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
8 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
9 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
10 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
11 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
Which is a circulant matrix. Another way of defining circulant graph is group theoretic one and which is defined below:

Consider the additive group \( \mathbb{Z}_n \) and a subset \( C \) not containing 0. The circulant graph \( G \) is a graph whose vertices are elements of \( \mathbb{Z}_n \) and two vertices \( i, j \in \mathbb{Z}_n \) are adjacent in \( G \) if and only if \( i - j \in C \). The graph \( G \) of the Figure 5 is a circulant graph on \( \mathbb{Z}_{12} \) and the set \( C = \{ \pm 1, \pm 2, \pm 3 \} \). The set \( C \) is called a connection set of the circulant graph. By the definition of circulant graph, the cycle \( C_n \) and complete graph \( K_n \) are circulant graphs with connection sets \( C_1 = \{ \pm 1 \} \) and \( C_2 = \{ \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2} \} \) where \( n \) is odd respectively.

In this section, we find the signed domatic number of a circulant graph \( G_n \) on \( \mathbb{Z}_n \) with the connection set \( C = \{ \pm 1, \pm 2 \} \), so that the circulant graph considered here is a regular graph of degree four.

We find the signed domatic number of circulant graph \( G_n \) on \( \mathbb{Z}_n \) with connection set \( C = \{ \pm 1, \pm 2 \} \) for various values of \( n \) in terms of following Lemmas:

**Lemma 5.4.1:** Let \( G_n \) be a circulant graph on \( \mathbb{Z}_n \) with connection set \( C = \{ \pm 1, \pm 2 \} \) and \( n = 5l \), for some integer \( l \), then \( d_s(G_n) = 5 \).

**Proof:** Let \( V(G_n) = \{0, 1, 2, \ldots, 5l - 1\} \) and define a functions \( f_j : V \rightarrow \{-1, 1\} \) for \( j = 1, 2, 3, 4, 5 \) as below:

- \( f_1(5i) = f_1(5i + 1) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_1(x) = 1 \), for \( x \neq 5i \) and \( 5i + 1 \);
- \( f_2(5i + 1) = f_2(5i + 2) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_2(x) = 1 \), for \( x \neq 5i + 1 \) and \( 5i + 2 \);
- \( f_3(5i + 2) = f_3(5i + 3) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_3(x) = 1 \), for \( x \neq 5i + 2 \).
and $5i + 3$;

$f_4(5i + 3) = f_4(5i + 4) = -1$, for $i = 0, 1, 2, ..., l - 1$ and $f_4(x) = 1$, for $x \neq 5i + 3$
and $5i + 4$ finally,

$f_5(5i + 4) = f_5(5i + 5) = -1$, for $i = 0, 1, 2, ..., l - 1$ and $f_5(x) = 1$, for $x \neq 5i + 4$
and $5i + 5$.

One can easily verify that each $f_j$, for $j = 1, 2, 3, 4, 5$ is a signed dominating
function in $G_n$ and also $\sum_{j=1}^{5} f_j(k) \leq 1$; for all $k \in \mathbb{Z}$. Thus, $d_s(G_n) \geq 5$. On the other
hand, $G_n$ is a regular graph of degree 4 and hence by the Proposition 5.2.9, we have

$d_s(G_n) \geq 4 + 1 = 5$, Thus, the Lemma is proved.

The following example explains the proof of the Lemma 5.4.1, Consider $G_{18}$
with $C = \{\pm 1, \pm 2\}$ as in the Figure-5.6

![Figure-5.6](image-url)
The following table gives the functional values of \( f_j \), for \( j = 1, 2, 3, 4, 5 \).

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\[ \sum_{j=1}^{5} f_j(k) \]

Table 1

**Lemma 5.4.2:** Let \( G_n \) be a circulant graph on \( Z_n \) with connection set \( C = \{\pm 1, \pm 2\} \) and \( n = 3l \), and \( n \equiv 0 \pmod{5} \), then \( d_s(G_n) = 3 \).

**Proof:** Define a functions \( f_1, f_2 \) and \( f_3 \) on the vertex set \( V(G_n) \) into \( \{-1, 1\} \) as below:

\( f_1(3i) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_1(x) = 1 \), for \( x \neq 3i; \)

\( f_2(3i + 1) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_2(x) = 1 \), for \( x \neq 3i + 1; \)

\( f_3(3i + 2) = -1 \), for \( i = 0, 1, 2, \ldots, l - 1 \) and \( f_3(x) = 1 \), for \( x \neq 3i + 2. \)

Clearly, \( f_j \) is a signed dominating function for \( j = 1, 2, 3 \) and also \( \sum_{j=1}^{3} f_j(k) \leq 1 \), for all \( k \in Z_n \). Thus, \( d_s(G_n) \geq 3 \) holds.

On the other hand, by the Theorem 2.3.1, \( \gamma_s(G) \geq \frac{n}{4 + 1} = \frac{9}{5} \), but 5 is not divisible by \( n \), therefore \( \gamma_s(G) > \frac{n}{5} \). By using the Proposition 5.2.2; we have
\[ \frac{n}{5}.d_s(G) < \gamma_s(G).d_s(G) \leq n. \]

Therefore, \( d_s(G) < 5 \) which implies \( d_s(G) \leq 3 \), since \( d_s(G) \) is always odd integer and hence the Lemma.

**LEMMA 5.4.3:** Let \( G_n \) be a circulant graph on \( Z_n \) with the connection set \( C = \{\pm 1, \pm 2\} \) and \( n \equiv 1, 2 (mod 3) \) and 5 is not divisible by \( n \) and \( n \neq 7 \), then \( d_s(G) = 3 \).

**PROOF:** The proof of this lemma, runs on the similar as in case of Lemma 5.4.2.

**LEMMA 5.4.4:** Let \( G_7 \) be a circulant graph on \( Z_7 \) with the connection set \( C = \{\pm 1, \pm 2\} \), then \( d_s(Z_7) = 1 \).

**PROOF:** Clearly, no signed dominating function \( f \) admits more than two negative, and therefore \( \gamma_s(G) \geq 3 \). But a function \( f : V \rightarrow \{-1, 1\} \) defined by \( f(0) = f(1) = -1, f(x) = 1 \) for \( x \neq 0, 1 \) gives us that \( \gamma_s(G) \geq 3 \). Therefore, \( \gamma_s(G) = 3 \) holds. Again by the Proposition 5.2.2, we have \( d_s(G_7) \leq \frac{7}{3} \).

This implies that \( d_s(G_7) \leq 1 \) and hence \( d_s(G_7) = 1 \).
REFERENCES


[12] Z.Zhang, B.Xu, Y.Li and Liu: *A note on the lower bounds of signed dom-


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