CHAPTER 3

CYLINDRICAL WAVE PROPAGATION

3.1 INTRODUCTION

The phase and amplitude of light propagating from cylindrical surface varies in space (with time) in an entirely different fashion compared to that from a plane surface. But still they fit well into the framework of diffraction theories for wave propagation from plane surface (Born and Wolf 1999). Hence wave propagation from cylindrical surfaces has very interesting properties. Generally, the complex amplitude of a wave propagating from a source surface to an observation surface can be calculated using the diffraction theories as explained in the previous chapter. If the source and observation surface are planes parallel to each other, then these diffraction theories can be simplified a lot. Moreover since such a system is shift-invariant the calculation speed can be dramatically improved by using fast Fourier transform. But if the source or observation surface is curved or tilted with respect to each other, then the complex amplitude can be calculated only using direct integration, which is time consuming. But if both the source and observation surface are curved such that the shift-invariance still holds, then fast Fourier transform can be used. Such a situation occurs when both the object and observation surface are concentric cylindrical surfaces. The diffraction theories then have to be expressed in cylindrical coordinates \((r, \theta, h)\), for fast Fourier transform to be used. So the diffraction formulae that were devised in Cartesian coordinates (in the previous chapter) has to be devised in cylindrical coordinates. The two near field and less approximated calculation methods, a) convolution method and b) angular
spectrum of plane waves method are of interest in cylindrical geometry. The convolution formula has already been devised for cylindrical coordinates and digital cylindrical holography based on convolution method has been demonstrated by Sando et al. (2005). But digital holography has not yet been demonstrated using the plane wave decomposition method (also known as angular spectrum method), which is much faster than the convolution method. This work is an attempt to do the same. In other words this work is an attempt to devise a formula for wave propagation from cylindrical surface in spectral domain and apply it to digital holography.

In order to appreciate the usefulness and challenges in this, a very short review of the evolution of computer generated cylindrical holography is presented below. In the beginning days, computer generated holograms were made only on planar surface due to the existence of the shift invariance relationship which allowed using FFT for calculations. Hence computer generated holograms were usually made for geometries where the object and hologram surfaces are planar and perpendicular to the optical axis. The other reason was the availability of planar hologram recording and display devices. Later on the method was improved and fast computation schemes using FFT were also developed for non shift invariant systems such as, a tilted plane geometry (Tommasi and Bianco 1992, 1993). But for a long time the cylindrical geometry was not considered for making a computer generated hologram. The main reason was the extreme difficulty in devising a fast computation method for a curved hologram surface.

Fast computation method is possible if the wave propagation is devised in cylindrical co-ordinates. For this the solution to the wave equation (Equation 2.27) is to be derived in cylindrical co-ordinates. The normal mode solutions to Equation (2.27) in cylindrical and spherical geometries were given by Stratton (1941). Berry (1975) discusses on the propagation of cylindrical waves and the occurrence of evanescent waves. The construction of Greens function for cylindrical and spherical geometries was given
by Marathay (1975). Even though the theories on cylindrical wave propagation were available, they were less attractive due to the unavailability of cylindrical shaped recording or display devices. However, the recent developments in technology and the possibility of producing advanced display and recording devices, has made people focus on cylindrical geometries for digital holography. As a result, papers describing cylindrical computer generated holography started to appear in the recent five years. Sakamoto and Tobise (2005) used the angular spectrum of plane waves method to generate a cylindrical hologram of a plane object. They employed the shift invariance in rotation between a planar and cylindrical surface and hence could use FFT. Then they improved on their method to generate the hologram of a volume object by slicing it into planar segments (Kashiwagi and Sakamoto 2007). This took 2.76 hrs to calculate the hologram of a 13×13×13 mm object. Yamaguchi et al. (2008) used the Fresnel transform and segmentation approach to generate cylindrical holograms. They approximated the cylindrical holographic surface into smaller plane surface to generate the hologram. Since it was a multiplexed hologram they had to use a large number of samples which demanded a calculation time of 81 hrs on a parallel computing machine for an object of size 15×15×15 mm. They also developed a computer generated cylindrical rainbow hologram using the same method (Yamaguchi et al. 2009). The calculation time for the rainbow hologram was 45 min on a single computer, but sacrifices vertical parallax. Then Sando et al. (2005) took a very different and smart approach by considering the object also to be a cylindrical surface. Both the object surface and the hologram surfaces are concentric cylinders. The most important significance of this approach is that, the shift-invariance relation is preserved in both horizontal and vertical directions and hence FFT can be used. However their calculation method was based on wave propagation in spatial domain using convolution. This method uses three FFT calculations and is faster than all the other existing methods for cylindrical holography. However if a method could be devised using wave propagation in spectral domain using
decomposition of plane waves theory, then it would be still faster because it uses only two FFT operations. So far no one has computed a hologram by considering wave propagation from a cylindrical surface in spectral domain. This research work attempts to achieve the same using a spectral method in cylindrical coordinates. In other words, this method can be considered analogous to the angular spectrum of plane waves in Cartesian coordinates. For any spectral propagation method the most important task is to find the proper transfer function and then characterize it for proper sampling conditions and if possible approximate it for easy computation. Hence this chapter will explain in detail the derivation of the transfer function and also discuss the proper sampling conditions required for loss free numerical computation. In more general terms, this chapter will devise the plane wave decomposition method in cylindrical coordinates. This can also be considered as an improvement or extension of the work by Sando et al. (2005).

3.2 HELICAL WAVE SPECTRUM

Since the diffraction theory for propagation from cylindrical surface fits into the frame work of that of plane surface, the same procedure followed in the previous chapter (explaining wave propagation in rectangular coordinates) will be followed in this section. Accordingly we start with the scalar wave equation expressed earlier as Equation (2.27), but now in cylindrical coordinates $(r, \theta, y, t)$.

\[ \nabla^2 E - \varepsilon \mu \frac{\partial^2 E}{\partial t^2} = 0 \]  

(3.1)

The cylindrical coordinate system is shown in Figure 3.1. In cylindrical coordinates the Laplace operator $\nabla^2$ is defined as
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial y^2} \]  

(3.2)

Figure 3.1: Coordinate system

As explained in Chapter 2, if the medium of propagation is linear, isotropic, homogeneous and nondispersive, Equation (3.3) can be represented as a scalar equation. Hence we drop the vectorial nature of the equation and look for a scalar function \( p(r, \phi, y, t) \) as the solution. Hence Equation (3.1) can be expressed as

\[ \nabla^2 p - \epsilon \mu \frac{\partial^2 p}{\partial t^2} = 0 \]  

(3.3)

The solution to Equation (3.3) can be found using separation of variables method. For this, the solution has to be written as a product of solutions of function of each coordinate and of time. That is

\[ p(r, \phi, y, t) = R(r)\Phi(\phi)Y(y)T(t) \]  

(3.4)

Substituting the solution in Equation (3.3) and dividing out by \( R\phi YT \) (for separation of variables), leads to
\[
\left(\frac{1}{R} \frac{d^2R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2\Phi} \frac{d^2\Phi}{d\phi^2}\right) + \left(\frac{1}{Y} \frac{d^2Y}{dy^2}\right) = \frac{1}{c^2T} \frac{d^2T}{dt^2}
\] (3.5)

The terms in the first set of brackets depend only on the variables \(r\) and \(\phi\) and the second set of brackets only on \(y\) and the right hand side only on \(t\). Thus since \(r, \phi, y\) and \(t\) are all independent of each other, each of these terms must be equal to a constant. We choose the following arbitrary constants, \(k\) and \(k_y\), satisfying the following equations.

\[
\frac{1}{c^2T} \frac{d^2T}{dt^2} = -k^2
\] (3.6)

\[
\frac{1}{Y} \frac{d^2Y}{dy^2} = -k_y^2
\] (3.7)

\[
\frac{1}{R} \left(\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr}\right) + \frac{1}{r^2\Phi} \frac{d^2\Phi}{d\phi^2} = -k^2 + k_y^2 = -k_r^2
\] (3.8)

where the constant

\[
k_r = \sqrt{k^2 - k_y^2}
\] (3.9)

Equation (3.8) can be written as

\[
\frac{r^2}{R} \left(\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr}\right) + k_r^2 r^2 = -\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2}
\] (3.10)

The left hand side of Equation (3.10) is a function of \(r\) alone and the right hand side of \(\phi\) alone. Hence the right and left hand sides must be equal to constants. Choosing \(n^2\) as one of the constants leads to
\[
\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -n^2 
\] (3.11)

and the left hand side turns out to be the Bessel’s equation,

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( k_r^2 - \frac{n^2}{r^2} \right) R = 0 
\] (3.12)

The solutions of Equation (3.12) are well known and are given by the Bessel functions of the first and second kinds \( J_n(k_r r) \) and \( Y_n(k_r r) \). \( Y_n \) is also called as the Neumann function. The solution to Equation (3.12) uses these two independent functions with arbitrary constants \( R_1 \) and \( R_2 \)

\[
R(r) = R_1 J_n(k_r r) + R_2 Y_n(k_r r) 
\] (3.13)

\( J_n \) and \( Y_n \) are called standing wave solutions of Equation (3.12) because of their asymptotic behavior. A linear combination of these functions is necessary for traveling wave solutions, and is given by the Hankel functions of the first and second kind.

\[
H_n^{(1)}(k_r r) = J_n(k_r r) + iY_n(k_r r) 
\] (3.14)

\[
H_n^{(2)}(k_r r) = J_n(k_r r) - iY_n(k_r r) 
\] (3.15)

With the time dependence \( e^{-i\omega t} \), \( H_n^{(1)}(k_r r) \) corresponds to a diverging outgoing wave and \( H_n^{(2)}(k_r r) \) to an incoming converging wave. The general traveling wave solution is then

\[
R(r) = R_1 H_n^{(1)}(k_r r) + R_2 H_n^{(2)}(k_r r) 
\] (3.16)
Similarly, since Equation (3.6), Equation (3.7) and Equation (3.11) are second order differential equations, each has a general solution with two arbitrary constants

\[
\Phi(\phi) = \Phi_1 e^{i n \phi} + \Phi_2 e^{-i n \phi}
\] (3.17)

\[
Y(y) = Y_1 e^{i k_y y} + Y_2 e^{-i k_y y}
\] (3.18)

\[
T(t) = T_1 e^{-i \omega t} + T_2 e^{-i \omega t}
\] (3.19)

with arbitrary constants \(\Phi_1, \Phi_2, Y_1, Y_2, T_1\) and \(T_2\). Further, the quantity \(n\) must be an integer because \(\Phi(\phi + 2\pi) = \Phi(\phi)\), and \(k = \omega/c\). Also we assume \(T_2 = 0\) for the convention of time.

Now, we combine the solutions given by Equations (3.16, 3.17, 3.18 and 3.19). There are six possible combinations with the two independent solutions for each coordinate.

\[
p(r, \phi, y, t) \propto H_n^{(1),(2)}(k, r) e^{\pm i n \phi} e^{\pm i k_y y} e^{-i \omega t}
\] (3.20)

All these six combinations can be included in the general solution by summing over all possible positive and negative values of \(n\) and \(k_y\) with arbitrary coefficient functions (functions of \(n, k_y\) and \(\omega\)) replacing the pairs of constants, \(Y_1, Y_2, \Phi_1, \Phi_2, R_1\) and \(R_2\). Thus the most general solution to Eq. (3.3) in the spectral domain is given by

\[
p(r, \phi, y, \omega) = \sum_{n=-\infty}^{\infty} e^{i n \phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A_n(k, \omega) e^{i k_y y} H_n^{(1)}(k, r) + B_n(k, \omega) e^{i k_y y} H_n^{(2)}(k, r) \right] dk_y
\] (3.21)
where \( A_n(k_y, \omega) \) and \( B_n(k_y, \omega) \) are the arbitrary constants replacing the constants \( Y_1, Y_2, \Phi_1, \Phi_2, R_1 \) and \( R_2 \).

The time domain solution of the wave equation (Equation 3.3) can be obtained from the inverse Fourier transform.

\[
p(r, \phi, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(r, \phi, y, \omega) e^{-i\omega t} d\omega
\]

(3.22)

\[\text{Figure 3.2: Region of validity for calculating cylindrical wave propagation with respect to radiating source positions}\]

Equation (3.21) represents the complete general solution to the wave equation in a source-free region. In order to determine the arbitrary coefficients, boundary conditions are to be specified on the coordinate surfaces, for example, \( r = \text{constant} \). Boundary conditions with \( y = \text{constant} \) leads to discrete solutions in \( k_y \) instead of continuous ones formulated above. The boundary condition on \( r \) alone leads to the solution that suits the problem discussed in this research work. Hence we proceed in finding the solution by imposing the boundary condition on \( r \).

Consider the case in which the boundary condition is specified at \( r = a \) and \( r = b \), as shown in Figure 3.2. In this case the sources are located in
the two regions labeled $\Sigma_1$ and $\Sigma_2$. The homogeneous wave equation is valid in the annular disk region shown in Figure 3.2. In this region Equation (3.21) can be used to solve for the wavefield. The boundary conditions on the surfaces at $r = a$ and $r = b$ yield unique solution (for all values of $y$ and $\phi$). Two boundary conditions are necessary because there are two unknown functions, $A_n$ and $B_n$ in the equations. No part of the source region is allowed to cross the infinite cylinder surfaces defining the annular disk region.

The two parts to the solution of Equation (3.21) can be explained with respect to the two Hankel functions. The first term represents an outgoing wave expressed in Equation (3.14) due to sources which must be on the interior of the volume of validity ($\Sigma_1$), causing the waves to diverge outward. $A_n$ provides the strength of these sources. The second Hankel function (Equation 3.15) represents incoming waves and is needed to account for the sources external to the annular region ($\Sigma_2$). Similarly, $B_n$ provides the strength of these sources.

![Figure 3.3: Region of validity when all radiating sources are outside the boundary](image)

Now, two other boundary conditions also arise which are shown in Figures 3.3 and 3.4 respectively. The first one is called the interior problem in which the sources are located completely outside the boundary surface $r = b$ (Figure 3.3). The second boundary problem is called the exterior problem because the boundary surface $r = a$ completely encloses all the sources (Figure 3.4). The research work reported in this thesis is also a problem of this
kind. Hence we proceed discussing with only the second boundary value problem shown in Figure 3.4. Now the solution to Equation (3.21) is to be found out based on this boundary condition. It turns out that the second term in Equation (3.21) represents an in-coming wave which cannot exist when all the sources are within the boundary. Thus we set the second coefficient function to zero i.e., $B_n = 0$. Now the general solution becomes

$$p(r, \phi, z, \omega) = \sum_{n=-\infty}^{\infty} e^{i n \phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} (A_n(k_y, \omega)e^{ik_y}H_n^{(1)}(k yr)) dk_y$$  \hspace{1cm} (3.23)$$

Now, if the wavefield on the boundary at $r = a$ is specified then $A_n$ can be determined and Equation (3.23) can be used to solve for the wavefield in the region from the surface at $r = a$ to $r = \infty$ (Figure 3.4). In this reported research work, the boundary surface at $r = a$ constitutes the object surface whose wavefield is already known and the hologram is another surface that is exterior to $r = a$. Hence this research work also demands a solution of the same kind. Hence now we proceed to determine the quantity $A_n$ using the known boundary values.
Since the time dependence of the propagation is known a priori (for a monochromatic wave), the time component ($\omega$, in spectral domain) can be neglected in Equation (3.23). It is also worth noting here that the wave equation with the time component dropped is nothing but the Helmholtz equation defined in Chapter 2 as Equation (2.36). Hence, in other words solution to Helmholtz equation is being found out as it was done in Chapter 2, but now in cylindrical coordinates. Hence Equation (3.23) reduces to

$$p(a, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_n(k_y) e^{ik_y y} H_n^{(1)}(k_y a) dk_y$$

(3.24)

Now, let us consider $P_n(r, k_y)$ to be the two-dimensional Fourier transform in $\phi$ and $y$ in cylindrical coordinates of the wavefield defined at $r$.

$$P_n(r, k_y) \equiv \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} p(r, \phi, y) e^{-in\phi} e^{-ik_y y} dy$$

(3.25)

The inverse relation for Eq. (3.25) is given by

$$p(r, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} P_n(r, k_y) e^{ik_y y} dk_y$$

(3.26)

where $n$ can take only integer values because the cylindrical surface is a closed one in the circumferential direction. Comparing Equation (3.26) at $r = a$ with Eq. (3.24) we get

$$P_n(a, k_y) = A_n(k_y) H_n^{(1)}(k_y a)$$

(3.27)
Using Eq. (3.27) to eliminate $A_n$ in Eq. (3.24) yields

$$p(r, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} P_n(a, k_y) e^{ik_y y} \frac{H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r a)} dk_y$$  \hspace{1cm} (3.28)

where,

$$P_n(a, k_y) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} p(a, \phi, y) e^{-in\phi} e^{-ik_y y} dy$$  \hspace{1cm} (3.29)

Equation (3.28) calculates the complex amplitude at any position $p(r, \phi, y)$, given the complex amplitude in another cylindrical surface $p(a, \phi, y)$ such that ($r > a$). The spectral solution in Equation (3.28) is similar in form to the plane wave expansion (angular spectrum of plane waves) defined in Chapter 2 as Equation (2.66).

$$U(x, y, z) = \int_{-\infty}^{\infty} A(k_x, k_y, 0) e^{i(k_x x + k_y y)} dk_x dk_y$$  \hspace{1cm} (3.30)

Hence Equation (3.28) can be represented by the term cylindrical wave expansion. For an easy understanding, the spectral solution in cylindrical coordinates (Equation 3.28) can be compared with the spectral solution in Cartesian coordinates (Equation 3.30). On comparison the following correspondences can be revealed.
\[ U(x, y, z) \Rightarrow p(r, \phi, y) \]
\[ A(k_x, k_y; 0) \Rightarrow P_n(a, k_y) \]
\[ A(k_x, k_y; z) \Rightarrow P_n(r, k_y) \]
\[ \exp(ik_z) \Rightarrow \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \]
\[ k_x \Rightarrow n/r \]
\[ k_y \Rightarrow k_y \]
\[ k_z \Rightarrow k_r \text{ where } k_r = \sqrt{k^2 - k_z^2} \]

Thus in view of the fact that \( A(k_x, k_y; z) \) is the plane wave (angular) spectrum, \( P_n(r, k_y) \) can be called as the helical wave spectrum.

Since the two-dimensional Fourier transform (Equation 3.25) of the left hand side of Equation (3.28) is \( P_n(r, k_z) \) then,

\[ P_n(r, k_y) = \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} P_n(a, k_y) \] (3.31)

Equation (3.31) provides the relationship between the helical wave spectrum at different cylindrical surfaces in the same way that \( \exp(ik_z) \) provided the relationship between the planar surfaces. The spectral component in Equation (3.28) is given by

\[ P_n(a, k_y) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} p(a, \phi, y)e^{-in\phi} e^{-ik_y y} dy \] (3.32)

which is nothing but a Fourier transform relation. The propagation component in Equation (3.28) (Transfer function) is given by

\[ T(a, k_r, r, k_r) = \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \] (3.33)
\[ k_r = \sqrt{k^2 - k_y^2} \quad \text{and} \quad k = 2\pi/\lambda \]

The propagation of helical wave spectrum is very difficult to visualize both in the axial direction and the radial direction. Some visual ideas on how the propagation of helical wave spectrum can be perceived, is given by Williams (1999). He also provides a discussion on the existence of evanescent wave and the necessary conditions. However these concepts are not important to the reported research work and hence are not mentioned here.

When ‘r’ in Equation (3.28) is kept constant, i.e., the measurement plane (hologram surface) is also a cylinder, then the system is shift invariant. Hence we can use FFT to evaluate Eq. (3.28) and hence fast calculation. Deriving out the analytical expression for the Transfer Function for propagation from cylindrical surface as shown in Equation (3.33), is the most important step in this work.

All the theories explained above are given with greater details by Lebedev (1965), Arfken (2001)

### 3.2.1 Sampling Conditions

Proper sampling at the object and hologram surface is required for loss free reconstruction. For this, the Nyquist sampling conditions must be satisfied. Consider the transfer function was generated using \( N \) samples which runs from \([-N/2...0...N/2]\). According to Nyquist theorem, the discrete transfer function’s rate of change should be less than or equal to \( \pi \) at \( N/2 \). From the analysis of Equation (3.33) one could understand that, the spatial rate of change of \( k_r r \) is higher than that of \( k_y a \). Hence as long as the sampling condition for \( k_r r \) is satisfied, the entire Transfer function also meets the sampling condition approximately. Accordingly the Nyquist sampling
condition can be expressed by the inequality as shown below

\[
\frac{\partial}{\partial n} \left| kr \sqrt{1 - (\lambda k_y)^2} \right|_{n=N/2} \leq \pi 
\]

(3.34)

From the above inequality with conditions, \( k_y = n \Delta k_y \), \( \Delta k_y = \frac{1}{\Delta L_0} \), and \( k = \frac{2\pi}{\lambda} \), (where \( \Delta L_0 \) is the height of the cylinder) we can obtain

\[
\left| \frac{k\lambda^2 rn}{\Delta L_0^2 \sqrt{1 - \frac{\Delta^2 n^2}{\Delta L_0^2}}} \right|_{n=N/2} \leq \pi 
\]

(3.35)

which again reduces to

\[
2nr\lambda \leq \Delta L_0 \sqrt{\Delta L_0^2 - \lambda^2 \left( \frac{N}{2} \right)^2} 
\]

(3.36)

As \( \Delta L_0^2 \gg \lambda^2 \left( \frac{N}{2} \right)^2 \) and is usually satisfied, a better approximation of the above inequality is

\[
\Delta L_0 \geq \sqrt{Nr\lambda} \quad \text{(or)} \quad N \geq \frac{\Delta L_0^2}{r\lambda} 
\]

(3.37)

Based on this sampling condition (Equation 3.37), the dimensions of the object and hologram were chosen. Accordingly, the object and hologram were assumed to be cylindrical surfaces with radius \( a = 1 \) and \( r = 10 \) respectively. The height of the cylindrical surface was assumed to be \( y = 10 \). \( \lambda = 180\mu m \) was chosen for the initial simulation trials in order to avoid harsh sampling requirements. When all these dimensions were substituted in the sampling condition, given by Equation (3.37), the required number of samples turned out to be \( N \approx 512 \). Hence the object and the transfer function will be generated as 512\times512 matrices for the simulation.
3.3 SUMMARY

The wave propagation from one cylindrical surface to another based on scalar diffraction theory was discussed. The theory was developed from the solution to wave equation (Equation 3.3) in spectral domain. The spectral propagation formula for cylindrical waves had the same architecture as that of spectral propagation for plane waves. In the case of cylindrical waves, the spectral expansion can be denoted by the term helical wave spectrum, similar to the term plane wave spectrum used for plane wave case. The transfer function was found to be the ratio of Hankel function of first kind (Equation 3.33). The whole formula can be computed using two FFT operations as in the case of plane waves, and hence is computationally inexpensive.