CHAPTER 2

ANALYSIS OF A BATCH ARRIVAL GENERAL BULK SERVICE QUEUEING SYSTEM WITH MULTIPLE VACATIONS, SETUP TIME AND SERVER’S CHOICE OF ADMITTING RE-SERVICE

2.1 INTRODUCTION

In many practical situations one can observe that the leaving batch of customers may request for re-service. Avi - Itzhak and Naor (1963) have analyzed an M/G/1 queueing model with repair of service station on request by a leaving customer. Rosenberg and Uri Yechiali (1993) analyzed a bulk arrival general service single server queue with single and multiple vacations under LIFO service regime. Huan et al (1995) discussed a computational analysis of M(n)/G/1/N queues with setup time. The interdeparture time distribution for each class in the $\sum M^X_1/G_1/1$ queue with setup times and repeated server vacations was studied by Frans (1999). Choudhury (2000) analyzed single server Poisson bulk arrival general service queue with a setup period and a vacation period.

Madan and Baklizi (2002) considered an M/G/1 queueing model, in which the server performs first essential service to all arriving customers. As soon as the first service is over, they may leave the system with the probability $(1-\theta)$ and second optional service is provided with probability $\theta$. Arumuganathan and Ramaswami (2003) analyzed a non-Markovian bulk

In the literature, queueing models with re-service, the server accepts all requests for re-service, irrespective of other constraints like number of customers in the queue, the cost of power and so on. But in reality, one can observe that, the server may reject the request for re-service with some probability. In many systems, before the commencement of service, the server will do some preparatory work such as warming up the machine or booting the computer, etc; in queueing terminology, such term is referred to as setup time. Server vacation models are useful for the system in which the server wants to utilize the idle time for different purposes. Application of vacation models can be found in production systems, designing local area networks and data communications systems. Addressing this, the proposed model M^{X/G(a,b)/1} queueing system with multiple vacations, setup time and server’s choice of admitting re-service is developed.

In this chapter, a bulk queueing system with server’s choice of admitting re-service, multiple vacations and setup time is considered. At a service completion, the leaving batch may request for re-service with
probability $\pi$ and it is not mandatory to accept it; the server admits this request with a probability $\alpha$. After the re-service or service completion without request for re-service, if the queue length is less than $a$, the server leaves for a secondary job (vacation) of random length. After this vacation, if the queue length is still less than ‘a’, the server leaves for another vacation and so on, until he finally finds at least ‘a’ customers waiting for service. At a vacation completion epoch, if the server finds at least ‘a’ customers waiting for service, he requires a setup time to start the service. After a setup time or on service completion or on re-service completion, if the server finds at least ‘a’ customers waiting for service say $\xi$, he serves a batch of min ($\xi, b$) customers, where $b \geq a$. Analytical treatment of this model is obtained by the supplementary variable technique. The model under study is schematically represented in Figure 2.1.

![Figure 2.1 Schematic Representation of the Queueing Model](Q – Queue Length)
The motivation of the model comes from a real life situation observed in the Environmental Sensor Networks (ESN). ESN system can potentially provide a new data for environmental science (eg. climate models) as well as vital hazard warnings (eg. flood alerts) etc. This is particularly important in remote or dangerous environments where many fundamental processes have rarely been studied due to their inaccessibility. A sensor network is designed to transmit the data from an array of sensors to a data repository on a server. Monitoring the behavior of ice caps and glaciers is an important part of our understanding of the Earth’s climate. The environmental sensor nodes gather data such as glaciers, movement of stones and sediment under the ice, temperature, pressure, vibration etc. autonomously and pass the data to the cluster heads. The cluster heads pass the gathered messages (customers) to the base station. After obtaining the required information, the base station operator generates various reports and transmits (service) the report to the application nodes. After getting the reports, the application node may request some more reports with the same data (re-service). The request may be accepted or rejected by the base station based on the importance of the current report generation at that instant. If the number of messages received by the base station to produce the report is inadequate, then the base station will do some other associated work such as antivirus running, backup process, etc., At the completion epoch of the associated work (vacations), if the number of messages is inadequate, then, the base station will repeat the associated works until the required number of information reaches to process it. When the operator returns from the associated work and finds the required number of messages available in the queue, then the operator begins some preparatory work, such as checking the application nodes, type of reports required, etc, for which some amount of time is required, called as setup time. The above situation can be modeled as $M^{X}/G(a,b)/1$ queueing system with multiple vacations, setup time and server’s choice of admitting re-service.
For the proposed model, the probability generating function (PGF) of the steady state queue size distribution at an arbitrary time epoch is obtained using supplementary variable technique. Particular cases and some special cases are discussed. Various performance measures are derived. A cost model for the queueing system is developed. Numerical solution for particular values of parameters is presented.

2.2 MATHEMATICAL MODEL

Let $X$ be the group size random variable of the arrival, $\lambda$ be the Poisson arrival rate, $g_k$ be the probability that ‘$k$’ customers arrive in a batch and $X(z)$ be its probability generating function (PGF). Let ‘$\pi$’ be the probability that a leaving batch request for re-service and ‘$\alpha$’ be the probability that the server accepts a re-service. Let $S(x)$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform}[remaining service time] of service. Let $V(x)$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform}[remaining vacation time] of vacation. Let $U(x)$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform}[remaining set up time] of set up. Let $R(x)$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform}[remaining re-service time] of re-service. $N_q(t)$ denotes the number of customers waiting for service at time $t$, $N_s(t)$ denotes the number of customers under service at time $t$. The different states of the server at time ‘$t$’ are defined as follows:
C(t) = \begin{cases} 
0, & \text{if the server is busy with service} \\
1, & \text{if the server is busy with re-service} \\
2, & \text{if the server is on vacation} \\
3, & \text{if the server is on setup work} 
\end{cases}

Z(t) = j \text{ if the server is on } j^{th} \text{ vacation starting from the idle period}

To obtain the system equations, the following state probabilities are define

\[
P_{i,j}(x,t) = \Pr \{N_s(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + dt, C(t) = 0\}, \quad a \leq i \leq b, \quad j \geq 0
\]

\[
R_n(x,t) = \Pr \{N_q(t) = n, x \leq R^0(t) \leq x + dt, C(t) = 1\}, \quad n \geq 0
\]

\[
Q_{j,n}(x,t) = \Pr \{N_q(t) = n, x \leq V^0(t) \leq x + dt, C(t) = 2, Z(t) = j\}, \quad j \geq 1, \quad n \geq 0 \text{ and}
\]

\[
U_n(x,t) = \Pr \{N_q(t) = n, x \leq U^0(t) \leq x + dt, C(t) = 3\}, \quad n \geq a
\]

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

\[
P_{i,0}(x - \Delta t, t + \Delta t) = P_{i,0}(x, t)(1-\lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{m,i}(0,t)s(x)\Delta t
\]

\[
+ \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,i}(0,t)s(x)\Delta t + R_i(0,t)s(x)\Delta t + U_i(0,t)s(x)\Delta t
\]

\[
a \leq i \leq b
\]

\[
P_{i,j}(x - \Delta t, t + \Delta t) = P_{i,j}(x, t)(1-\lambda \Delta t) + \sum_{k=1}^{j} P_{i,j-k}(x, t)\lambda g_k \Delta t, \quad a \leq i \leq b - 1, \quad j \geq 1
\]

\[
P_{b,j}(x - \Delta t, t + \Delta t) = P_{b,j}(x, t)(1-\lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{m,b+j}(0,t)s(x)\Delta t
\]

\[
+ \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,b+j}(0,t)s(x)\Delta t + \sum_{k=1}^{j} P_{b,j-k}(x, t)\lambda g_k \Delta t
\]

\[
+ R_{b+j}(0,t)s(x)\Delta t + U_{b+j}(0,t)s(x)\Delta t; \quad j \geq 1
\]
\[ Q_{1,0}(x - \Delta t, t + \Delta t) = Q_{1,0}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=0}^{b} P_{m,0}(0, t) v(x) \Delta t \]
\[ + \pi(1 - \alpha) \sum_{m=0}^{b} P_{m,0}(0, t) v(x) \Delta t + R_0(0, t) v(x) \Delta t \]

\[ Q_{1,n}(x - \Delta t, t + \Delta t) = Q_{1,n}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=0}^{b} P_{m,n}(0, t) v(x) \Delta t \]
\[ + \pi(1 - \alpha) \sum_{m=0}^{b} P_{m,n}(0, t) v(x) \Delta t + \sum_{k=1}^{n} Q_{1,n-k}(x, t) \lambda g_k \Delta t \]
\[ + R_n(0, t) v(x) \Delta t, \quad 1 \leq n \leq a - 1 \]

\[ Q_{1,n}(x - \Delta t, t + \Delta t) = Q_{1,n}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{1,n-k}(x, t) \lambda g_k \Delta t, \quad n \geq a \]

\[ Q_{j,0}(x - \Delta t, t + \Delta t) = Q_{j,0}(x, t)(1 - \lambda \Delta t) + Q_{j-1,0}(0, t) v(x) \Delta t, \quad j \geq 2 \]

\[ Q_{j,n}(x - \Delta t, t + \Delta t) = Q_{j,n}(x, t)(1 - \lambda \Delta t) + Q_{j-1,n}(0, t) v(x) \Delta t \]
\[ + \sum_{k=1}^{n} Q_{j,n-k}(x, t) \lambda g_k \Delta t, \quad 1 \leq n \leq a - 1, j \geq 2 \]

\[ Q_{j,n}(x - \Delta t, t + \Delta t) = Q_{j,n}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{j,n-k}(x, t) \lambda g_k \Delta t, n \geq a, j \geq 2 \]

\[ R_0(x - \Delta t, t + \Delta t) = R_0(0, t)(1 - \lambda \Delta t) + \pi \alpha \sum_{m=0}^{b} P_{m,0}(0, t) r(x) \Delta t \]

\[ R_n(x - \Delta t, t + \Delta t) = R_n(0, t)(1 - \lambda \Delta t) + \pi \alpha \sum_{m=0}^{b} P_{m,n}(0, t) r(x) \Delta t + \sum_{k=1}^{n} R_{n-k}(x, t) \lambda g_k \Delta t \]
\[ n \geq 1 \]

\[ U_n(x - \Delta t, t + \Delta t) = U_n(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n-a} U_{n-k}(x, t) \lambda g_k \Delta t + \sum_{l=1}^{\infty} Q_{l,0}(0, t) u(x) \Delta t, n \geq a \]
2.3 STEADY STATE QUEUE SIZE DISTRIBUTION

From the above equations, the steady state queue size equations are obtained as follows:

\[ \frac{d}{dx} P_{i,0}(x) = -\lambda P_{i,0}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m,i}(0)s(x) + \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,i}(0)s(x) + R_{i}(0)s(x) + U_{i}(0)s(x); \quad a \leq i \leq b \]  \hspace{1cm} (2.1)

\[ \frac{d}{dx} P_{i,j}(x) = -\lambda P_{i,j}(x) + \lambda \sum_{k=1}^{i} P_{i,j-k}(x)g_{k}, \quad a \leq i \leq b - 1, \quad j \geq 1 \]  \hspace{1cm} (2.2)

\[ \frac{d}{dx} P_{b,j}(x) = -\lambda P_{b,j}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m,b+j}(0)s(x) + \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,b+j}(0)s(x) + R_{b,j}(0)s(x) + U_{b,j}(0)s(x); \quad j \geq 1 \]  \hspace{1cm} (2.3)

\[ \frac{d}{dx} Q_{1,0}(x) = -\lambda Q_{1,0}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m,0}(0)v(x) + \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,0}(0)v(x) \]  \hspace{1cm} (2.4)

\[ \frac{d}{dx} Q_{1,n}(x) = -\lambda Q_{1,n}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m,n}(0)v(x) + \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,n}(0)v(x) + R_{1,n}(0)v(x); \quad 1 \leq n \leq a - 1 \]  \hspace{1cm} (2.5)

\[ \frac{d}{dx} Q_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda \sum_{k=1}^{n} Q_{1,n-k}(x)g_{k}, \quad n \geq a \]  \hspace{1cm} (2.6)

\[ \frac{d}{dx} Q_{j,0}(x) = -\lambda Q_{j,0}(x) + Q_{j-1,0}(0)v(x); \quad j \geq 2 \]  \hspace{1cm} (2.7)

\[ \frac{d}{dx} Q_{j,n}(x) = -\lambda Q_{j,n}(x) + Q_{j-1,n}(0)v(x) + \lambda \sum_{k=1}^{n} Q_{j,n-k}(x)g_{k}, \quad 1 \leq n \leq a - 1, \quad j \geq 2 \]  \hspace{1cm} (2.8)
\[ \frac{d}{dx} Q_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda \sum_{k=1}^{n} Q_{j,n-k}(x)g_k, \quad n \geq a, \ j \geq 2 \quad (2.9) \]

\[ \frac{d}{dx} R_0(x) = -\lambda R_0(x) + \pi \alpha \sum_{m=a}^{b} P_{m,0}(0)r(x) \quad (2.10) \]

\[ \frac{d}{dx} R_n(x) = -\lambda R_n(x) + \pi \alpha \sum_{m=a}^{b} P_{m,n}(0)r(x) + \lambda \sum_{k=1}^{n} R_{n-k}(x)g_k, \quad n \geq 1 \quad (2.11) \]

\[ \frac{d}{dx} U_n(x) = -\lambda U_n(x) + \sum_{k=1}^{n-a} U_{n-k}(x)\lambda g_k + \sum_{l=1}^{\infty} Q_{l,n}(0)u(x), \quad n \geq a \quad (2.12) \]

The Laplace–Stieltjes transforms (LST) of \( P_{j,n}(x) \), \( Q_{j,n}(x) \), \( U_n(x) \) and \( R_n(x) \) are defined as

\[ \tilde{P}_{j,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} P_{j,n}(x)dx, \quad \tilde{Q}_{j,n}(\theta) = \int_{0}^{\infty} e^{-\theta x} Q_{j,n}(x)dx, \]

\[ \tilde{U}_n(\theta) = \int_{0}^{\infty} e^{-\theta x} U_n(x)dx \quad \text{and} \quad \tilde{R}_n(\theta) = \int_{0}^{\infty} e^{-\theta x} R_n(x)dx \]

Taking LST on both sides of the Equations (2.1) through (2.12), we have

\[ \theta \tilde{P}_{i,0}(\theta) - P_{i,0}(0) = \lambda \tilde{P}_{i,0}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{m,i}(0)\tilde{S}(\theta) - \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,i}(0)\tilde{S}(\theta) \]

\[ = -R_{i,0}(0)\tilde{S}(\theta) - U_{i,0}(0)\tilde{S}(\theta), \quad a \leq i \leq b \quad (2.13) \]

\[ \theta \tilde{P}_{i,j}(\theta) - P_{i,j}(0) = \lambda \tilde{P}_{i,j}(\theta) - \lambda \sum_{k=1}^{j} \tilde{P}_{i,j-k}(\theta)g_k, \quad a \leq i \leq b-1, \ j \geq 1 \quad (2.14) \]

\[ \theta \tilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda \tilde{P}_{b,j}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}(\theta) \]

\[ = -\pi(1 - \alpha) \sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}(\theta) - \lambda \sum_{k=1}^{j} \tilde{P}_{b,j-k}(\theta)g_k \]

\[ = -R_{b+j}(0)\tilde{S}(\theta) - U_{b+j}(0)\tilde{S}(\theta), \quad j \geq 1 \quad (2.15) \]
\[ 0Q_{1,0}(0) - Q_{1,0}(0) \]
\[ \lambda \tilde{Q}_{1,0}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{m,0}(0) \tilde{V}(\theta) - \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,0}(0) \tilde{V}(\theta) - R_{0}(0) \tilde{V}(\theta) \]  
(2.16)

\[ 0Q_{1,n}(0) - Q_{1,n}(0) \]
\[ \lambda \tilde{Q}_{1,n}(\theta) - (1 - \pi) \sum_{m=a}^{b} P_{m,n}(0) \tilde{V}(\theta) - \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,n}(0) \tilde{V}(\theta) - \lambda \sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta)g_{k} - R_{n}(0) \tilde{V}(\theta), \quad 1 \leq n \leq a - 1 \]  
(2.17)

\[ 0Q_{j,0}(0) - Q_{j,0}(0) \]
\[ \lambda \tilde{Q}_{j,0}(\theta) - Q_{j-1,0}(0) \tilde{V}(\theta) \]  
(2.19)

\[ 0Q_{j,n}(0) - Q_{j,n}(0) \]
\[ \lambda \tilde{Q}_{j,n}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda g_{k} - Q_{j-1,n}(0) \tilde{V}(\theta), \quad 1 \leq n \leq a - 1; j \geq 2 \]  
(2.20)

\[ 0Q_{j,n}(0) - Q_{j,n}(0) \]
\[ \lambda \tilde{Q}_{j,n}(\theta) - \lambda \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)g_{k}, \quad n \geq a; j \geq 2 \]  
(2.21)

\[ 0R_{0}(0) - R_{0}(0) \]
\[ \tilde{R}_{0}(\theta - \pi \alpha) \sum_{m=a}^{b} P_{m,0}(0) \tilde{R}(\theta) \]  
(2.22)

\[ 0R_{n}(0) - R_{n}(0) = \lambda \tilde{R}_{n}(0) - \pi \alpha \sum_{m=a}^{b} P_{m,n}(0) \tilde{R}(\theta) - \sum_{k=1}^{n} \tilde{R}_{n-k}(\theta)\lambda g_{k}, \quad n \geq 1 \]  
(2.23)

\[ 0U_{n}(0) - U_{n}(0) = \lambda \tilde{U}_{n}(\theta) - \lambda \sum_{k=1}^{n} \tilde{U}_{n-k}(\theta)\lambda g_{k} - \sum_{l=1}^{\infty} Q_{l,n}(0) \tilde{U}(\theta); \quad n \geq a \]  
(2.24)

### 2.3.1 Probability Generating Function

Lee (1991) developed a new technique to find the steady state probability generating function (PGF) of the number of customers in the queue at an arbitrary time epoch. To apply the technique, first the following probability generating functions are defined.
\[ \tilde{P}_i(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{i,j}(z) j^j \quad \text{and} \quad P_i(z,0) = \sum_{j=0}^{\infty} P_{i,j}(0) j^j \quad a \leq i \leq b \]

\[ \tilde{Q}_j(z, \theta) = \sum_{n=0}^{\infty} \tilde{Q}_{j,n}(z) n^j \quad \text{and} \quad Q_j(z,0) = \sum_{n=0}^{\infty} Q_{j,n}(0) n^j \quad j \geq 1 \]

\[ \tilde{R}_n(z, \theta) = \sum_{n=0}^{\infty} \tilde{R}_n(z) n^n \quad \text{and} \quad R(z,0) = \sum_{n=0}^{\infty} R_n(0) n^n \]

\[ \tilde{U}_n(z, \theta) = \sum_{n=a}^{\infty} \tilde{U}_n(z) n^n \quad \text{and} \quad U(z,0) = \sum_{n=a}^{\infty} U_n(0) n^n \quad (2.25) \]

where \( \sum_{i=a}^{b} \tilde{P}_i(z,0) \) is the PGF of joint transform of the number of customers in the queue during busy period. At \( \theta = 0 \), one can get the PGF of the number of customers in the queue during busy period at an arbitrary time. Similarly, the other joint transforms \( \tilde{Q}_j(z, \theta) \), \( \tilde{R}_n(z, \theta) \) and \( \tilde{U}_n(z, \theta) \) are defined.

The probability generating function \( P(z) \) of the number of customers in the queue at an arbitrary time of the proposed model can be obtained using the following equation.

\[ P(z) = \sum_{i=a}^{b-1} P_i(z,0) + P_b(z,0) + \sum_{j=1}^{\infty} \tilde{Q}_j(z,0) + \tilde{R}(z,0) + \tilde{U}(z,0) \quad (2.26) \]

In order to find \( \tilde{P}_i(z,0), \tilde{P}_b(z,0), \tilde{Q}_j(z,0), \tilde{R}(z,0) \) and \( \tilde{U}(z,0) \), the following sequence of operations are done.

Multiplying (2.16) by \( z^0 \), (2.17) by \( z^n \) \((1 \leq n \leq a-1)\), (2.18) by \( z^n \) \((n \geq a)\), summing up from \( n = 0 \) to \( \infty \) and using (2.25), we get

\[ (\theta - \lambda + \lambda X(z)) \tilde{Q}_1(z,0) = Q_1(z,0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} \left( 1 - \pi \alpha \right) \sum_{m=a}^{b} P_{m,n}(0) n^n + R_n(0) n^n \quad (2.27) \]
Multiplying (2.19) by \( z^0 \), (2.20) by \( z^n \) \((1 \leq n \leq a-1)\), (2.21) by \( z^n \) \((n \geq a)\), summing up from \( n = 0 \) to \( \infty \) and using (2.25), we get

\[
(\theta - \lambda + \lambda X(z))\tilde{Q}_j(z, \theta) = Q_j(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n, \quad j \geq 2
\]

(2.28)

Multiplying (2.22) by \( z^0 \), (2.23) by \( z^n \) \((n \geq 1)\), summing up from \( n = 0 \) to \( \infty \) and using (2.25), we get

\[
(\theta - \lambda + \lambda X(z))\tilde{R}(z, \theta) = R(z, 0) - \pi \alpha \tilde{R}(\theta) \sum_{m=a}^{b} P_m(z, 0)
\]

(2.29)

Multiplying (2.24) by \( z^n \) \((n \geq a)\), summing up from \( n = a \) to \( \infty \) and using (2.25), we get

\[
(\theta - \lambda + \lambda X(z))\tilde{U}(z, \theta) = U(z, 0) - \tilde{U}(\theta) \sum_{l=1}^{\infty} \left( Q_1(z, 0) - \sum_{n=0}^{a-1} Q_{1,n}(0)z^n \right)
\]

(2.30)

Multiplying (2.13) by \( z^0 \), (2.14) by \( z^j \) \((j \geq 1)\), summing up from \( j = 0 \) to \( \infty \) and using (2.25), we get

\[
(\theta - \lambda + \lambda X(z))\tilde{P}_i(z, \theta) = P_i(z, 0) - \tilde{S}(\theta) \{ (1 - \pi) \sum_{m=a}^{b} P_{m,i}(0) \}
\]

\[+ \pi (1 - \alpha) \sum_{m=a}^{b} P_{m,i}(0) + R_i(0) + U_i(0) \}, \quad a \leq i \leq b-1
\]

(2.31)

Multiplying (2.13) by \( z^0 \) with \( i = b \), (2.15) by \( z^j \) \((j \geq 1)\), summing up from \( j = 0 \) to \( \infty \) and using (2.25), we get
(θ - λ + λX(z))z^b \tilde{P}_b(z, 0) = z^b P_b(z, 0)

By substituting \( \theta = \lambda - \lambda X(z) \) in the Equations (2.27) through (2.32), we get

\[
Q_1(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \left( (1 - \alpha) \sum_{m=a}^{b} P_{m,n}(0)z^n + R_n(0)z^n \right)
\]

(2.33)

\[
Q_j(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n \quad j \geq 2
\]

(2.34)

\[
R(z, 0) = \pi \alpha \tilde{R}(\lambda - \lambda X(z)) \sum_{m=a}^{b} P_m(z, 0)
\]

(2.35)

\[
U(z, 0) = \tilde{U}(\lambda - \lambda X(z)) \sum_{i=1}^{\infty} \left( Q_1(z, 0) - \sum_{n=0}^{a-1} Q_{1,n}(0)z^n \right)
\]

(2.36)

\[
P_i(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left( (1 - \pi) \sum_{m=a}^{b} P_{m,i}(0) + \pi(1 - \alpha) \sum_{m=a}^{b} P_{m,i}(0) + R_i(0) + U_i(0) \right)
\]

\[ a \leq i \leq b-1 \]

(2.37)

and

\[
z^b P_b(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left( (1 - \pi) \sum_{m=a}^{b} P_{m,z}(0) - \sum_{j=0}^{b-1} P_{m,j}(0)z^j \right)
\]

\[
+ \pi(1 - \alpha) \sum_{m=a}^{b} \left( P_{m,z}(0) - \sum_{j=0}^{b-1} P_{m,j}(0)z^j \right)
\]

\[
+ \left( R(z, 0) - \sum_{n=0}^{b-1} R_n(0)z^n \right) + \left( U(z, 0) - \sum_{n=0}^{b-1} U_n(0)z^n \right)
\]

(2.38)
Let 
\[ p_i = \sum_{m=a}^{b} P_{m,i}(0), \quad q_i = \sum_{l=1}^{\infty} Q_{1,i}(0), \quad u_i = U_i(0), \quad r_i = R_i(0) \]
and
\[ k_i = (1 - \pi \alpha) p_i + q_i + r_i, \quad c_i = (1 - \pi \alpha) p_i + u_i + r_i. \]

From the Equation (2.38), we get
\[ P_b(z,0) = \frac{\tilde{S}(\lambda - \lambda X(z)) f(z)}{z^b - (1 - \pi \alpha) \tilde{S}(\lambda - \lambda X(z)) - \pi \alpha \tilde{S}(\lambda - \lambda X(z)) \tilde{R}(\lambda - \lambda X(z))} \quad (2.39) \]
where \[ f(z) = \sum_{m=a}^{b-1} \left( \left( (1 - \pi \alpha) + \pi \alpha \tilde{R}(\lambda - \lambda X(z)) \right) \tilde{S}(\lambda - \lambda X(z)) \right) c_m \]
\[ + \sum_{m=0}^{a-1} \left( \tilde{U}(\lambda - \lambda X(z)) \left( \tilde{V}(\lambda - \lambda X(z)) k_m - q_m \right) - \left( (1 - \pi \alpha) p_m + r_m \right) \right) z^m \]

From the Equations (2.27) and (2.33), we get
\[ \tilde{Q}_1(z,\theta) = \frac{1}{(0 - \lambda + \lambda X(z))} \left( \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta) \right) a \sum_{n=0}^{a-1} \left( (1 - \pi \alpha) p_n + r_n \right) z^n \quad (2.40) \]

From the Equations (2.28) and (2.34), we get
\[ \tilde{Q}_j(z,\theta) = \frac{1}{(0 - \lambda + \lambda X(z))} \left( \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta) \right) a \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \quad j \geq 2 \quad (2.41) \]

From the Equations (2.29) and (2.35), we get
\[ \tilde{R}(z,\theta) = \frac{1}{(0 - \lambda + \lambda X(z))} \left( \tilde{R}(\lambda - \lambda X(z)) - \tilde{R}(\theta) \right) \pi \alpha \sum_{n=a}^{b-1} P_m(z,0) \quad (2.42) \]

From the Equations (2.30) and (2.36), we get
\[ \tilde{U}(z,\theta) = \frac{1}{(0 - \lambda + \lambda X(z))} \left( \tilde{U}(\lambda - \lambda X(z)) - \tilde{U}(\theta) \right) \left( \sum_{i=1}^{\infty} Q_{1,i}(z,0) - \sum_{n=0}^{a-1} q_n z^n \right) \quad (2.43) \]
From the Equations (2.31) and (2.37), we get

\[
\bar{P}_i(z,0) = \frac{1}{(\theta - \lambda + \lambda X(z))} \left( \tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(0) \right) c_i, \quad a \leq i \leq b - 1 \quad (2.44)
\]

From the Equations (2.32) and (2.39), we get

\[
\bar{P}_b(z,0) = \frac{(\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(0)) f(z)}{(\theta - \lambda + \lambda X(z)) \left( z^b - (1 - \pi\alpha)\tilde{S}(\lambda - \lambda X(z)) - \pi\alpha \tilde{S}(\lambda - \lambda X(z)) R(\lambda - \lambda X(z)) \right)} \quad (2.45)
\]

Substituting \( \bar{P}_i(z,0), \bar{P}_b(z,0), \bar{Q}_j(z,0), R(z,0) \) and \( \bar{U}(z,0) \) from the Equations (2.40) – (2.45) in the Equation (2.26), the probability generating function of the queue size \( P(z) \) at an arbitrary time epoch is obtained as

\[
P(z) = \frac{\left( \tilde{S}(\lambda - \lambda X(z)) - 1 \right) \sum_{i=a}^{b-1} c_i}{-\lambda + \lambda X(z)} + \frac{f(z) \left( \tilde{S}(\lambda - \lambda X(z)) - 1 \right)}{h(z)(-\lambda + \lambda X(z))} + \\
\frac{\left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) \sum_{i=0}^{a-1} k_i z^i}{-\lambda + \lambda X(z)} + \frac{\pi\alpha \left( \tilde{R}(\lambda - \lambda X(z)) - 1 \right) \sum_{m=a}^{b-1} \tilde{S}(\lambda - \lambda X(z)) c_m}{(-\lambda + \lambda X(z))} \quad (2.46)
\]

\[
+ \frac{\pi\alpha \left( \tilde{R}(\lambda - \lambda X(z)) - 1 \right) \tilde{S}(\lambda - \lambda X(z)) f(z)}{h(z)(-\lambda + \lambda X(z))} + \\
\frac{\left( \tilde{U}(\lambda - \lambda X(z)) - 1 \right) \left( \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} k_n z^n - \sum_{n=0}^{a-1} q_n z^n \right)}{(-\lambda + \lambda X(z))}
\]
On further simplification, we get

\[
P(z) = \frac{\left( \tilde{S}(\lambda - \lambda X(z)) - 1 \right) + \sum_{m=a}^{b-1} \left( z^b - z^m \right) c_m}{h(z)(\lambda + \lambda X(z))} \quad (2.46)
\]

where \( h(z) = z^b - (1 - \pi \alpha) \tilde{S}(\lambda - \lambda X(z)) - \pi \alpha \tilde{S}(\lambda - \lambda X(z)) \tilde{R}(\lambda - \lambda X(z)) \)

The probability generating function \( P(z) \) has to satisfy the condition \( P(1) = 1 \). In order to satisfy the condition, applying L’Hospital’s rule in and evaluating \( \lim_{z \to 1} P(z) \) and equating the expression to 1, we have to satisfy

\[
(2\lambda E(X)(E(S) + \pi \alpha E(R))) \left( \sum_{m=a}^{b-1} (b - m) c_m \right)
- \varphi(X,S,R) \sum_{m=0}^{a-1} ((1 - \pi \alpha)p_m + r_m) + \psi(X,S,R) \sum_{m=0}^{a-1} (q_m - k_m)
+ \sum_{m=0}^{a-1} 2b\lambda E(X)E(V)k_m = 2\lambda E(X)B - \lambda E(X)(E(S) + \pi \alpha E(R))
\]

where

\[
\varphi(X,S,R) = 2\lambda mE(S)E(X) + \lambda^2 E^2(X)E(S^2) + \lambda E(S)E(X^2)
+ \pi \alpha \left( 2\lambda mE(R)E(X) + 2\lambda^2 E^2(X)E(R) + \lambda E(R)E(X^2) + \lambda^2 E(R^2)E^2(X) \right)
\]

and
\begin{equation}
40
\psi \in X, S, R)
\begin{align*}
2m & (b - \lambda \mathbb{E}(X) \mathbb{E}(S) - \lambda \pi \alpha \mathbb{E}(X) \mathbb{E}(R)) + b(b - 1) - \lambda \mathbb{E}(X^2) \mathbb{E}(S) \\
- \lambda^2 & \mathbb{E}^2(X) \mathbb{E}(S^2) - \lambda^2 \pi \alpha \mathbb{E}^2(X) \mathbb{E}(R^2) - 2 \lambda^2 \pi \alpha \mathbb{E}^2(X) \mathbb{E}(S) \mathbb{E}(R) \\
- \lambda \pi \alpha & \mathbb{E}(X^2) \mathbb{E}(R) + m(m - 1) - (m + b)(m + b - 1) - 2b \lambda \mathbb{E}(X) \mathbb{E}(G)
\end{align*}
\end{equation}

Since \( p_i, q_i, r_i \) and \( u_i \) are the probabilities of ‘i’ customers being in the queue at service completion epoch, vacation completion epoch, re-service completion epoch and set up time completion epoch, respectively, it follows that the left hand side of the above expression must be positive. Thus \( P(1) = 1 \) is satisfied if and only if \( b - \lambda \mathbb{E}(X)(\mathbb{E}(S) + \pi \mathbb{E}(R)) > 0 \).

Define ‘\( \rho \)’ as \( \frac{\lambda \mathbb{E}(X)(\mathbb{E}(S) + \pi \mathbb{E}(R))}{b} \). Thus \( \rho < 1 \) is the condition to be satisfied for the existence of steady state for the model under consideration.

\subsection{2.3.2 Computational Aspects of Unknown Probabilities}

Equation (2.46) gives the probability generating function \( P(z) \) of the number of customers in the queue at an arbitrary time epoch, which involves \( b + 2a \) unknown probabilities namely, \( p_0, p_1, p_2, \ldots, p_{a-1}, r_0, r_1, r_2, \ldots, r_{a-1}, q_0, q_1, q_2, \ldots, q_{a-1}, c_a, c_{a+1}, \ldots, c_{b-1} \). Using Lemma (2.1) and theorems (2.1) and (2.2), \( q_i \) and \( r_i \); \( i = 0 \) to \( a-1 \) are expressed in terms of \( p_i \); \( i = 0 \) to \( a-1 \).

Now, the Equation (2.46) contains only \( 'b' \) unknowns \( p_0, p_1, p_2, \ldots, p_{a-1}, c_a, c_{a+1}, \ldots, c_{b-1} \). By Rouche’s theorem, the expression \( z^b - (1 - \pi \alpha) \tilde{S}(\lambda - \lambda X(z)) - \pi \alpha \tilde{S}(\lambda - \lambda X(z)) \tilde{R}(\lambda - \lambda X(z)) \) has \( b-1 \) zeros inside and one on the unit circle \(|z|=1\). Since \( P(z) \) is analytic within...
and on the unit circle, the numerator of (2.46) must vanish at these points, which gives ‘b’ equations and ‘b’ unknowns. These equations can be solved by suitable numerical techniques.

Lemma 2.1

Let \( \beta_i \) be the probability that ‘i’ customers arrive during a vacation. The probability generating function of \( \beta_i \) is given by

\[
\sum_{i=0}^{\infty} \beta_i z^i = \bar{V}(\lambda - \lambda X(z))
\]

Proof:

Conditioning on the actual vacation length, number of arrivals and the group size, we get

\[
\beta_i = \int_0^\infty \left( \sum_{m=0}^i \frac{(e^{-\lambda t})(\lambda t)^m g_i^{(m)}}{m!} \right) dV(t)
\]

where \( g_i^{(m)} \) is the m – fold convolution of \( g_i \) with itself (i.e., total of m arrivals make ‘i’ customers)

Multiplying the above equation by \( z^i \) and taking the summation from \( i = 0 \) to \( \infty \), we get

\[
\sum_{i=0}^{\infty} \beta_i z^i = \int_0^\infty e^{-\lambda t} \left( \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \sum_{i=m}^{\infty} g_i^{(m)} z^i \right) dV(t)
\]
\[
\int_{0}^{\infty} e^{-\lambda t} \left( \sum_{m=0}^{\infty} \frac{(\lambda t)^m [X(z)]^m}{m!} \right) dV(t)
\]

\[
= \hat{V}(\lambda - \lambda X(z))
\]

(2.47)

Hence the Lemma \( \square \)

**Theorem: 2.1**

The constants \( r_n \) involved in \( P(z) \) are expressed in terms of \( p_n \) as,

\[
r_n = \pi \alpha \sum_{i=0}^{n} \omega_i p_{n-i} \text{, } n=0,1,2,\ldots a-1, \text{ where } \omega_i \text{ is the probability that ‘i’ customers arrive during a re-service period.}
\]

**Proof:**

From the Equation (2.35), we have

\[
R(z,0) = \pi \alpha \sum_{m=a}^{b} P_m(z,0) \tilde{R}(\lambda - \lambda X(z))
\]

\[
\sum_{n=0}^{\infty} R_n(0) z^n = \pi \alpha \tilde{R}(\lambda - \lambda X(z)) \sum_{m=a}^{b} \sum_{j=0}^{\infty} P_{m,j}(0) z^j
\]

\[
\sum_{n=0}^{\infty} r_n z^n = \pi \alpha \tilde{R}(\lambda - \lambda X(z)) \left( \sum_{j=0}^{\infty} p_j z^j \right)
\]

which, using Lemma 2.1, gives

\[
\sum_{n=0}^{\infty} r_n z^n = \pi \alpha \left( \sum_{n=0}^{\infty} \omega_n z^n \right) \left( \sum_{j=0}^{\infty} p_j z^j \right)
\]

Equating the coefficients of \( z^n \), \( n=0,1,2,\ldots a-1 \), we get
\[ r_n = \pi a \sum_{i=0}^{n} \omega_i p_{n-i} \quad ; \quad n=0,1,\ldots,a-1 \]  (2.48)

Hence the theorem \( \Box \)

**Theorem: 2.2**

The constants \( q_n \) involved in \( P(z) \) are expressed in terms of \( p_n \) as,

\[ q_n = \frac{\sum_{i=0}^{n} b_{n-i} p_i}{\cdot n = 0,1,2,3,\ldots,a-1, \text{ where } b_0 = \frac{\beta_0(1-\pi\alpha) + \pi\alpha\beta_0\omega_0}{1-\beta_0} \]

and

\[ b_n = \frac{(1-\pi\alpha)\beta_n + \pi\alpha\sum_{i=0}^{n} \beta_{n-i} \omega_i + \sum_{i=1}^{n} \beta_i b_{n-i}}{1-\beta_0}; \quad n = 1,2,3,\ldots,a-1 \quad \text{and} \quad \beta_i \]

is the probability that ‘i’ customers arrive during a vacation period.

**Proof:**

From the Equations (2.33) and (2.34), we have

\[ \sum_{j=1}^{\infty} Q_{j}(z,0) = \hat{V}(\lambda - \lambda X(z)) \left( a_1 \sum_{n=0}^{a-1} \left( (1-\pi\alpha)p_n + r_n \right) z^n + \sum_{n=0}^{a-1} q_n z^n \right) \]

\[ \sum_{n=0}^{\infty} q_n z^n = \hat{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \left( (1-\pi\alpha)p_n + r_n + q_n \right) z^n \]

which, using Lemma 2.1, gives

\[ \sum_{n=0}^{\infty} q_n z^n = \left( \sum_{n=0}^{\infty} \beta_n z^n \right) \sum_{n=0}^{a-1} \left( (1-\pi\alpha)p_n + r_n + q_n \right) z^n \]

\[ = a_1 \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \beta_{n-i} \left( (1-\pi\alpha)p_i + r_i + q_i \right) \right) z^n + \sum_{n=a}^{\infty} \left( \sum_{i=0}^{a-1} \beta_{n-i} \left( (1-\pi\alpha)p_i + r_i + q_i \right) \right) z^n \]
Equating the coefficients of $z^n$ on both sides of the above equation for $n = 0, 1, 2, 3, \ldots a-1$, we get

$$q_n = \sum_{i=0}^{n} \beta_{n-i} ((1-\pi\alpha)p_i + r_i + q_i)$$

Substituting $r_i$ and solving it for $q_0$, we get

$$q_n = \frac{\sum_{i=0}^{n} \beta_{n-i} p_i + \pi\alpha \sum_{i=0}^{n} \beta_{n-k-i} \omega_i p_k + \sum_{i=0}^{n-1} \beta_{n-i} q_i}{1-\beta_0}$$

Coefficient of $p_n$ in $q_n$ is

$$\frac{\beta_0 (1-\pi\alpha) + \pi\alpha \beta_0 \omega_0}{1-\beta_0} = b_0$$

Coefficient of $p_{n-1}$ in $q_n$ is $[\beta_1 + \beta_1 \text{Coefficient of } p_{n-1} \text{ in } q_{n-1}]/1-\beta_0$

$$= \frac{\beta_1 (1-\pi\alpha) + \pi\alpha (\beta_1 \omega_0 + \beta_0 \omega_1) + \beta_1 b_0}{1-\beta_0} = b_1$$

Coefficient of $p_{n-2}$ in $q_n$ is $[\beta_2 + \beta_1 \text{Coefficient of } p_{n-2} \text{ in } q_{n-1}] + \beta_2 \text{Coefficient of } p_{n-2} \text{ in } q_{n-2}/1-\beta_0$

$$= \frac{\beta_2 (1-\pi\alpha) + \pi\alpha (\beta_2 \omega_0 + \beta_1 \omega_1 + \beta_0 \omega_2) + (\beta_1 b_1 + \beta_2 b_0)}{1-\beta_0} = b_2$$

Proceeding like this, we get

Coefficient of $p_0$ in $q_0$ is

$$\frac{(1-\pi\alpha)\beta_0 + \pi\alpha \sum_{i=0}^{n} \beta_{n-i} \omega_1 + \sum_{i=1}^{n} \beta_1 b_{n-i}}{1-\beta_0} = b_n$$

Therefore, $q_n = \sum_{i=0}^{n} b_{n-i} p_i$; \quad $n = 0, 1, 2, 3, \ldots a-1$ (2.49)

Hence the theorem \square
2.4 PERFORMANCE MEASURES

In this section, some useful performance measures of the proposed model like, expected number of customers in the queue $E(Q)$, expected length of idle period $E(I)$, expected length of busy period $E(B)$ are derived which are useful to find the total average cost of the system. Also, probability that the server is on setup work $P(U)$, probability that the server is on vacation $P(V)$ and probability that the server is busy $P(B)$ are derived.

2.4.1 Expected Queue Length

The expected queue length $E(Q)$ (i.e. mean number of customers waiting in the queue) at an arbitrary time epoch, is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$E(Q) = \lim_{z \to 1} P(z) = E(Q)$$

$$E(Q) = \frac{1}{24\lambda^2 E^2(X)T^2} \left\{ \sum_{m=a}^{b-1} f_1(X,S,R)c_m + \sum_{m=0}^{a-1} f_2(X,S,R,U,V)k_m \right\}$$

where $S1 = \lambda E(X)E(S), \ S2 = \lambda X^*(1)E(S) + \lambda^2 E^2(X)E(S^2), \ S3 = \lambda E^2(X)E(S)E(R), \ S4 = \lambda X^*(1)E(S), \ S5 = \lambda^2 X^*(1)E(X)E(S^2), \ S6 = \lambda^3 E^3(X)E(S^3), \ S7 = \lambda X^*(1)E(S), \ S8 = X^*(1)E(S^2) + \pi\alpha X^*(1)E(R^2), \ S9 = [E(S) + \pi\alpha E(R)](X^*(1))^2, \ R1 = \lambda E(X)E(R), \ R2 = \lambda X^*(1)E(R) + \lambda^2 E^2(X)E(R^2), \ R3 = \lambda^2 E^2(X)E(X)E(S)E(R) \ R4 = \lambda X^*(1)E(R), \ R5 = \lambda^2 X^*(1)E(X)E(R^2), \ R6 = \lambda^3 E^3(X)E(R^3), \ R7 = \lambda X^*(1)E(R), \ R8 = E(R^2)E(S) + E(R)E(S^2)$.
\[ V_1 = \lambda E(X)E(V) \quad V_2 = \lambda X^* E(V) + \lambda^2 E^2(X)E(V^2) \]

\[ U_1 = \lambda E(X)E(U) \quad U_2 = \lambda X^* E(U) + \lambda^2 E^2(X)E(U^2) \quad U_3 = \lambda^2 E^2(X)E(U)E(V) \]

\[ T_1 = \beta - S_1 - \pi \alpha R_1 \quad T_2 = S_1 + \pi \alpha R_1 \quad T_3 = S_2 + \pi \alpha R_2 \quad T_6 = S_6 + \pi \alpha R_6 \]

\[ T_7 = S_6 + S_7 + 3S_5 \quad T_8 = 3R_5 + 6R_3 + R_4 + R_6 + 3\lambda^3 E^3(X)R_8 + 6mS_3 \]

\[ f_1 = 6\lambda E(X)[\beta(b - 1) - S_2 - \pi \alpha R_2 - 2\pi \alpha S_3 + 6\lambda X^*(1)T_1] \quad f_2 = 2m T_2 + T_3 + 2\pi \alpha S_3 \]

\[ f_3 = 4\lambda b(b - 1)(b - 2)E(X) - 12\lambda^2 \pi \alpha E(X)\left(3R_3 + \lambda^3 E^3(X)R_8\right) - 4\lambda E(X)T_6 - 8\lambda X^*(1)T_2 \\
+ 4\lambda X^*(1)b - 6\lambda^2 S_9 + 6\lambda b(b - 1)X^*(1) - 18\lambda^3 E^2(X)S_8 \]

E1 = (b^2 - b + m - m^2)T_2 + (b - m)(T_3 + 2\pi \alpha S_3), \quad E2 = 2(b - m)f_1 T_2

E3 = 3(m^2 - m)T_2 + 3mT_3 + T_7 + \pi \alpha T_8

E4 = 3m(m - 1)T_1 + 3m(b - 1) - \lambda S_2 - \pi \alpha (\lambda R_2 + 2S_3)

E5 = b(b - 1)(b - 2) + \lambda S_6 - S_4 - 3S_5 + \pi \alpha (R_6 - 3\lambda^3 E^3(X)R_8 - R_4 - 6R_3 - 3R_5)

E6 = (m + b)(m + b - 1)(m + b - 2) - m(m - 1)(m - 2)

E7 = 2mT_1 + b(b - 1) - S_2 - \pi \alpha (R_2 + 2S_3)

E8 = (m + b)(m + b - 1) - m(m - 1), \quad E9 = 2(2mb + b^2 - b)V_1 + +3b(V_2 + 2U_3)

f_1(X, S, R) = 12\lambda E(X)T_1E_1 - E_2; \quad f_2(X, S, R, V, U) = 4\lambda E(X)T_1E_9 - f_1(2b) V_1

f_3(X, S, R) = f_1 f_2 + 3f_3 T_2 - 4\lambda E(X)T_1E_3 \quad \text{and}

f_4(X, S, R, U) = 4\lambda E(X)T_1\left(E_4 + E_5 - E_6 - U_1 3(2mb + b^2 - b) - 3b U_2 \right) - f_1(E_7 - 2b U_1 - E_8) - 3f_3(T_1 - b)
2.4.2 Expected Length of Idle Period

Let I be the idle period random variable, then the expected length of the idle period is given by,

\[ E(I) = E(I_1) + E(U), \]

where \( I_1 \) is the random variable denoting ‘idle period due to multiple vacation process’ and \( E(U) \) is the expected length of setup time.

To find \( E(I_1) \), another random variable \( U_1 \) is defined as,

\[ U_1 = \begin{cases} 0, & \text{if the server finds at least ‘a’ customers after the first vacation} \\ 1, & \text{if the server finds less than ‘a’ customers after the first vacation} \end{cases} \]

Now, the expected length of idle period due to multiple vacations \( E(I_1) \) is given by

\[ E(I_1) = E(I_1 / U_1 = 0)P(U_1 = 0) + E(I_1 / U_1 = 1)P(U_1 = 1) \]

\[ = E(V)P(U_1 = 0) + (E(V) + E(I_1))P(U_1 = 1) \]

where \( E(V) \) is the expected vacation time.

Solving for \( E(I_1) \), we have

\[ E(I_1) = \frac{E(V)}{P(U_1 = 0)} \quad (2.51) \]

To find \( P(U_1 = 0) \), we do some algebra using the Equation (2.33), then

\[ Q_1(z, 0) = \sum_{n=0}^{\infty} Q_{1,n}(0)z^n = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} ((1 - \pi \alpha)p_n + r_n)z^n \]

\[ = \left( \sum_{n=0}^{\infty} \beta_n z^n \right) \left( \sum_{n=0}^{a-1} ((1 - \pi \alpha)p_n + r_n)z^n \right) \]
Equating the coefficients of $z^n$ ($n=0,1,2,\ldots,a-1$) on both sides, we get

$$Q_{1,n}(0) = \sum_{i=0}^{n} \beta_i \left( (1-\pi\alpha)p_{n-i} + r_{n-i} \right)$$

Therefore,

$$P(U_1 = 0) = 1 - \sum_{n=0}^{a-1} Q_{1,n}(0)$$

$$= 1 - \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \beta_i \left( (1-\pi\alpha)p_{n-i} + r_{n-i} \right) \right)$$

Using (2.51) and (2.52), the expected length of idle period $E(I)$ is obtained as

$$E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \beta_i \left( (1-\pi\alpha)p_{n-i} + r_{n-i} \right) \right)} + E(U)$$

(2.53)

where $E(V)$ is the expected vacation time

### 2.4.3 Expected Length of Busy Period

Let $B$ be the busy period random variable. Let ‘$T$’ be the residence time that the server is rendering service or under re-service. Therefore, $T=S$ with probability $(1-\pi\alpha)$ and $T=S+R$ with probability $\pi\alpha$.

Another random variable $J$ is defined as,

$$J = \begin{cases} 0, & \text{if the server finds less than \textit{a} customers after a residence time} \\ 1, & \text{if the server finds at least \textit{a} customers after a residence time} \end{cases}$$
Now, the expected length of busy period $E(B)$ is given by

$$E(B) = E(B / J = 0)P(J = 0) + E(B / J = 1)P(J = 1)$$

Solving for $E(B)$, we get

$$E(B) = E(S) + \pi \alpha E(R)$$

Thus, the expected length of busy period is obtained as

$$E(B) = \frac{E(S)}{\sum_{n=0}^{a-1} ((1 - \pi \alpha)p_n + r_n)} + \pi \alpha E(R) \quad (2.54)$$

where $E(R)$ is the expected re-service time and $E(S)$ is the expected service time.

2.4.4 Probability that the Server is on Setup Work

Let $U$ be the setup period random variable and $P(U)$ be the probability that the server is on setup work.

From the Equation (2.43), we have

$$\tilde{U}(z, 0) = \frac{1}{(-\lambda + \lambda X(z))}\left(\tilde{U}(\lambda - \lambda X(z)) - 1\right)\left(\tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} (k_n - q_n)z^n\right)$$

Now, the probability that the server is on setup work is given by

$$P(U) = \lim_{z \to 1} \tilde{U}(z, 0)$$

$$= \lim_{z \to 1} \left(\frac{(\tilde{U}(\lambda - \lambda X(z)) - 1)(\tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} (k_n - q_n)z^n)}{(-\lambda + \lambda X(z))}\right)$$
\[ P(U) = E(U) \sum_{n=0}^{a-1} \left( (1-\pi a)p_n + r_n \right) \] (2.55)

where \( E(U) \) is the expected setup period.

### 2.4.5 Probability that the Server is on Vacation

Let \( V \) be the random variable for multiple vacations and \( P(V) \) be the probability that the server is on multiple vacations at time \( t \).

From the Equations (2.40) and (2.41), we have

\[
\sum_{j=1}^{\infty} \tilde{Q}_j(z,0) = \frac{\tilde{V}(\lambda - \lambda X(z))^{-1}}{\lambda (\lambda + \lambda X(z))} \left( \sum_{n=0}^{a-1} \left( (1-\pi a)p_n + r_n \right) z^n + \sum_{j=2}^{\infty} \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n \right)
\]

\[
= \frac{\tilde{V}(\lambda - \lambda X(z))^{-1}}{\lambda (\lambda + \lambda X(z))} \left( \sum_{n=0}^{a-1} \left( (1-\pi a)p_n + r_n \right) z^n + \sum_{n=0}^{a-1} q_n z^n \right)
\]

\[
= \frac{\tilde{V}(\lambda - \lambda X(z))^{-1}}{\lambda (\lambda + \lambda X(z))} \left( \sum_{n=0}^{a-1} k_n z^n \right), \text{ where } k_n = (1-\pi a)p_n + q_n + r_n
\]

Now, the probability that the server is on vacation is given by

\[
P(V) = \lim_{z \rightarrow 1} \sum_{j=1}^{\infty} \tilde{Q}_j(z,0)
\]

\[
= \lim_{z \rightarrow 1} \left( \frac{\tilde{V}(\lambda - \lambda X(z))^{-1}}{\lambda (\lambda + \lambda X(z))} \left( \sum_{n=0}^{a-1} k_n z^n \right) \right)
\]

\[
P(V) = E(V) \left( \sum_{n=0}^{a-1} k_n \right) \] (2.56)

where \( E(V) \) is the expected vacation period.
2.4.6 Probability that the Server is Busy

Let $B$ be the busy period random variable and $P(B)$ be the probability that the server is busy at time $t$.

From the Equations (2.42), (2.44) and (2.45), we have

$$P(B) = \lim_{z \to 1} \left( \sum_{i=a}^{b} \tilde{P}_i(z,0) + \tilde{R}(z,0) \right)$$

$$= \lim_{z \to 1} \left( \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \tilde{R}(z,0) \right)$$

Now, the probability that the server is busy is given by

$$P(B) = E(S) \left[ \sum_{i=a}^{b-1} c_i f'(1) + \frac{\pi a E(R)}{T1} \left( f'(1) + \sum_{i=a}^{b-1} (T1)c_i \right) \right]$$

(2.57)

where

$$f'(1) = \sum_{m=a}^{b-1} \left( \lambda E(X)E(S) + \lambda \pi a E(R)E(X) - m \right)c_m + \sum_{m=0}^{a-1} \left( (1 - \pi a)p_m + r_m \right)\lambda E(X)E(U)$$

and $T1 = b \cdot \lambda E(X) \left( E(S) + \pi a E(R) \right)$

2.5 PARTICULAR CASES

In this section, some of the existing models are deduced as a particular case of the proposed model.

Case (i): If no request for re-service (i.e. $\pi = 0$) and if there is no setup time (i.e., $\tilde{U}(\lambda - \lambda X(z)) = 1$), then the Equation (2.46) reduces to

$$P(z) = \frac{1}{h(z)(-\lambda + \lambda X(z))} \left\{ \begin{array}{c} [\tilde{S}(\lambda - \lambda X(z)) - 1] \sum_{m=a}^{b-1} (z^b - z^m)^k m \\ + [\tilde{V}(\lambda - \lambda X(z)) - 1] \sum_{m=0}^{a-1} (z^b - 1)^k m z^m \end{array} \right\}$$
where \( h(z) = z^b - \tilde{S}(\lambda - \lambda X(z)) \) and \( k_m = p_m + q_m \), which exactly coincides with the result \( M^X / G(a,b)/1 \) and multiple vacations without setup time and N-policy of Krishna Reddy et al (1998).

**Case (ii):** Considering single service (i.e. \( a = b = 1 \)), no request for re-service (i.e. \( \pi = 0 \)) and if there is no setup time (i.e. \( \bar{U}(\lambda - \lambda X(z)) = 1 \)), then (2.46) becomes

\[
P(z) = \frac{\left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) (z - 1) k_0}{z - \tilde{S}(\lambda - \lambda X(z)) (\lambda X(z) - \lambda)} , \text{ where } k_0 = p_0 + q_0
\]

which coincides with the result \( M^X / G/1 \) queueing system and multiple vacations without N-Policy of Lee et al (1994).

**2.5.1 Special Cases**

The model so developed is general in nature as the service time, set-up time and vacation time are arbitrary. But for practical purposes, service time, set-up time and vacation time with particular distribution is required. In this section, some special cases of the proposed model by specifying vacation time random variable as exponential distribution, setup time random variable as exponential distribution and bulk service time random variable as hyper exponential distribution are discussed.

**Case (i):** \( M^X/G(a,b)/1 \) queueing system with setup time, server's choice of admitting re-service and **exponential vacation time**

If the vacation time is assumed as exponential with probability density function \( v(x) = \gamma e^{-\gamma x} \), where \( \gamma \) is the parameter, then,

\[
\tilde{V}(\lambda - \lambda X(z)) = \frac{\gamma}{\gamma + \lambda (1 - X(z))} . \text{ Substituting this expression for } \tilde{V}(\lambda - \lambda X(z)) \text{ in}
\]
(2.46) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

\[
P(z) = \frac{\left( \frac{\tilde{S}(\lambda - \lambda X(z)) - 1}{h(z)(\lambda + \lambda X(z))} \right)}{a_{m=0}^{\infty} \left( \sum_{m=a}^{b} \frac{(z^b - z^m) c_m}{m!} \right)}
\]

where \( h(z) = z^b - (1 - \pi \alpha) \tilde{S}(\lambda - \lambda X(z)) - \pi \alpha \tilde{S}(\lambda - \lambda X(z)) \tilde{R}(\lambda - \lambda X(z)) \)

**Case (ii):** \( M^{\lambda X}/G(a,b)/1 \) queueing system with multiple vacations, server’s choice of admitting re-service and **exponential setup time**

In case of exponential setup time random variable with probability density function \( u(x) = u e^{-ux} \), where ‘u’ is the parameter, then,

\[
\tilde{U}(\lambda - \lambda X(z)) = \frac{u}{u + \lambda (1 - X(z))}.
\]

Substituting this expression for \( \tilde{U}(\lambda - \lambda X(z)) \) in (2.46) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,
\[
\begin{align*}
P(z) &= \frac{\left( (\tilde{S}(\lambda - \lambda X(z)) - 1) + \pi_\alpha \left( \tilde{R}(\lambda - \lambda X(z)) - 1 \right) \tilde{S}(\lambda - \lambda X(z)) \right) h^{-1}}{h(z)(-\lambda + \lambda X(z))} \\
&\quad \left( \sum_{m=a}^{b-1} (z - z^m)c_m \right) \\
&\quad + \pi_\alpha \left( \tilde{R}(\lambda - \lambda X(z)) - 1 \right) \tilde{S}(\lambda - \lambda X(z)) \\
&\quad \left( \sum_{m=0}^{a-1} \left( (1 - \pi_\alpha)p_m + r_m \right) z^m \right) \\
&\quad + \frac{\sum_{m=0}^{a-1} (z - 1)^a \left( \frac{u}{u + \lambda(1 - X(z))} \right) \left( \tilde{V}(\lambda - \lambda X(z))k_m - q_m \right) z^m}{h(z)(-\lambda + \lambda X(z))} \\
&\quad \left( \sum_{m=0}^{a-1} \left( k_m - q_m \right) z^m \right)
\end{align*}
\]

and \( h(z) = z^b - (1 - \pi_\alpha)\tilde{S}(\lambda - \lambda X(z)) - \pi_\alpha \tilde{S}(\lambda - \lambda X(z))\tilde{R}(\lambda - \lambda X(z)) \)

**Case (iii):** \( M^X/G(a,b)/1 \) queueing system with **hyper exponential bulk service time**, setup time and server's choice of admitting re-service

If the service time is assumed as hyper-exponential service time with the probability density function \( s(x) = y e^{-yX} + (1-c)w e^{-wx} \), where \( y \) and \( w \) are the parameters, then, \( \tilde{S}(\lambda - \lambda X(z)) = \left( \frac{yc}{y + \lambda(1 - X(z))} \right) + \left( \frac{w(1-c)}{w + \lambda(1 - X(z))} \right) \).

Substituting this expression for \( \tilde{S}(\lambda - \lambda X(z)) \) in (2.46) and after some algebra, the PGF of the queue size distribution of this special case of the queueing model is obtained as,

\[
\begin{align*}
P(z) &= \left( \sum_{m=0}^{a-1} \left( (1 - \pi_\alpha)p_m + r_m \right) z^m \right) \\
&\quad + \frac{\sum_{m=0}^{a-1} (z - 1)^a \left( \frac{u}{u + \lambda(1 - X(z))} \right) \left( \tilde{V}(\lambda - \lambda X(z))k_m - q_m \right) z^m}{h(z)(-\lambda + \lambda X(z))} \\
&\quad \left( \sum_{m=0}^{a-1} \left( k_m - q_m \right) z^m \right) \\
&\quad + \frac{\sum_{m=0}^{a-1} (z - 1)^a \left( \frac{u}{u + \lambda(1 - X(z))} \right) \left( \tilde{V}(\lambda - \lambda X(z))k_m - q_m \right) z^m}{h(z)(-\lambda + \lambda X(z))} \\
&\quad \left( \sum_{m=0}^{a-1} \left( k_m - q_m \right) z^m \right)
\end{align*}
\]
where

\[ h(z) = z^b(1 - \alpha) \left( \frac{yc}{y + \lambda(1 - X(z))} \right) + \left( \frac{w(1-c)}{w + \lambda(1 - X(z))} \right) \]

\[ -\alpha \left( \frac{yc}{y + \lambda(1 - X(z))} + \frac{w(1-c)}{w + \lambda(1 - X(z))} \right) \hat{R}(\lambda - \lambda X(z)) \]

### 2.6 COST MODEL

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost, setup cost, re-service cost and reward cost. It is quite natural that the management of the system desires to minimize the total average cost and to optimize the cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

- \( C_s \): Startup cost per cycle
- \( C_h \): Holding cost per customer per unit time
- \( C_o \): Operating cost per unit time
- \( C_r \): Reward cost per cycle due to vacation
- \( C_{rs} \): Re-service cost per unit time
- \( C_s \): Setup cost per cycle

Since the length of the cycle is the sum of the idle period and busy period, from the Equations (2.53) and (2.54), the expected length of cycle, \( E(T_c) \) is obtained as
\[ E(T_c) = E(\text{length of the Idle Period}) + E(\text{length of the Busy Period}) \]

\[
E(T_c) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \beta_i \left( (1-\pi\alpha) p_{n-i} + r_{n-i} \right) \right)} + E(U) \\
+ \frac{E(S)}{\sum_{n=0}^{a-1} \left( (1-\pi\alpha) p_n + r_n \right)} + \pi\alpha E(R)
\]

Now, the total average cost (TAC) per unit time is obtained as

**Total Average Cost** = Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time \(*\ \rho\ +\ \text{re-service cost per unit time}\ +\ \text{Setup cost per cycle} \ -\ \text{reward due to vacation per cycle}.\)

\[
TAC = \left( C_s - C_r \left( \frac{E(V)}{P(U = 0)} \right) \right) + C_u E(U) + C_{rs} E(R) \pi \alpha \frac{1}{E(T_c)} + C_h E(Q) + C_o \rho
\]

(2.58)

It is difficult to have a direct analytical result for the optimal value \(a^*\) (minimum batch size in \(M^X/G(a,b)/1\) queueing system) to minimize the total average cost. The simple direct search method to find optimal policy for a threshold value \(a^*\) to minimize the total average cost, is defined.

**Step 1:** Fix the value of maximum batch size ‘\(b\)’

**Step 2:** Select the value of ‘\(a\)’ which will satisfy the following relation

\[
TAC(a^*) \leq TAC(a), \quad 1 \leq a \leq b
\]

**Step 3:** The value \(a^*\) is optimum, since it gives minimum total average cost.
Using the above procedure, the optimal value of ‘a’ can be obtained, which minimizes the total average cost function. Some numerical example to illustrate the above procedure is presented in the next section.

2.7 NUMERICAL ILLUSTRATION

In this section, the consistency of the theoretical results obtained in the sections 2.3 – 2.4 are justified numerically with the following assumptions and notations:

Service time distribution is $2 – \text{Erlang with parameter } \mu$
Batch size distribution of the arrival is geometric with mean 2
Probability of requesting re-service $\pi$
Probability of admitting re-service $\alpha$
Re-service time is exponential with parameter $\eta$
Vacation time is exponential with parameter $\gamma$
Setup time is exponential with parameter $u$
Minimum service capacity $a$
Maximum service capacity $b$

2.7.1 Effects of Various Parameters on the Performance Measures

The effects of various parameters such as arrival rates, expected queue length, expected idle period, expected busy period, probability that the server is on vacation, probability that the server is busy, probability of request for re-service, probability of server’s choice of admitting customers for re-service and threshold value ‘a’ are analyzed numerically and presented in Tables 2.1 – 2.3 and represented in Figure 2.2. All numerical results are obtained using Mat Lab software.
The effects of different probabilities of server’s choice of admitting re-service ‘α’ for a fixed ‘a’ and ‘b’ on various performance measures are obtained numerically. These results are tabulated in Table 2.1. From the table, the following observations are made:

If the probability of server’s choice of admitting re-service increases, then

- the probability that the server is on vacation decreases
- the probability that the server is busy increases
- the expected queue length increases
- the expected idle period decreases
- the expected busy period increases

The effects of different arrival rates for a fixed ‘a’ and ‘b’ are compared with various performance measures and are presented in Table 2.2. From the table, it is clear that, if the arrival rate increases, then

- the probability that the server is on vacation decreases
- the probability that the server is busy increases
- the expected queue length increases
- the expected idle period decreases
- the expected busy period increases

In Table 2.3 and Figure 2.2, for various threshold values ‘a’, the expected queue length, the expected idle period and the expected busy period are compared with different probabilities of server’s choice of admitting re-service. From the table and the figure, the following points are observed:
- When the probability of admitting re-service increases, the mean queue size and the mean busy period increase
- When the threshold value increases, the mean queue size and mean busy period increase whereas the mean idle period decreases

### 2.7.2 Optimal Cost

In this section, a numerical example is analyzed to illustrate how the management of an Environmental Sensor Networks (ESN) can effectively use the results obtained in the sections 2.3, 2.4 and 2.6 to make the decision regarding the threshold value to minimize the total average cost.

It is assumed that, the maximum capacity of the ESN is 10 units (i.e. \( b = 10 \) messages). If the ESN operator starts the process even for a single message (i.e. \( a = 1 \)) without waiting for further arrival, clearly, the operating cost will increase. On the other hand, if they start the process until all 10 messages arrive, the holding cost may increase; hence, there must be some value between 1 and 10 that will optimize the cost. An optimal policy regarding the threshold value ‘\( a \)’ which will minimize the total average cost is wished to be obtained.

The total average costs are obtained numerically with the following assumptions:

- Startup cost : `3.00`
- Operating cost per unit time : `5.00`
- Holding cost per customer : `0.25`
- Re-service cost per unit time : `2.00`
- Reward cost per unit time due to vacation : `1.00`
- Setup cost per unit time : `1.00`
The effects of threshold value ‘a’ on the total average cost with $b = 10$ are reported in Tables 2.3 and 2.4 and represented in Figures 2.3 and 2.4.

From the Table 2.3 and the Figure 2.3, it is clear that, for an environmental sensor networks center with the capacity of 10 messages at a time, **the management has to fix the threshold value $a = 5$ to minimize the total average cost** for the probability of requesting re-service 0.6 (i.e. $\pi = 0.6$) and the probability of server’s choice of admitting re-service 0.2 (i.e. $\alpha = 0.2$). Similarly, **the management has to fix the threshold value $a = 4$ to minimize the total average cost** for the probability of requesting re-service 0.6 (i.e. $\pi = 0.6$) and the probability of server’s choice of admitting re-service 0.4 (i.e. $\alpha = 0.4$).

Similarly, the management has to fix the threshold value ‘a’ to minimize the total average cost for various probabilities of requesting re-service and different probabilities of server’s choice of admitting re-service. These values are presented in Table 2.4 and represented in Figure 2.4.

2.8 CONCLUSION

In this chapter, a “$\text{M}^X/\text{G}(a,b)/1$ queueing system with server’s choice of admitting re-service, multiple vacations and setup time” is analyzed. The probability generating function for the queue size at an arbitrary time epoch is derived. Various performance measures are also obtained. Some particular cases and special cases are also discussed. The theoretical development of the model is justified with numerical results. The results so obtained in this chapter can be used for managerial decision to optimize the overall cost and search for the best operating policy in a waiting line system.
Table 2.1  Probability of Server’s Choice of Admitting Customers (Vs)

Performance Measures

(For \( \lambda = 2.5 \), \( a = 3 \), \( b = 4 \), \( \mu = 2.0 \), \( \pi = 0.6 \), \( \eta = 6 \), \( \gamma = 5 \), \( u = 4 \))

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( P(V) )</th>
<th>( P(B) )</th>
<th>( E(Q) )</th>
<th>( E(I) )</th>
<th>( E(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.3062</td>
<td>0.5995</td>
<td>4.2853</td>
<td>0.5319</td>
<td>1.3258</td>
</tr>
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<td>0.2</td>
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<td>0.6269</td>
<td>4.7467</td>
<td>0.5228</td>
<td>1.4617</td>
</tr>
<tr>
<td>0.4</td>
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<td>0.6528</td>
<td>5.2338</td>
<td>0.5175</td>
<td>1.5626</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2445</td>
<td>0.6796</td>
<td>5.8252</td>
<td>0.5108</td>
<td>1.7060</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2239</td>
<td>0.7065</td>
<td>6.5247</td>
<td>0.5043</td>
<td>1.8749</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2031</td>
<td>0.7334</td>
<td>7.3580</td>
<td>0.4982</td>
<td>2.0689</td>
</tr>
</tbody>
</table>

\( P(V) \) - Probability that the server is on multiple vacations; \( P(B) \) - Probability that the server is busy; \( E(Q) \) – Expected queue length; \( E(I) \) – Expected idle period; \( E(B) \) – Expected busy period

Table 2.2 Arrival Rate (Vs) Performance Measures

(For \( \mu = 2.0 \), \( a = 3 \), \( b = 4 \), \( \alpha = 0.4 \), \( \pi = 0.6 \), \( \eta = 6 \), \( \gamma = 5 \), \( u = 4 \))

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( P(V) )</th>
<th>( P(B) )</th>
<th>( E(Q) )</th>
<th>( E(I) )</th>
<th>( E(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.5295</td>
<td>0.3659</td>
<td>2.6919</td>
<td>0.5625</td>
<td>1.2346</td>
</tr>
<tr>
<td>2.0</td>
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<td>0.5081</td>
<td>3.5863</td>
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<td>1.2866</td>
</tr>
<tr>
<td>2.5</td>
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<td>0.6528</td>
<td>5.2338</td>
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</tr>
<tr>
<td>3.0</td>
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<td>2.3810</td>
</tr>
<tr>
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<td>0.0414</td>
<td>0.9418</td>
<td>44.3704</td>
<td>0.4597</td>
<td>7.5073</td>
</tr>
</tbody>
</table>

\( P(V) \) - Probability that the server is on multiple vacations; \( P(B) \) - Probability that the server is busy; \( E(Q) \) – Expected queue length; \( E(I) \) – Expected idle period; \( E(B) \) – Expected busy period
### Table 2.3 Threshold Value (Vs) Performance Measures and Total Average Cost
(For $\lambda = 0.5, \mu = 2.0, b = 10, \eta = 3, \gamma = 8, u = 7$)

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\alpha = 0.2; \pi = 0.6$</th>
<th>$\alpha = 0.4; \pi = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$E(Q)$</td>
<td>$E(I)$</td>
</tr>
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<td>1</td>
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</tr>
<tr>
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<td>0.3115</td>
</tr>
<tr>
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<td>0.2961</td>
</tr>
<tr>
<td>5</td>
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<td>0.2917</td>
</tr>
<tr>
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</tr>
<tr>
<td>7</td>
<td>2.9823</td>
<td>0.2860</td>
</tr>
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<td>8</td>
<td>3.5578</td>
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</tr>
<tr>
<td>9</td>
<td>4.1997</td>
<td>0.2829</td>
</tr>
<tr>
<td>10</td>
<td>4.9706</td>
<td>0.2822</td>
</tr>
</tbody>
</table>

$E(Q)$ – Expected queue length; $E(I)$ – Expected idle period; $E(B)$ – Expected busy period; $TAC$ – Total average cost

### Table 2.4 Threshold Value (Vs) Total Average Cost for various $\alpha$ and $\pi$
(For $\lambda = 0.5, \mu = 2.0, b = 10, \eta = 3, \gamma = 8, u = 7$)

<table>
<thead>
<tr>
<th>$a$</th>
<th>Total Average Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.7; \pi = 1.0$</td>
</tr>
<tr>
<td>1</td>
<td>2.0087</td>
</tr>
<tr>
<td>2</td>
<td>1.8872</td>
</tr>
<tr>
<td>3</td>
<td>1.8229</td>
</tr>
<tr>
<td>4</td>
<td><strong>1.7998</strong></td>
</tr>
<tr>
<td>5</td>
<td>1.8091</td>
</tr>
<tr>
<td>6</td>
<td>1.8456</td>
</tr>
<tr>
<td>7</td>
<td>1.9063</td>
</tr>
<tr>
<td>8</td>
<td>1.9935</td>
</tr>
<tr>
<td>9</td>
<td>2.1130</td>
</tr>
<tr>
<td>10</td>
<td>2.2833</td>
</tr>
</tbody>
</table>
Figure 2.2 Threshold Value (Vs) Expected Queue Length
(For various probability of admitting re-service ‘α’)

Figure 2.3 Threshold Value (Vs) Total Average Cost
(For various probability of admitting re-service ‘α’)

Figure 2.4 Threshold Value (Vs) Total Average Cost for $\alpha = 0.7$, $\pi = 1.0$
(For $\lambda = 0.5$, $\mu = 2.0$, $b = 10$, $\eta = 3$, $\gamma = 8$, $u = 7$)