Chapter 3

ALMOST CONTRA $G\delta S$-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

In 1996, Dontchev [29] introduced the notion of contra continuity and strong $S$-closedness in topological spaces. In this chapter, the concepts of almost contra $g\delta s$-continuous functions are introduced and investigated some of their properties and characterizations.

3.1 Introduction

A function is said to be almost contra $g\delta s$-continuous if inverse image every regular open set is $g\delta s$-closed set. It is proved that a function $f$ is almost contra $g\delta s$-continuous if and only if for each $x \in X$ and each regular closed set $F$ of $Y$ containing $f(x)$, there exists $g\delta s$-open $U$ containing $x$ such that $f(U) \subseteq F$.

The graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be contra $g\delta s$-closed if for each $(x,y) \in (X,Y) - G(f)$, there exist $U \in G\delta SO(X,x)$ and $V \in C(Y,y)$ such that $(U \times V) \cap G(f) = \emptyset$. It is proved that, if $f : X \rightarrow Y$ is contra $g\delta s$-continuous and $Y$ is Urysohn, then $G(f)$ contra $g\delta s$-closed in $X \times Y$ and if $f : X \rightarrow Y$ is almost weakly $g\delta s$-
continuous and \( Y \) is Urysohn, then \( G(f) \) strongly contra \( g \delta s \)-closed in \( X \times Y \).

### 3.2 Preliminaries

Throughout this chapter, \( (X, \tau) \), \( (Y, \sigma) \) and \( (Z, \eta) \) (or simply \( X \), \( Y \) and \( Z \)) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset \( A \) of a space \( X \) the closure and interior of \( A \) with respect to \( \tau \) are denoted by \( Cl(A) \) and \( Int(A) \) respectively. A subset \( A \) of \( X \) is called regular open (resp. regular closed) if \( A = Int(Cl(A)) \) (resp. \( A = Cl(Int(A)) \)). The \( \delta \)-interior [74] of a subset \( A \) of \( X \) is the union of all regular open sets of \( X \) contained in \( A \) and is denoted by \( \delta - Int(A) \) and the set \( A \) of \( X \) is called \( \delta \)-open [74] if \( A = \delta - Int(A) \). The complement of a \( \delta \)-open set is called \( \delta \)-closed.

**Definition 3.2.1.** A subset \( A \) of a space \( X \) is called
(i) a semiopen set [42] if \( A \subset Cl(Int(A)) \).
(ii) an \( \alpha \)-open set [52] if \( A \subset Int(Cl(Int(A))) \).
(iii) a regular open set [70] if \( A = Int(Cl(A)) \).

The complements of the above mentioned sets are called their respective closed sets. The semi-closure [20] of a subset \( A \) of a space \( X \) is the intersection of all semiclosed sets that contain \( A \) and is denoted
by $s\text{Cl}(A)$. The semi interior $[20]$ of a subset $A$ of space $X$ is the union of all semiopen sets contained in $A$ and is denoted by $s\text{Int}(A)$.

**Definition 3.2.2.** [9] A subset $A$ of $X$ is $g\delta s$-closed if $s\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\delta$-open in $X$. The family of all $g\delta s$-closed subsets of the space $X$ is denoted by $G\delta SC(X)$.

**Definition 3.2.3.** [9] The intersection of all $g\delta s$-closed sets containing a set $A$ is called $g\delta s$-closure of $A$ and is denoted by $g\delta s\text{-Cl}(A)$.

A set $A$ is $g\delta s$-closed set if and only if $g\delta s\text{-Cl}(A) = A$.

**Definition 3.2.4.** [9] The union of all $g\delta s$-open sets contained in $A$ is called $g\delta s$-interior of $A$ and is denoted by $g\delta s\text{-Int}(A)$.

A set $A$ is $g\delta s$-open if and only if $g\delta s\text{-Int}(A) = A$.

**Definition 3.2.5.** [9] A topological space $X$ is called
(i) $sT_{3/4}$[9] if every $g\delta s$-closed subset of $X$ is $\delta$-closed.
(ii) $g\delta sT_{1/2}$[9] if every $g\delta s$-closed subset of $X$ is semiclosed.
(iii) $Tg\delta s$[10] if every $g\delta s$-closed subset of $X$ is closed.

**Definition 3.2.6.** [10] A function $f : X \to Y$ is called $g\delta s$-continuous, if the inverse image of every closed set in $Y$ is $g\delta s$-closed in $X$. 
3.3 Almost contra $g\delta s$-continuous functions

In this section, new type of continuity called an almost contra $g\delta s$-continuity, which is weaker than contra $g\delta s$-continuity is introduced and studied some of their properties and characterizations.

Definition 3.3.1. A function $f : X \to Y$ is said to be almost contra $g\delta s$-continuous if $f^{-1}(V)$ is $g\delta s$-closed in $X$ for each regular open set $V$ in $Y$.

Definition 3.3.2. [57] A function $f : X \to Y$ is said to be almost continuous if $f^{-1}(V)$ is open in $X$ for each regular open set $V$ of $Y$.

Definition 3.3.3. [58] A function $f : X \to Y$ is said to be $(\theta, s)$-continuous if $f^{-1}(V)$ is closed in $X$ for each regular open set $V$ of $Y$.

Theorem 3.3.4. If $X$ is $Tg\delta s$-space and $f : X \to Y$ is almost contra $g\delta s$ continuous, then it is $(\theta, s)$-continuous.

Proof. Let $U$ be a regular open set in $Y$. Since $f$ is almost contra $g\delta s$-continuous $f^{-1}(U)$ is $g\delta s$-closed set in $X$ and $X$ is $Tg\delta s$-space, which implies $f^{-1}(U)$ is closed set in $X$. Therefore $f$ is contra almost continuous.

Definition 3.3.5. [15] A space $X$ is called locally $g\delta s$-indiscrete if every $g\delta s$-open set is closed in $X$. 

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Theorem 3.3.6. If a function $f : X \rightarrow Y$ is almost contra $g\delta s$-continuous and $X$ is locally $g\delta s$-indiscrète space, then $f$ is almost continuous.

Proof. Let $U$ be a regular open set in $Y$. Since $f$ is almost contra $g\delta s$-continuous $f^{-1}(U)$ is $g\delta s$-closed set in $X$ and $X$ is locally $g\delta s$-indiscrète space, which implies $f^{-1}(U)$ is an open set in $X$. Therefore $f$ is almost continuous.

Definition 3.3.7. [13] A function $f : X \rightarrow Y$ is said to be contra $g\delta s$-continuous if $f^{-1}(V)$ is $g\delta s$-closed in $X$ for each open set $V$ in $Y$.

Theorem 3.3.8. If $f : X \rightarrow Y$ is contra $g\delta s$-continuous then it is almost contra $g\delta s$-continuous.

Proof. Obvious, because every regular open set is open set.

Remark 3.3.9. Converse of the above theorem need be true in general as seen from the following example.

Example 3.3.10. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \rightarrow Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then $f$ is almost contra $g\delta s$-continuous function but not contra $g\delta s$-continuous, because for the open set $\{a, b\}$ in $Y$ and $f^{-1}(\{a, b\}) = \{a, b\} = \{a, b\}$ is not $g\delta s$-closed in $X$: 
Theorem 3.3.11. The following are equivalent for a function $f : X \to Y$

(i) $f$ is almost contra $g\delta s$-continuous.

(ii) for every regular closed set $F$ of $Y$, $f^{-1}(F)$ is $g\delta s$-open set of $X$.

(iii) for each $x \in X$ and each regular closed set $F$ of $Y$ containing $f(x)$, there exists $g\delta s$-open $U$ containing $x$ such that $f(U) \subseteq F$.

(iv) for each $x \in X$ and each regular open set $V$ of $Y$ not containing $f(x)$, there exists $g\delta s$-closed set $K$ not containing $x$ such that $f^{-1}(V) \subseteq K$.

Proof. (i)⇒ (ii) Let $F$ be a regular closed set in $Y$, then $Y - F$ is a regular open set in $Y$. By (i), $f^{-1}(Y - F) = X - f^{-1}(F)$ is $g\delta s$-closed set in $X$. This implies $f^{-1}(F)$ is $g\delta s$-open set in $X$. Therefore (ii) holds.

(ii)⇒(i) Let $G$ be a regular open set of $Y$. Then $Y - G$ is a regular closed set in $Y$. By (ii), $f^{-1}(Y - G)$ is $g\delta s$-open set in $X$. This implies $X - f^{-1}(G)$ is $g\delta s$-open set in $X$, which implies $f^{-1}(G)$ is $g\delta s$-closed set in $X$. Therefore (i) hold.

(ii)⇒(iii) Let $F$ be a regular closed set in $Y$ containing $f(x)$, which implies $x \in f^{-1}(F)$. By (ii), $f^{-1}(F)$ is $g\delta s$-open in $X$ containing $x$. Set $U = f^{-1}(F)$, which implies $U$ is $g\delta s$-open in $X$ containing $x$ and $f(U) = f(f^{-1}(F)) \subseteq F$. Therefore (iii) holds.

(iii)⇒(ii) Let $F$ be a regular closed set in $Y$ containing $f(x)$, which
implies $x \in f^{-1}(F)$. From (iii), there exists $g\delta s$-open $U_x$ in $X$ containing $x$ such that $f(U_x) \subset F$. That is $U_x \subset f^{-1}(F)$. Thus $f^{-1}(F) = \bigcup \{U_x : x \in f^{-1}(F)\}$, which is union of $g\delta s$-open sets. Therefore $f^{-1}(F)$ is $g\delta s$-open set of $X$.

(iii) $\Rightarrow$ (iv) Let $V$ be a regular open set in $Y$ not containing $f(x)$. Then $Y - V$ is a regular closed set in $Y$ containing $f(x)$. From (iii), there exists a $g\delta s$-open set $U$ in $X$ containing $x$ such that $f(U) \subset Y - V$. This implies $U \subset f^{-1}(Y - V) = X - f^{-1}(V)$. Hence, $f^{-1}(V) \subset X - U$.

Set $K = X - U$, then $K$ is $g\delta s$-closed set not containing $x$ in $X$ such that $f^{-1}(V) \subset K$.

(iv) $\Rightarrow$ (iii) Let $F$ be a regular closed set in $Y$ containing $f(x)$. Then $Y - F$ is a regular open set in $Y$ not containing $f(x)$. From (iv), there exists $g\delta s$-closed set $K$ in $X$ not containing $x$ such that $f^{-1}(Y - F) \subset K$. This implies $X - f^{-1}(F) \subset K$. Hence, $X - K \subset f^{-1}(F)$, that is $f(X - K) \subset F$. Set $U = X - K$, then $U$ is $g\delta s$-open set containing $x$ in $X$ such that $f(U) \subset F$.

**Theorem 3.3.12.** The following are equivalent for a function $f : X \rightarrow Y$:

(i) $f$ is almost contra $g\delta s$-continuous.

(ii) $f^{-1}(\text{Int}(\text{Cl}(G)))$ is $g\delta s$-closed set in $X$ for every open subset $G$ of $Y$. 

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(iii) \( f^{-1}(\text{Cl}(\text{Int}(F))) \) is \( \text{g}\delta s \)-open set in \( X \) for every closed subset \( F \) of \( Y \).

**Proof.** (i)\( \Rightarrow \) (ii) Let \( G \) be an open set in \( Y \). Then \( \text{Int}(\text{Cl}(G)) \) is regular open set in \( Y \). By (i), \( f^{-1}(\text{Int}(\text{Cl}(G))) \in G\delta SC(X) \).

(ii)\( \Rightarrow \) (i) Proof is obvious.

(i)\( \Rightarrow \) (iii) Let \( F \) be a closed set in \( Y \). Then \( \text{Cl}(\text{Int}(G)) \) is regular closed set in \( Y \). By (i), \( f^{-1}(\text{Cl}(\text{Int}(G))) \in G\delta SO(X) \).

(iii)\( \Rightarrow \) (i) Proof is obvious.

**Definition 3.3.13.** [17] A space \( X \) is said to be weakly Hausdorff if each element of \( X \) is an intersection of regular closed sets.

**Definition 3.3.14.** [11] A function \( f : X \rightarrow Y \) is said to be strongly \( g\delta s \)-open (resp. strongly \( g\delta s \)-closed) if image of every \( g\delta s \)-open (resp. \( g\delta s \)-closed) set of \( X \) is \( g\delta s \)-open (resp. \( g\delta s \)-closed) set in \( Y \).

**Definition 3.3.15.** [11] A topological space \( X \) is said to be \( g\delta s \)-\( T_1 \) space if for any pair of distinct points \( x \) and \( y \), there exist a \( g\delta s \)-open sets \( G \) and \( H \) such that \( x \in G, y \notin G \) and \( x \notin H, y \in H \).

**Theorem 3.3.16.** If \( f : X \rightarrow Y \) is an almost contra \( g\delta s \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( g\delta s \)-\( T_1 \).

**Proof.** Suppose \( Y \) is weakly Hausdorff. For any distinct points \( x \) and \( y \) in \( X \), there exist \( V \) and \( W \) regular closed sets in \( Y \) such that
$f(x) \in V, f(y) \notin V, f(y) \in W$ and $f(x) \notin W$. Since $f$ is almost contra $g\delta s$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $g\delta s$-open subsets of $X$ such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $y \in f^{-1}(W)$ and $x \notin f^{-1}(W)$. This shows that $X$ is $g\delta s$-$T_1$.

**Corollary 3.3.17.** If $f : X \to Y$ is a contra $g\delta s$-continuous injection and $Y$ is weakly Hausdorff, then $X$ is $g\delta s$-$T_1$.

**Definition 3.3.18.** [67] A topological space $X$ is called Ultra Hausdroff space, if for every pair of distinct points $x$ and $y$ in $X$, there exist disjoint clopen sets $U$ and $V$ in $X$ containing $x$ and $y$ respectively.

**Definition 3.3.19.** [11] A topological space $X$ is said to be $g\delta s$-$T_2$ space if for any pair of distinct points $x$ and $y$, there exist disjoint $g\delta s$-open sets $G$ and $H$ such that $x \in G$ and $y \in H$.

**Theorem 3.3.20.** If $f : X \to Y$ is an almost contra $g\delta s$-continuous injective function from space $X$ into a Ultra Hausdroff space $Y$, then $X$ is $g\delta s$-$T_2$.

**Proof.** Let $x$ and $y$ be any two distinct points in $X$. Since $f$ is an injective $f(x) \neq f(y)$ and $Y$ is Ultra Hausdroff space, there exist disjoint clopen sets $U$ and $V$ of $Y$ containing $f(x)$ and $f(y)$ respectively. Then $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\delta s$-open sets in $X$. Therefore $X$ is $g\delta s$-$T_2$. 

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Definition 3.3.21. [67] A topological space $X$ is called Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.

Definition 3.3.22. [11] A topological space $X$ is said to be $g\delta s$-normal if each pair of disjoint closed sets can be separated by disjoint $g\delta s$-open sets.

Theorem 3.3.23. If $f : X \rightarrow Y$ is an almost contra $g\delta s$-continuous closed injection and $Y$ is ultra normal, then $X$ is $g\delta s$-normal.

Proof. Let $E$ and $F$ be disjoint closed subsets of $X$. Since $f$ is closed and injective $f(E)$ and $f(F)$ are disjoint closed sets in $Y$. Since $Y$ is ultra normal there exists disjoint clopen sets $U$ and $V$ in $Y$ such that $f(E) \subset U$ and $f(F) \subset V$. This implies $E \subset f^{-1}(U)$ and $F \subset f^{-1}(V)$. Since $f$ is an almost contra $g\delta s$-continuous injection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $g\delta s$-open sets in $X$. This shows $X$ is $g\delta s$-normal.

Definition 3.3.24. Let $A$ be a subset of $X$. Then $g\delta s-\text{Cl}(A) - g\delta s-\text{Int}(A)$ is called $g\delta s$-frontier of $A$ and is denoted by $g\delta s-\text{Fr}(A)$.

Theorem 3.3.25. The set of all points $x$ of $X$ at which $f : X \rightarrow Y$ is not almost contra $g\delta s$-continuous is identical with the union of $g\delta s$-frontier of the inverse images of closed sets of $Y$ containing $f(x)$. 

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**Proof.** Assume that \( f \) is not almost contra \( g\delta s \)-continuous at \( x \in X \). Then, there exists \( F \in RC(Y, f(x)) \) such that \( f(U) \cap (Y - F) \neq \phi \) for every \( U \in G\delta SO(X, x) \). This implies \( U \cap f^{-1}(Y - F) \neq \phi \) for every \( U \in G\delta SO(X, x) \). Therefore \( x \in g\delta s-Cl(f^{-1}(Y - F)) = g\delta s-Cl(X - f^{-1}(F)) \) and also \( x \in f^{-1}(F) \subset g\delta s-Cl(f^{-1}(F)) \). Thus, \( x \in g\delta s-Cl(f^{-1}(F)) \cap g\delta s-Cl(X - f^{-1}(F)) \). This implies, \( x \in g\delta s-Cl(f^{-1}(F)) - g\delta s-Int(f^{-1}(F)) \). Therefore \( x \in g\delta s-Fr(f^{-1}(F)) \).

Conversely, suppose \( x \in g\delta s-Fr(f^{-1}(F)) \) for some \( F \in RC(Y, f(x)) \) and \( f \) is almost contra \( g\delta s \)-continuous at \( x \in X \), then there exists \( U \in G\delta SO(X, x) \) such that \( f(U) \subset F \). Therefore \( x \in U \subset f^{-1}(F) \) and hence \( x \in g\delta s-Int(f^{-1}(F)) \subset X - g\delta s-Fr(f^{-1}(F)) \). This contradicts that \( x \in g\delta s-Fr(f^{-1}(F)) \). Therefore \( f \) is not almost contra \( g\delta s \)-continuous.

**Definition 3.3.26.** [13] A space \( X \) is called \( g\delta s \)-connected provided that \( X \) is not the union of two disjoint nonempty \( g\delta s \)-open sets.

**Theorem 3.3.27.** If \( f : X \to Y \) is an almost contra \( g\delta s \)-continuous surjection and \( X \) is \( g\delta s \)-connected space, then \( Y \) is connected.

**Proof.** Let \( f : X \to Y \) be an almost contra \( g\delta s \)-continuous surjection and \( X \) is \( g\delta s \)-connected space. Suppose \( Y \) is a not connected space. Then there exist disjoint open sets \( U \) and \( V \) such that \( Y = U \cup V \). Therefore \( U \) and \( V \) are clopen in \( Y \). Since \( f \) is almost contra \( g\delta s-\)
continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( g\delta s \)-open sets in \( X \). Moreover \( f^{-1}(U) \) and \( f^{-1}(V) \) are non empty disjoint and \( X = f^{-1}(U) \cup f^{-1}(V) \).

This is contradiction to the fact that \( X \) is \( g\delta s \)-connected space. Therefore \( Y \) is connected.

**Definition 3.3.28.** [18] A function \( f : X \to Y \) is said to be R-map if \( f^{-1}(V) \) is regular open in \( X \) for each regular open set \( V \) of \( Y \).

**Definition 3.3.29.** [55] A function \( f : X \to Y \) is said to perfectly continuous if \( f^{-1}(V) \) is clopen in \( X \) for each open set \( V \) of \( Y \).

**Theorem 3.3.30.** For two functions \( f : X \to Y \) and \( g : Y \to Z \), let \( g \circ f : X \to Z \) is a composition function. Then, the following properties hold.

(i) if \( f \) is almost contra \( g\delta s \)-continuous and \( g \) is an R-map, then \( g \circ f \) is almost contra \( g\delta s \)-continuous.

(ii) if \( f \) is almost contra \( g\delta s \)-continuous and \( g \) is perfectly continuous, then \( g \circ f \) is \( g\delta s \)-continuous and contra \( g\delta s \)-continuous.

(iii) if \( f \) is contra \( g\delta s \)-continuous and \( g \) is almost continuous, then \( g \circ f \) is almost contra \( g\delta s \)-continuous.

**Proof.** (i) Let \( V \) be any regular open set in \( Z \). Since \( g \) is an R-map, \( g^{-1}(V) \) is regular open in \( Y \). Since \( f \) is an almost contra \( g\delta s \)-continuous \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( g\delta s \)-closed set in \( X \). Therefore \( g \circ f \) is almost contra
\( g\delta s\)-continuous.

(ii) Let \( V \) be any open set in \( Z \). Since \( g \) is perfectly continuous, \( g^{-1}(V) \) is clopen in \( Y \). Since \( f \) is an almost contra \( g\delta s\)-continuous \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( g\delta s\)-open and \( g\delta s\)-closed set in \( X \). Therefore \( g \circ f \) is \( g\delta s\)-continuous and contra \( g\delta s\)-continuous.

(iii) Let \( V \) be any regular open set in \( Z \). Since \( g \) is almost continuous, \( g^{-1}(V) \) is open in \( Y \). Since \( f \) is contra \( g\delta s\)-continuous \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( g\delta s\)-closed set in \( X \). Therefore \( g \circ f \) is almost contra \( g\delta s\)-continuous.

**Theorem 3.3.31.** Let \( f : X \to Y \) is a contra \( g\delta s\)-continuous and \( g : Y \to Z \) is \( g\delta s\)-continuous. If \( Y \) is \( Tg\delta s\)-space, then \( g \circ f : X \to Z \) is an almost contra \( g\delta s\)-continuous.

**Proof.** Let \( V \) be any regular open and hence open set in \( Z \). Since \( g \) is \( g\delta s\)-continuous \( g^{-1}(V) \) is \( g\delta s\)-open in \( Y \) and \( Y \) is \( Tg\delta s\)-space implies \( g^{-1}(V) \) open in \( Y \). Since \( f \) is contra \( g\delta s\)-continuous \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( g\delta s\)-closed set in \( X \). Therefore \( g \circ f \) is an almost contra \( g\delta s\)-continuous.

**Theorem 3.3.32.** If \( f : X \to Y \) is surjective strongly \( g\delta s\)-open (or strongly \( g\delta s\)-closed) and \( g : Y \to Z \) is a function such that \( g \circ f : X \to Z \) is an almost contra \( g\delta s\)-continuous, then \( g \) is an almost contra \( g\delta s\)-continuous.

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Proof. Let $V$ be any regular closed (resp. regular open) set in $Z$. Since $g \circ f$ is an almost contra $g\delta s$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $g\delta s$-open (resp. $g\delta s$-closed) in $X$. Since $f$ is surjective and strongly $g\delta s$-open (or strongly $g\delta s$-closed), $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $g\delta s$-open (or $g\delta s$-closed). Therefore $g$ is an almost contra $g\delta s$-continuous.

Definition 3.3.33. A topological space $X$ is said to be $g\delta s$-ultra-connected if every two nonempty $g\delta s$-closed subsets of $X$ intersect.

Definition 3.3.34. [68] A topological space $X$ is said to be hyperconnected if every open set is dense.

Theorem 3.3.35. If $X$ is $g\delta s$-ultra-connected and $f : X \to Y$ is an almost contra $g\delta s$-continuous surjection, then $Y$ is hyperconnected.

Proof. Let $X$ be a $g\delta s$-ultra-connected and $f : X \to Y$ is an almost contra $g\delta s$-continuous surjection. Suppose $Y$ is not hyperconnected. Then there exists an open set $V$ such that $V$ is not dense in $Y$. Therefore there exist nonempty regular open subsets $B_1 = \text{Int}(\text{Cl}(V))$ and $B_2 = Y - \text{Cl}(V)$ in $Y$. Since $f$ is an almost contra $g\delta s$-continuous surjection, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint $g\delta s$-closed sets in $X$. This is contrary to the fact that $X$ is $g\delta s$-ultra-connected. Therefore $Y$ is hyperconnected.
Definition 3.3.36. [13] A function $f : X \to Y$ is called weakly $g\delta s$-continuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in G\delta SO(X, x)$ such that $f(U) \subseteq Cl(V)$.

Theorem 3.3.37. If a function $f : X \to Y$ is an almost contra $g\delta s$-continuous, then $f$ is weakly $g\delta s$-continuous function.

Proof. Let $x \in X$ and $V$ be an open set in $Y$ containing $f(x)$. Then $Cl(V)$ is regular closed in $Y$ containing $f(x)$. Since $f$ is an almost contra $g\delta s$-continuous function by theorem 3.3.12. (ii), $f^{-1}(Cl(V))$ is $g\delta s$-open set in $X$ containing $x$. Set $U = f^{-1}(Cl(V))$, then $f(U) \subseteq f(f^{-1}(Cl(V))) \subseteq Cl(V)$. This shows that $f$ is almost weakly $g\delta s$-continuous function.

Definition 3.3.38. A space $X$ is said to be
(i) $g\delta s$-compact if every $g\delta s$-open cover of $X$ has a finite subcover.
(ii) $G\delta S$-closed compact [13] if every $g\delta s$-closed cover of $X$ has a finite subcover.
(iii) nearly compact [63] if every regular open cover of $X$ has a finite subcover.
(iv) countably $g\delta s$-compact if every countable cover of $X$ by $g\delta s$-open sets has a finite subcover.
(v) countably $G\delta S$-closed compact [13] if every countable cover of $X$ by $g\delta s$-closed sets has a finite subcover.
(vi) nearly countably compact [63] if every countable cover of $X$ by regular open sets has a finite subcover.

(vii) $g\delta s$-Lindelof if every $g\delta s$-open cover of $X$ has a countable subcover.

(viii) $G\delta S$-Lindelof [13] if every $g\delta s$-closed cover of $X$ has a countable subcover.

(ix) nearly Lindelof [63] if every regular open cover of $X$ has a countable subcover.

(x) $S$-Lindelof [34] if every cover of $X$ by regular closed sets has a countable subcover.

(xi) countably $S$-closed [26] if every countable cover of $X$ by regular closed sets has a finite subcover.

(xii) $S$-closed [1] if every regular closed cover of $X$ has a finite subcover.

(xiii) mildly $g\delta s$-compact if every $g\delta s$-clopen cover of $X$ has a finite subcover.

(xiv) mildly countably $g\delta s$-compact if every countable cover of $X$ by $g\delta s$-clopen sets has a finite subcover.

(xv) mildly $g\delta s$-Lindelof if every $g\delta s$-clopen cover of $X$ has a countable subcover.

**Theorem 3.3.39.** Let $f : X \to Y$ be an almost contra $g\delta s$-continuous surjection. Then, the following properties hold

(i) if $X$ is $G\delta S$-closed compact, then $Y$ is nearly compact.
(ii) if $X$ is countably $G\delta S$-closed compact, then $Y$ is nearly countably compact.

(iii) if $X$ is $G\delta S$-Lindelof, then $Y$ is nearly Lindelof.

**Proof.** (i) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\delta s$-closed cover of $X$. Since $X$ is $G\delta S$-closed compact, there exists a finite subset $I_0$ of $I$ such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \cup \{V_\alpha : \alpha \in I_0\}$, which is finite subcover for $Y$. Therefore $Y$ is nearly compact.

(ii) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable $g\delta s$-closed cover of $X$. Since $X$ is countably $G\delta S$-closed compact, there exists a finite subset $I_0$ of $I$ such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \cup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore $Y$ is nearly countably compact.

(iii) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\delta s$-closed cover of $X$. Since $X$ is $G\delta S$-Lindelof, there exists a countable subset $I_0$ of $I$ such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \cup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore $Y$ is nearly Lindelof.
Theorem 3.3.40. Let $f : X \to Y$ be an almost contra $g\delta s$-continuous surjection. Then, the following properties hold

(i) if $X$ is $g\delta s$-compact, then $Y$ is $S$-closed.

(ii) if $X$ is countably $g\delta s$-closed, then $Y$ is countably $S$-closed.

(iii) if $X$ is $g\delta s$-Lindelof, then $Y$ is $S$-Lindelof.

Proof. (i) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore, $Y$ is $S$-closed.

(ii) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular closed cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable $g\delta s$-open cover of $X$. Since $X$ is countably $g\delta s$-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore, $Y$ is countably $S$-closed.

(iii) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\delta s$-open cover of $X$. Since $X$ is $g\delta s$-Lindelof, there exists a countable subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore, $Y$ is $S$-Lindelof.
Definition 3.3.41. A function $f : X \rightarrow Y$ is said to be almost $g\delta s$-continuous if $f^{-1}(V)$ is $g\delta s$-open in $X$ for each regular open set $V$ of $Y$.

Theorem 3.3.42. Let $f : X \rightarrow Y$ be an almost contra $g\delta s$-continuous and almost $g\delta s$-continuous surjection. Then, the following properties hold

(i) if $X$ is mildly $g\delta s$-closed, then $Y$ is nearly compact.

(ii) if $X$ is mildly countably $G\delta S$-closed, then $Y$ is nearly countably compact.

(iii) if $X$ is mildly $g\delta s$-Lindelof, then $Y$ is nearly Lindelof.

Proof. (i) Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous and almost $g\delta s$ surjection, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is $g\delta s$-clopen cover of $X$. Since $X$ is mildly $\delta s$-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$, which is finite subcover for $Y$. Therefore $Y$ is nearly compact.

(ii) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous and almost $g\delta s$ surjection, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable $g\delta s$-closed cover of $X$. Since $X$ is mildly countably $g\delta s$-compact, there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \bigcup \{V_\alpha : \alpha \in I_0\}$ is nearly Lindelof.
finite subcover for $Y$. Therefore $Y$ is nearly countably compact.

(iii) Let \( \{V_\alpha : \alpha \in I\} \) be any regular open cover of $Y$. Since $f$ is almost contra $g\delta s$-continuous and almost $g\delta s$ surjection, \( \{f^{-1}(V_\alpha) : \alpha \in I\} \) is $g\delta s$-closed cover of $X$. Since $X$ is mildly $g\delta s$-Lindelöf, there exists a countable subset $I_0$ of $I$ such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since $f$ is surjective, $Y = \cup \{V_\alpha : \alpha \in I_0\}$ is finite subcover for $Y$. Therefore $Y$ is nearly Lindelöf.

3.4 Contra-closed graphs.

In this section, $g\delta s$-regular graphs and contra $g\delta s$-closed graphs are defined and investigated the relationships between the graphs and contra functions.

Recall that for a function $f : X \to Y$, the subset \( \{(x, f(x)) : x \in X\} \subset X \times Y \) is called the graph of $f$ and is denoted by $G(f)$.

**Theorem 3.4.1.** Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of $f$, defined by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is almost contra $g\delta s$-continuous function, then $f$ is an almost contra $g\delta s$-continuous.

**Proof.** Let $V \in RC(Y)$, then $X \times V = X \times Cl(\text{Int}(V)) = Cl(\text{Int}(X)) \times Cl(\text{Int}(V)) = Cl(\text{Int}(X \times V))$. Therefore $X \times V \in RC(X \times Y)$. Since $g$ is almost contra $g\delta s$-continuous, $f^{-1}(V) = g^{-1}(X \times V) \in G\delta SO(X)$. 

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Thus, $f$ is an almost contra $g\delta s$-continuous.

**Definition 3.4.2.** The graph $G(f)$ of a function $f : X \to Y$ is said to be contra $g\delta s$-closed if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in G\delta SO(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

**Lemma 3.4.3.** [34] Let $G(f)$ be the graph of $f$, for any subset $A \subseteq X$ and $B \subseteq Y$, we have $f(A) \cap B = \emptyset$ if and only if $(A \times B) \cap G(f) = \emptyset$.

**Lemma 3.4.4.** The graph $G(f)$ of $f : X \to Y$ is contra $g\delta s$-closed in $X \times Y$ if and only if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in G\delta SO(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

**Proof.** This is a direct consequences of definition 3.4.2. and lemma 3.4.3.

**Theorem 3.4.5.** If $f : X \to Y$ is contra $g\delta s$-continuous and $Y$ is Urysohn, then $G(f)$ is contra $g\delta s$-closed in $X \times Y$.

**Proof.** Let $(x, y) \in (X, Y) - G(f)$. Then $y \neq f(x)$. Since $Y$ is Urysohn, there exist open sets $V$ and $W$ such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \emptyset$. Since $f$ is contra $g\delta s$-continuous, there exists $U \in G\delta SO(X, x)$ such that $f(U) \subset Cl(V)$. Therefore $(x, y) \in U \times Cl(W) \subset X \times Y - G(f)$. This shows that $G(f)$ contra $g\delta s$-closed in $X \times Y$. 

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Theorem 3.4.6. If \( f : X \to Y \) is \( g\delta_s \)-continuous and \( Y \) is \( T_1 \), then \( G(f) \) is contra \( g\delta_s \)-closed in \( X \times Y \).

Proof. Let \((x, y) \in (X, Y) - G(f)\). Then \( y \neq f(x) \) and there exists open set \( V \) of \( Y \) such that \( f(x) \in V, y \notin V \). Since \( f \) is \( g\delta_s \)-continuous there exists \( U \in G\delta SO(X, x) \) such that \( f(U) \subset V \). Therefore \( f(U) \cap (Y - V) = \emptyset \). Thus, for each \((x, y) \in (X, Y) - G(f)\), there exist \( U \in G\delta SO(X, x) \) and \( Y - V \in C(Y, y) \) such that \( f(U) \cap Y - V = \emptyset \). Therefore \( G(f) \) is contra \( g\delta_s \)-closed in \( X \times Y \).

Definition 3.4.7. The graph \( G(f) \) of a function \( f : X \to Y \) is said to be \( g\delta_s \)-regular (resp. strongly contra \( g\delta_s \)-closed) if for each \((x, y) \in (X, Y) - G(f)\), there exist \( g\delta_s \)-closed (resp. \( g\delta_s \)-open) set \( U \) in \( X \) containing \( x \) and \( V \in RO(Y, y) \) (resp. \( V \in RC(Y, y) \)) such that \((U \times V) \cap G(f) = \emptyset \).

Lemma 3.4.8. The graph \( G(f) \) of \( f : X \to Y \) is \( g\delta_s \)-regular (resp. strongly contra \( g\delta_s \)-closed) in \( X \times Y \) if and only if for each \((x, y) \in (X, Y) - G(f)\), there exist \( g\delta_s \)-closed (resp. \( g\delta_s \)-open) set \( U \) in \( X \) containing \( x \) and \( V \in RO(Y, y) \) (resp. \( V \in RC(Y, y) \)) such that \( f(U) \cap V = \emptyset \).

Proof. Follows from lemma 3.4.3.

Theorem 3.4.9. If \( f : X \to Y \) is almost \( g\delta_s \)-continuous and \( Y \) is \( T_2 \), then \( G(f) \) is \( g\delta_s \)-regular in \( X \times Y \).
Proof. Let \((x, y) \in (X, Y) - G(f)\). Then \(y \neq f(x)\). Since \(Y\) is \(T_2\), there exists regular open sets \(V\) and \(W\) in \(Y\), such that \(f(x) \in V\), \(y \in W\) and \(V \cap W = \emptyset\). Since \(f\) is almost \(g\delta s\)-continuous \(f^{-1}(V)\) is \(g\delta s\)-closed set in \(X\) containing \(x\). Set \(U = f^{-1}(V)\), then \(f(U) \subset V\). Therefore \(f(U) \cap W = \emptyset\) and \(G(f)\) is \(g\delta s\)-regular in \(X \times Y\).

**Theorem 3.4.10.** Let \(f : X \to Y\) have a \(g\delta s\)-regular \(G(f)\). If \(f\) is injective, then \(X\) is \(g\delta s\)-\(T_0\).

**Proof.** Let \(x\) and \(y\) be any two distinct points of \(X\). Then, \((x, f(y)) \in (X, Y) - G(f)\). Since \(G(f)\) is \(g\delta s\)-regular, there exists \(g\delta s\)-closed set \(U\) in \(X\) containing \(x\) and \(V \in RO(Y, f(y))\) such that \(f(U) \cap V = \emptyset\) by lemma 3.4.8. and hence \(U \cap f^{-1}(V) = \emptyset\). Therefore \(y \notin U\). Thus, \(y \in X - U\) and \(x \notin X - U\) and \(X - U\) is \(g\delta s\)-open set in \(X\). This implies \(X\) is \(g\delta s\)-\(T_0\).

**Theorem 3.4.11.** Let \(f : X \to Y\) have a \(g\delta s\)-regular graph \(G(f)\). If \(f\) is surjective, then \(Y\) is weakly Hausdorff.

**Proof.** Let \(y_1\) and \(y_2\) be any two distinct points of \(Y\). Since \(f\) is surjective, \(f(x) = y_1\) for some \(x \in X\) and \((x, y_2) \in (X, Y) - G(f)\). Since \(G(f)\) is \(g\delta s\)-regular, there exist \(g\delta s\)-closed set \(U\) in \(X\) containing \(x\) and \(F \in RO(Y, y_2)\) such that \(f(U) \cap F = \emptyset\) by lemma 3.4.8. and hence \(y_1 \notin F\). Then \(y_1 \in Y - F\) and \(y_2 \notin Y - F\) and \(Y - F\) is regular closed set in \(Y\). This implies \(Y\) is weakly Hausdorff.
Theorem 3.4.12. Let \( f : X \to Y \) have a strongly contra \( g\delta s \)-closed graph \( G(f) \). If \( f \) is an almost contra \( g\delta s \)-continuous injection, then \( X \) is \( g\delta s - T_2 \).

**Proof.** Let \( x \) and \( y \) be any two distinct points of \( X \). Since \( X \) is injective, \( f(x) \neq f(y) \). Then, \( (x, f(y)) \in (X, Y) - G(f) \). Since \( G(f) \) is strongly contra \( g\delta s \)-closed, by lemma 3.4.8 there exist \( g\delta s \)-open set \( U \) in \( X \) containing \( x \) and \( V \in RC(Y, y) \) such that \( f(U) \cap V = \emptyset \) and hence \( U \cap f^{-1}(V) = \emptyset \). Since \( f \) is an almost contra \( g\delta s \)-continuous, \( f^{-1}(V) \) is \( g\delta s \)-open in \( X \) containing \( y \). This shows that \( X \) is \( g\delta s - T_2 \).

Definition 3.4.13. A function \( f : X \to Y \) is called almost weakly \( g\delta s \)-continuous if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in G\delta SO(X, x) \) such that \( f(U) \subseteq Cl(V) \).

Theorem 3.4.14. If \( f : X \to Y \) is almost weakly \( g\delta s \)-continuous and \( Y \) is Urysohn, then \( G(f) \) strongly contra \( g\delta s \)-closed in \( X \times Y \).

**Proof.** Let \( (x, y) \in (X, Y) - G(f) \) implies, \( y \neq f(x) \). Since \( Y \) is Urysohn there exist open sets \( V \) and \( W \) in \( Y \) such that \( y \in V \), \( f(x) \in W \) and \( Cl(V) \cap Cl(W) = \emptyset \). Since \( f \) is almost weakly \( g\delta s \)-continuous, by definition 4.13 there exists \( U \in G\delta SO(X, x) \) such that \( f(U) \subseteq Cl(W) \). This shows that \( f(U) \cap Cl(V) = f(U) \cap Cl(Int(V)) = \emptyset \), where \( Cl(Int(V)) \in RC(Y) \) and hence by lemma 3.4.8. we have \( G(f) \) strongly contra \( g\delta s \)-closed in \( X \times Y \).
Theorem 3.4.15. If $f : X \rightarrow Y$ is almost contra $g\delta_s$-continuous, then $f$ is almost weakly $g\delta_s$-continuous.

Proof. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then $Cl(V)$ is a regular closed set of $Y$ containing $f(x)$. Since $f$ is almost contra $g\delta_s$-continuous by theorem 3.3.11. there exists $g\delta_s$-open set in $X$ containing $x$ such that $f(U) \subset Cl(V)$. By definition 3.4.13. $f$ is almost weakly $g\delta_s$-continuous.

Corollary 3.4.16. If $f : X \rightarrow Y$ is almost contra $g\delta_s$-continuous and $Y$ is Urysohn, then $G(f)$ strongly contra $g\delta_s$-closed in $X \times Y$.

We recall that a topological space $X$ is said to be extremely disconnected (E.D) if the closure of every open set of $X$ is open in $X$.

Theorem 3.4.17. Let $Y$ be E.D. Then a function $f : X \rightarrow Y$ is almost contra $g\delta_s$-continuous if and only if it is almost $g\delta_s$-continuous.

Proof. Let $x \in X$ and $V$ be any regular open set of $Y$ containing $f(x)$. Since $Y$ is E.D then $V$ is clopen and hence $V$ is regular closed set of $Y$ containing $f(x)$. Since $f$ is almost contra $g\delta_s$-continuous by theorem there exists $g\delta_s$-open set in $X$ containing $x$ such that $f(U) \subset V$. Then $f$ is almost $g\delta_s$-continuous.

Conversely, let $F$ be any regular closed set of $Y$. Since $Y$ is E.D, $F$ is also regular open and $f^{-1}(F)$ is $g\delta_s$-open in $X$. This shows that $f$ is almost contra $g\delta_s$-continuous.