Chapter 5

ON WEAKLY URYSOHN SPACES

In this chapter, we have studied and investigated weakly Urysohn spaces and their properties.

A topological space $X$ is said to be weakly Urysohn if for each $x, y$ in $X$ with distinct closures there are neighbourhoods $U$ and $V$ of $x$ and $y$ respectively such that the closures of $U$ and $V$ are disjoint. In this chapter, a characterization of $R_0$-spaces is given. It is noticed that a regular $R_0$-space is weakly Urysohn and a weakly Urysohn space is $R_1$. Also a weakly Urysohn $T_0$-space is a Urysohn space.

5.1 Introduction

A topological space $X$ is said to be a Urysohn space \cite{21} if for each pair of distinct points $x, y$ in $X$ there are neighbourhoods $U, V$ of $x$ and $y$ respectively such that $\overline{U} \cap \overline{V} = \emptyset$. It is known that every regular $T_1$-space is Urysohn and that every Urysohn space is Hausdorff. In the present chapter a weaker form of Urysohn spaces is proposed. It is observed that a regular $R_0$-space is weakly Urysohn and a weakly Urysohn space is $R_1$. Also a weakly Urysohn $T_0$-space is a Urysohn space.
space. If $X$ is any topological space and $x \in X$ then $\bar{x}$ stands for the closure of $\{x\}$.

### 5.2 Weakly Urysohn spaces

**Definition 5.2.1.** A topological space $X$ is said to be weakly Urysohn if for each pair of points $x, y$ in $X$ with $\bar{x} \neq \bar{y}$ there are neighbourhoods $U$ and $V$ of $x$ and $y$ respectively such that $\bar{U} \cap \bar{V} = \phi$.

Every Urysohn space is weakly Urysohn, but not conversely.

**Example 5.2.2.** Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ be a topology on $X$. Then $\{X, \phi, \{b, c\}, \{a\}\}$ is the family of all closed sets. Note that $X$ is a weakly Urysohn space. We have $\bar{a} = \{a\}, \bar{b} = \{b, c\}$ and $\bar{c} = \{b, c\}$. Now $\bar{a} \neq \bar{b}$. Let $U = \{a\}$ and $V = \{b, c\}$. Clearly $U$ and $V$ are neighbourhoods of $a$ and $b$ respectively such that $\bar{U} \cap \bar{V} = \{a\} \cap \{b, c\} = \{a\} \cap \{b, c\} = \phi$. Also $\bar{a} \neq \bar{c}$. Let $U = \{a\}$ and $V = \{b, c\}$. Then $U$ and $V$ are neighbourhoods of $a$ and $c$ respectively such that $\bar{U} \cap \bar{V} = \phi$. Therefore $X$ is weakly Urysohn. $X$ is not a Urysohn Space; for $b, c \in X$ with $b \neq c$ there do not exist neighbourhoods $U$ and $V$ of $b$ and $c$ respectively with $\bar{U} \cap \bar{V} = \phi$. Therefore $X$ is not Urysohn.

**Theorem 5.2.3.** The following statements about a topological space $X$ are equivalent

(i) for each open set $G$ in $X$, if $x \in G$ then $\bar{x} \subset G$. 

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(ii) for each $x, y \in X$ either $\overline{x} = \overline{y}$ or $\overline{x} \cap \overline{y} = \emptyset$.

(iii) for each $x, y \in X$ if $x \notin \overline{y}$ then $y \notin \overline{x}$.

**Proof.** (i) ⇒ (iii) Suppose (i) holds. Let $x, y \in X$ and $x \notin \overline{y}$. Then $x \in X - \overline{y}$. Now $X - \overline{y}$ is an open set and $x \in X - \overline{y}$. From (i) it follows that $\overline{x} \subset X - \overline{y}$. Therefore $\overline{x} \cap \overline{y} = \emptyset$ and hence $y \notin \overline{x}$. Thus (iii) holds.

(iii) ⇒ (i) Suppose (iii) holds. Let $G$ be an open set in $X$ and $x \in G$. Then $x \notin X - G$. Let $y \in X - G$. Then $\overline{y} \subset X - G$. Therefore, $x \notin \overline{y}$. From (iii), $y \notin \overline{x}$. Therefore $y \in X - \overline{x}$. Thus $X - G \subset X - \overline{x}$. Therefore $\overline{x} \subset G$.

Thus (i) holds. Thus (i) and (iii) are equivalent. The equivalence of (ii) and (i) is proved in [22].

**Definition 5.2.4.** [21] A topological space $X$ is said to be $R_0$ if for each $x, y \in X$, $x \notin \overline{y}$ implies $y \notin \overline{x}$.

**Definition 5.2.5.** [51] A topological space $X$ is said to be $R_1$ if for every pair of points $x, y$ of $X$ with $\overline{x} \neq \overline{y}$ implies $x$ and $y$ have disjoint neighbourhoods.

**Theorem 5.2.6.** The following statements about a topological space $X$ are true:

(i) a regular $R_0$-space is weakly Urysohn.
(ii) a weakly Urysohn space is $R_1$.

(iii) a weakly Urysohn $T_0$-space is Urysohn.

**Proof.** (i) Let $X$ be regular $R_0$-space. Let $x, y \in X$ with $x \neq y$. From theorem 5.2.3. (ii), $\bar{x} \cap \bar{y} = \emptyset$ which implies that $x \notin \bar{y}$. Therefore there are open sets $U(x)$ and $V(y)$ containing $x$ and $y$ respectively such that $U(x) \cap V(y) = \emptyset$.

Therefore $U(x) \cap \bar{V}(y) = \emptyset$. Now $x \in U(x)$ and $X$ is regular. Therefore there exists an open set $W$ containing $x$ such that $x \in W \subset \bar{W} \subset U(x)$. Therefore $\bar{W} \cap \bar{V}(y) = \emptyset$. Hence $X$ is weakly Urysohn.

(ii) Let $X$ be weakly Urysohn. Let $x, y \in X$ with $x \neq y$. Then by definition there are neighbourhoods $U$ and $V$ of $x$ and $y$ respectively such that $\bar{U} \cap \bar{V} = \emptyset$.

Therefore $\bar{U} \cap \bar{V} = \emptyset$. Hence $X$ is a $R_1$-space.

(iii) Let $X$ be a weakly Urysohn $T_0$-space. Let $x, y \in X$ with $x \neq y$. Since $X$ is $T_0$, $x \neq y$. Therefore there are neighbourhoods $U$ and $V$ of $x$ and $y$ respectively such that $\bar{U} \cap \bar{V} = \emptyset$. Hence $X$ is a Urysohn space.