CHAPTER-5
STRUCTURE OF DUO Γ-SEMIGROUPS

DOROFEEVA, MANNEPALLI and SATYANARAYANA [16] established several equivalent characterizations of commutative cancellative chained semigroups. SATYANARAYANA [44] characterized commutative chained semigroups using TAMURA’s concentric condition \(<a>^w = \emptyset \text{ for all } a \in S>\)[52]. He proved that noetherian cancellative chained semigroups without idempotents has this property. Also he [44] characterized commutative cancellative chained semigroups. Further in [45], an ideal theory for commutative semigroups which are unions of a finite number of principal ideals has been developed and an analogue of HILBERT basis theorem was proved. ANJANEYULU [4] made a study on the structure of duo chained semigroups and duo noetherian semigroups. He showed that every noetherian cancellative duo chained semigroup without idempotents which is not globally idempotent is cyclic and satisfies TAMURA’s concentric condition [52]. He characterized archimedian duo chained semigroups without idempotents. Further he proved that in a duo semigroup which is a union of finite number of principal ideals, if every proper prime ideal is principal, then every ideal is an intersection of a principal ideal and an S-primary ideal, and if this semigroup is globally idempotent, then every proper ideal is prime. Also he proved that every noetherian cancellative duo semigroup without identity is finitely generated. Finally he showed that every cancellative noetherian duo semigroup is a direct product of the additive semigroup of nonnegative integers and a group and also he extend ‘The analogue of HILBERT basis theorem’ to duo semigroups. In this thesis we made a study on the structure of duo chained Γ-semigroups and duo noetherian Γ-semigroups. We obtained some characterizations of duo chained Γ-semigroups and duo noetherian Γ-semigroups. Also we extend ‘The analogue of HILBERT basis theorem’ to duo Γ-semigroups.

This chapter is divided into 2 sections. In section 1, the terms; chained Γ-semigroup and Γ-group are introduced. It is proved that if \(P\) is a prime Γ-ideal of a duo chained Γ-semigroup \(S\) and \(x \notin P\) then \(P = \bigcap_{n=1}^{\infty} (x\Gamma)^nP\). It is also proved that every duo chained Γ-semigroup is a semiprimary Γ-semigroup. It is proved that (1) if \(a \in S\) is a semisimple element of a duo chained Γ-semigroup \(S\), then \(<a>^w \neq \emptyset\), (2) if a duo chained Γ-semigroup \(S\) has no Γ-idempotent elements, then for any \(a \in S\), \(<a>^w = \emptyset\) or \(<a>^w\) is
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a prime $\Gamma$-ideal. In a duo chained $\Gamma$-semigroup $S$ if $S \neq S \Gamma S$ then $S \backslash S \Gamma S = \{ \{x\} \}$ for some $x \in S$. Further it is proved that in a duo chained $\Gamma$-semigroup $S$, if $S \neq S \Gamma S$ such that $S \backslash S \Gamma S = \{ \{x\} \}$ for some $x \in S$, then (1) $S = x \Gamma S^1 = S^1 \Gamma x$ and $S \backslash S \Gamma S = x \backslash S \Gamma S = S \backslash \Gamma x$ is the unique maximal $\Gamma$-ideal of $S$, (2) If $a \in S$ and $a \notin <x>^w$ then $a \in (x \Gamma)^{n-1} x$ for some natural number $n > 1$, and (3) If $S$ contains strongly $\Gamma$-cancellable elements then $x$ is a strongly $\Gamma$-cancellable element and $<x>^w$ is either empty or a prime $\Gamma$-ideal of $S$. It is proved that, if $S$ is a duo chained $\Gamma$-semigroup, then $S$ is an Archimedean $\Gamma$-semigroup without $\Gamma$-idempotents if and only if $<a>^w = \emptyset$ for every $a \in S$. It is proved that if $S$ is a strongly $\Gamma$-cancellative Archimedean duo chained $\Gamma$-semigroup with $<a>^w \neq \emptyset$ for some $a \in S$, then $S$ is a $\Gamma$-group. Also it is proved that, if $S$ is a duo chained $\Gamma$-semigroup containing strongly $\Gamma$-cancellative elements and $<a>^w = \emptyset$ for every $a \in S$, then $S$ is a strongly $\Gamma$-cancellative $\Gamma$-semigroup.

The contents of section 1 of chapter 5 are published in “International Journal of Mathematical Sciences, Technology and Humanities” under the title ‘Duo chained $\Gamma$-semigroups’ [19]

In section 2, the terms; noetherian $\Gamma$-semigroup, $\Gamma$-closed $\Gamma$-semigroup and centre of a $\Gamma$-semigroup are introduced. It is proved that if $S$ is a noetherian $\Gamma$-semigroup containing proper $\Gamma$-ideals, then $S$ has a maximal $\Gamma$-ideal. It is proved that if $H$ is the collection of all $\Gamma$-ideals in a $\Gamma$-closed duo $\Gamma$-semigroup $S$ which are not principal and $H \neq \emptyset$, then there exists a prime $\Gamma$-ideal of $S$ which is not a principal $\Gamma$-ideal. It is proved that if every prime $\Gamma$-ideal including $S$ is principal in a $\Gamma$-closed duo $\Gamma$-semigroup $S$, then every $\Gamma$-ideal in $S$ is principal. It is proved that if $S$ is a $\Gamma$-closed duo $\Gamma$-semigroup, which is a union of finite number of principal $\Gamma$-ideals and every proper prime $\Gamma$-ideal is principal, then every $\Gamma$-ideal is an intersection of a principal $\Gamma$-ideal and an $S$-Primary $\Gamma$-ideal. Also it is proved that if $S$ is a $\Gamma$-closed duo $\Gamma$-semigroup, which is a union of finite number of principal $\Gamma$-ideals and every proper prime $\Gamma$-ideal of $S$ is principal and $S = S \backslash S \Gamma S$ then every proper $\Gamma$-ideal is principal.

If $S$ is a duo $\Gamma$-semigroup such that $S \neq S \Gamma S$ and every maximal $\Gamma$-ideal is principal then it is proved that (1) $S$ has at most two maximal $\Gamma$-ideals and (2) if $P$ is a proper prime $\Gamma$-ideal of $S$ then either $P$ is a principal $\Gamma$-ideal or $P = x \Gamma P$ for some $x \in S$. If every maximal $\Gamma$-ideal in a $\Gamma$-closed duo $\Gamma$-semigroup $S$ is principal and $S \neq S \Gamma S$, $<x>^w = \emptyset$ for every $x \in S$, then it is proved that $S$ is a union of two principal $\Gamma$-ideals and every $\Gamma$-ideal is an intersection of a prime $\Gamma$-ideal and an $S$-primary $\Gamma$-ideal. If $S$ is a

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noetherian or Archimedean duo $\Gamma$-semigroup such that $S = \bigcup_{i=1}^{n} <x_i>$ and suppose $a \not\in <x_i\Gamma a>$ for all $\alpha \in S$, which is not a product of power of $x_i$s, then it is proved that $S$ is finitely generated and in particular if $S$ is noetherian strongly $\Gamma$-cancellative $\Gamma$-semigroup without identity then $S$ is finitely generated. If $S$ is a duo $\Gamma$-semigroup which is a union of finite number of principal $\Gamma$-ideals and if $S = S\Gamma S$, then it is proved that $S$ contains $\Gamma$-idempotent elements. If $S$ is a strongly $\Gamma$-cancellable duo $\Gamma$-semigroup which is a union of finite number of principal $\Gamma$-ideals, then it is proved that $S$ contains identity if and only if $S = S\Gamma S$.

In an Archimedean duo $\Gamma$-semigroup $S$, if $S$ is a union of finite number of principal $\Gamma$-ideals or $S$ contains a maximal $\Gamma$-ideal which is finitely generated, then it is proved that every proper $\Gamma$-ideal is principal and $S$ is a union of at most two principal $\Gamma$-ideals. It is proved that if $A$ is a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$, $A = A\Gamma B$ for some $\Gamma$-ideal $B$ and $\alpha \in A$ then $\alpha \in a\Gamma b$ for some $b \in B$. If $S$ is a duo $\Gamma$-semigroup containing no $\Gamma$-idempotents except perhaps the identity 1 and $P$ is a finitely generated prime $\Gamma$-ideal contained properly in $x\Gamma S$ for some $x \in S$ and $x\Gamma S \neq S$, then it is proved that (1) $P$ does not contain any strongly $\Gamma$-cancellative element and (2) if $A$ is finitely generated $\Gamma$-ideal containing a strongly $\Gamma$-cancellable element then $A \neq A\Gamma B$ for any proper $\Gamma$-ideal $B$. It is proved that if $A$ is a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$ and $A^w = B$ such that $A \Gamma B = \bigcap Q_\alpha$ where $Q_\alpha$ is a primary $\Gamma$-ideal, then $A\Gamma B = B$. If $S$ is a noetherian duo $\Gamma$-semigroup without $\Gamma$-idempotents except perhaps identity, then it is proved that for any $\Gamma$-ideal $A$, $A^w \subseteq Z$ where $Z$ is the set of all nonstrongly $\Gamma$-cancellable elements and $A^w = \emptyset$ if $S$ is strongly $\Gamma$-cancellative. If $S$ is a noetherian $\Gamma$-closed duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = <m>$ for some $m \in S$ and if $x \in M$ then it is proved that $x = (m\Gamma)^u$, $u$ is a unit or $x \in M^w$ with $x = m\Gamma x \Gamma s$. If $S$ is a noetherian duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = <m>$ for some $m \in S$ and if $P$ is a proper prime $\Gamma$-ideal of $S$ such that $P \neq M$, then it is proved that $P \subseteq M^w$.

If $S$ is a noetherian duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = <m>$ for some $m \in S$ and if $S$ has no $\Gamma$-idempotents except 1, then it is proved that $M^w$ is a prime $\Gamma$-ideal and also if $Z \neq M$ where $Z$ is the set of all non cancellable elements of $S$, then $Z = M^w$. If $T$ is a $\Gamma$-closed duo $\Gamma$-semigroup and $S$ is a duo $\Gamma$-semigroup such that $S$ is a
The structure and ideal theory of duo $\Gamma$-semigroups. A $\Gamma$-subsemigroup of $T$ and $T = x\Gamma S^1$ for some $x \in T$ and if $S$ is noetherian then it is proved that $T$ is noetherian. Further an analogue of HILBERT basis theorem is obtained for duo $\Gamma$-semigroups.

The contents of section 2 of chapter 5 are published in “International eJournal of Mathematics and Engineering” under the title ‘Duo Noetherian $\Gamma$-semigroups’ [20].

5.1. DUO CHAINED $\Gamma$-SEMIGROUPS

In this section, the terms; chained $\Gamma$-semigroup and $\Gamma$-group are introduced. It is proved that if $P$ is a prime $\Gamma$-ideal of a duo chained $\Gamma$-semigroup $S$ and $x \notin P$ then $P = \cap_{n=1}^{\infty} (x\Gamma)^n P$. It is also proved that every duo chained $\Gamma$-semigroup is a semiprimary $\Gamma$-semigroup. It is proved that (1) if $a \in S$ is a semisimple element of a duo chained $\Gamma$-semigroup $S$, then $<a>^w \neq \emptyset$, (2) if a duo chained $\Gamma$-semigroup $S$ has no $\Gamma$-idempotent elements, then for any $a \in S$, $<a>^w = \emptyset$ or $<a>^w$ is a prime $\Gamma$-ideal. In a duo chained $\Gamma$-semigroup $S$ if $S \neq \Gamma S$ then $S\Gamma S = \{x\}$ for some $x \in S$. Further it is proved that in a duo chained $\Gamma$-semigroup $S$, if $S \neq \Gamma S$ such that $S\Gamma S = \{x\}$ for some $x \in S$, then (1) $S = x \Gamma S^1 = S^1 \Gamma x$ and $S \Gamma S = x \Gamma S = S \Gamma x$ is the unique maximal $\Gamma$-ideal of $S$, (2) If $a \in S$ and $a \notin <x>^w$ then $a \in (x\Gamma)^{n+1}x$ for some natural number $n > 1$, and (3) If $S$ contains strongly $\Gamma$-cancellable elements then $x$ is a strongly $\Gamma$-cancellable element and $<x>^w$ is either empty or a prime $\Gamma$-ideal of $S$. It is proved that, if $S$ is a duo chained $\Gamma$-semigroup, then $S$ is an Archimedean $\Gamma$-semigroup without $\Gamma$-idempotents if and only if $<a>^w = \emptyset$ for every $a \in S$. It is proved that if $S$ is a strongly $\Gamma$-cancellative Archimedean duo chained $\Gamma$-semigroup with $<a>^w \neq \emptyset$ for some $a \in S$, then $S$ is a $\Gamma$-group. Also it is proved that, if $S$ is a duo chained $\Gamma$-semigroup containing strongly $\Gamma$-cancellative elements and $<a>^w = \emptyset$ for every $a \in S$, then $S$ is a strongly $\Gamma$-cancellative $\Gamma$-semigroup.

We now introduce the notion of a chained $\Gamma$-semigroup.

**DEFINITION 5.1.1**: A $\Gamma$-semigroup $S$ is said to be a **chained $\Gamma$-semigroup** if the $\Gamma$-ideals in $S$ are linearly ordered by set inclusion.

Now we characterize chained $\Gamma$-semigroup.

**THEOREM 5.1.2**: Let $S$ be a duo chained $\Gamma$-semigroup and $x \in S$. If $P$ is a prime $\Gamma$-ideal of $S$ and $x \notin P$ then $P = \cap_{n=1}^{\infty} (x\Gamma)^n P$. 

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Proof : Since \( x \notin P \) and \( P \) is prime, \((x \Gamma)^{n-1} x \hat{\bigcup} P \) for all natural numbers \( n \). Since \((x \Gamma)^{n-1} x \subseteq S \) and \( P \) is a \( \Gamma \) -ideal of \( S \), it follows that \((x \Gamma)^{n-1} x \Gamma P \subseteq P \) for all natural numbers \( n \) and hence \((x \Gamma)^{n} P \subseteq P \) for all natural numbers \( n \). Therefore \( \bigcap_{n=1}^{\infty} (x \Gamma)^{n} P \subseteq P \).

Since \( S \) is a duo chained \( \Gamma \)-semigroup, \((x \Gamma)^{n} S^1 \) is a \( \Gamma \) -ideal of \( S \). Since \((x \Gamma)^{n-1} x \hat{\bigcup} P \), we get \((x \Gamma)^{n} S^1 \hat{\bigcup} P \) and since \( S \) is a chained \( \Gamma \)-semigroup, \( P \subseteq (x \Gamma)^{n} S^1 \) for all natural numbers \( n \). Let \( y \in P \). Then \( y \in (x \Gamma)^{n} S^1 \). Therefore \( y \in (x \Gamma)^{n} z \) for some \( z \in S^1 \). Therefore \( y = x\alpha_1 x\alpha_2 \ldots x\alpha_n z \) for some \( z \in S^1 \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma \). Since \( P \) is prime, \( y = x\alpha_1 x\alpha_2 \ldots x\alpha_n z \in P \), \( x \notin P \), we get \( z \in P \). Therefore \( y \in (x \Gamma)^{n} P \) for all natural numbers \( n \). Hence \( P \subseteq \bigcap_{n=1}^{\infty} (x \Gamma)^{n} P \). Therefore \( P = \bigcap_{n=1}^{\infty} (x \Gamma)^{n} P \).

THEOREM 5.1.3 : If \( S \) is a duo chained \( \Gamma \)-semigroup, then \( S \) is a semiprimary \( \Gamma \)-semigroup.

Proof : Let \( A \) be a \( \Gamma \) -ideal of \( S \). We have \( \sqrt{A} = \bigcap_{\alpha \in \Delta} P_{\alpha} \) = Intersection of all prime \( \Gamma \)-ideals of \( S \) containing \( A \). Since \( S \) is a duo chained \( \Gamma \)-semigroup, we have \( \{P_{\alpha} : \alpha \in \Delta\} \) forms a chain. By Zorn’s lemma \( \{P_{\alpha} : \alpha \in \Delta\} \) has a minimal element say \( P_{\beta} \). Therefore \( \sqrt{A} = P_{\beta} \) and \( P_{\beta} \) is a prime \( \Gamma \)-ideal of \( S \) and hence \( \sqrt{A} \) is prime. Therefore \( A \) is a semiprimary \( \Gamma \)-ideal of \( S \) and hence \( S \) is a semiprimary \( \Gamma \)-semigroup.

NOTE 5.1.4 : If \( S \) is a \( \Gamma \)-semigroup and \( a \in S \) then we denote \( < a >^{w} = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a \).

NOTE 5.1.5 : If \( S \) is a duo \( \Gamma \)-semigroup then \( < a >^{w} = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a S^1 \).

THEOREM 5.1.6 : Let \( S \) be duo chained \( \Gamma \)-semigroup. If \( a \in S \) is a semisimple element of \( S \), then \( < a >^{w} \neq \emptyset \).

Proof : Suppose that \( a \) is a semisimple element of \( S \). Therefore \( a \in < a > \Gamma < a > \), implies that \( < a > = < a > \Gamma < a > \). Therefore \( a \in < a > = (a \Gamma)^{n-1} < a > \) for all natural numbers \( n \). Hence \( a \in \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a = < a >^{w} \) and hence \( < a >^{w} \neq \emptyset \).
THEOREM 5.1.7: Let S be a duo chained \( \Gamma \)-semigroup. If \( <a>^w = \emptyset \) for all \( a \in S \), then S has no semisimple elements.

Proof: Suppose that \( <a>^w = \emptyset \) for all \( a \in S \). Suppose if possible S has a semisimple element \( a \). By theorem 5.1.6, \( <a>^w \neq \emptyset \). It is a contradiction. Therefore S has no semisimple elements.

THEOREM 5.1.8: Let S be a duo chained \( \Gamma \)-semigroup. If S has no \( \Gamma \)-idempotents elements, then for any \( a \in S \), \( <a>^w = \emptyset \) or \( <a>^w \) is a prime \( \Gamma \)-ideal of S.

Proof: Suppose that S has no \( \Gamma \)-idempotent elements and \( a \in S \). We have \( <a>^w = \bigcap_{n=1}^{\infty} (<a>^\Gamma)^{n-1} <a>^w \). Assume that \( <a>^w \neq \emptyset \). If possible suppose that \( <a>^w \) is not prime. Then there exists \( x, y \in S \) such that \( x\Gamma y \subseteq <a>^w \), \( x \not\in <a>^w \) and \( y \not\in <a>^w \).

By theorem 3.2.7, \( x \Gamma <y> = <x\Gamma y> \subseteq <a>^w \).

Now \( x, y \not\in <a>^w \), implies that there exists natural numbers \( n, m \) such that

\[
x \not\in (<a>^\Gamma)^{n-1} <a>^w, \ y \not\in (<a>^\Gamma)^{m-1} <a>^w.
\]

Consider \( k = \min \{n, m\} \).

Then \( x, y \not\in (<a>^\Gamma)^{k-1} <a>^w \). Since S is a duo chained \( \Gamma \)-semigroup, we have \( (<a>^\Gamma)^{k-1} <a>^w \subseteq <x> \Gamma <y> \subseteq <a>^w \subseteq (<a>^\Gamma)^{2k-1} <a>^w \subseteq (<a>^\Gamma)^{2k-1} <a> \).

Therefore \( a^{2k} \) is a semisimple element of S. By theorem 3.5.8, \( a^{2k} \) is a regular element of S. Therefore \( a^{2k} = a^{2k} \Gamma x a^{2k} \) for some \( x \in S \), implies that \( (a^{2k} \Gamma x) \Gamma (a^{2k} \Gamma x) = a^{2k} \Gamma x \) and hence \( a^{2k} \Gamma x \) is a \( \Gamma \)-idempotent of S. So S has \( \Gamma \)-idempotent elements.

It is a contradiction. Hence \( <a>^w \) is a prime \( \Gamma \)-ideal of S.

THEOREM 5.1.9: If S is a duo chained strongly \( \Gamma \)-cancellative \( \Gamma \)-semigroup with an identity then for every nonunit \( a \), \( <a>^w \) is either empty or a prime \( \Gamma \)-ideal of S.

Proof: Suppose that \( a \) is a nonunit in S. If \( <a>^w = \emptyset \) then the proof is trivial.

Let \( <a>^w \neq \emptyset \). If possible suppose that \( <a>^w \) is not a prime \( \Gamma \)-ideal of S.

Then there exists \( x, y \in S \) such that \( x\Gamma y \subseteq <a>^w \) and \( x, y \not\in <a>^w \).
By theorem 3.2.7, \(<x>y = \langle x\Gamma y \rangle \leq \langle a \rangle^w\). Now \(x, y \not\in \langle a \rangle^w\), implies that there exists natural numbers \(n, m\) such that \(x \not\in \langle a \rangle^m \) and \(y \not\in \langle a \rangle^n \). Consider \(k = \min\{n, m\}\). Then \(x, y \not\in \langle a \rangle^{k-1} \). Since \(S\) is duo chained \(\Gamma\)-semigroup, we have \(\langle a \rangle^{k-1} < a \rangle \leq \langle x \rangle \) and \(\langle a \rangle^{k-1} < a \). Therefore \(\langle a \rangle^{k-1} < a \rangle \subseteq \langle x \rangle \). Then \(\langle a \rangle^{k-1} < a \rangle \subseteq \langle a \rangle^{4k-1} \langle a \rangle \). Hence \(a^{2k} \in \langle a \rangle^{2k-1} \langle a \rangle \). Therefore \(a^{2k}\) is a semisimple element of \(S\).

By theorem 3.5.8, \(a^{2k}\) is a regular element of \(S\).

Therefore \(a^{2k} = a^{2k} \Gamma x \Gamma a^{2k}\) for some \(x \in S\) implies that \((a^{2k} \Gamma x) \Gamma (a^{2k} \Gamma x) = a^{2k} \Gamma x\) and hence \(\Gamma\) has \(\Gamma\)-idempotent elements. Since \(S\) is strongly cancellative and \(a^{2k} \Gamma x \Gamma e = (a^{2k} \Gamma x) \Gamma (a^{2k} \Gamma x)\) implies that \(a^{2k} \Gamma x = e\) and hence \(a^{2k} \Gamma x = e\).

Hence \(a\) is a unit in \(S\). It is a contradiction. Thus \(\langle a \rangle^w\) is a prime \(\Gamma\)-ideal of \(S\).

Hence \(\langle a \rangle^w = \emptyset\) or \(\langle a \rangle^w\) is a prime \(\Gamma\)-ideal of \(S\).

**THEOREM 5.1.10** : Let \(S\) be a duo chained \(\Gamma\)-semigroup. If \(S \not= S \Gamma S\) then \(S S \Gamma S = \{x\}\) for some \(x \in S\).

**Proof** : Suppose if possible \(x, y \in S S \Gamma S\) and \(x \neq y\). Since \(S\) is a chained \(\Gamma\)-semigroup, \(\langle x \rangle \leq \langle y \rangle\) or \(\langle y \rangle \leq \langle x \rangle\). If \(\langle x \rangle \leq \langle y \rangle\) then \(x \in \langle y \rangle\) and hence \(x \in y \Gamma s\) for some \(s \in S\). Therefore \(x \in S \Gamma S\), which is not true. If \(\langle y \rangle \leq \langle x \rangle\), then \(y \in \langle x \rangle\) and hence \(y \in x \Gamma s\) for some \(s \in S\). Therefore \(y \in S \Gamma S\), which is not true. It is a contradiction. Therefore \(x = y\). So there exists unique \(x \in S\) such that \(x \not\in S \Gamma S\).

Therefore \(S \backslash S \Gamma S = \{x\}\) for some \(x \in S\).

**THEOREM 5.1.11** : Let \(S\) be a duo chained \(\Gamma\)-semigroup with \(S \backslash S \Gamma S = \{x\}\) for some \(x \in S\). Then \(S \backslash \{x\}\) is a \(\Gamma\)-ideal of \(S\).

**Proof** : Let \(a \in S \backslash \{x\}\) and \(s \in S\). Since \(\{x\} \cup S \Gamma S\) we have \(a \Gamma s \not= \{x\}\) and hence \(a \Gamma s \subseteq S \backslash \{x\}\). Therefore \(S \backslash \{x\}\) is a right \(\Gamma\)-ideal of \(S\). Since \(S\) is a duo \(\Gamma\)-semigroup, \(S \backslash \{x\}\) is a \(\Gamma\)-ideal of \(S\).
THEOREM 5.1.12: Let $S$ be a duo chained $\Gamma$-semigroup. If $S \neq S \Gamma S$ such that $SS \Gamma S = \{ x \}$ for some $x \in S$ then $S = x \Gamma S = S \Gamma x$ and $S \Gamma S = S = S \Gamma$ is the unique maximal $\Gamma$-ideal of $S$.

Proof: Since $S \setminus S \Gamma S = \{ x \}$, $S \Gamma S = S \setminus \{ x \}$. Now $x \Gamma S$ is a $\Gamma$-ideal of $S$ and $S \Gamma S$ is a $\Gamma$-ideal of $S$. Since $\{ x \} \cup S \Gamma S$ and since $S$ is a chained $\Gamma$-semigroup, $S \Gamma S \subseteq x \Gamma S$. So $x \Gamma S \subseteq S \Gamma S \cup \{ x \}$ and $\{ x \} \cup S \Gamma S$. Thus $S \Gamma S = x \Gamma S$. So $S = x \Gamma S = S \Gamma x$ and $S \Gamma S = S = S \Gamma$. Since $S \Gamma S$ is trivial, $S \Gamma S = S \Gamma x = \Gamma$ is a maximal $\Gamma$-ideal.

$\therefore S$ is a chained $\Gamma$-semigroup, $S \Gamma S = x \Gamma S = \Gamma x \Gamma$ is the unique maximal $\Gamma$-ideal of $S$.

THEOREM 5.1.13: Let $S$ be a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $SS \Gamma S = \{ x \}$ for some $x \in S$. If $a \in S$ and $a \neq < x >^w$ then $a \in (x \Gamma)^{n-1} x$ for some natural number $n > 1$.

Proof: Since $S$ is a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$, by theorem 5.1.12, $S \Gamma S = x \Gamma S = S \Gamma x = S \setminus \{ x \}$. Since $a \neq < x >^w = \bigcap_{n=1}^{\infty} < (x \Gamma)^{n-1} x >$, there exists a natural number $k$ such that $a \neq < (x \Gamma)^{k-1} x >$.

Let $n$ be the least positive integer such that $a \neq < (x \Gamma)^{n-1} x >$ and $a \in < (x \Gamma)^{n-2} x >$. Now $a \in (x \Gamma)^{n-2} x \Gamma S$ and $a \notin (x \Gamma)^{n-1} x \Gamma S$.

Now $a \in (x \Gamma)^{n-2} x \Gamma S \Rightarrow a \in (x \Gamma)^{n-2} x \Gamma S$ for some $s \in S$. $s \in S$, $s \neq x \Rightarrow s \in x \Gamma S$ Therefore $a \in (x \Gamma)^{n-2} x \Gamma x \Gamma S = (x \Gamma)^{n-1} x \Gamma S$. It is a contradiction and hence $s = x$. Therefore $a \in (x \Gamma)^{n-2} x \Gamma x = (x \Gamma)^{n-1} x$.

THEOREM 5.1.14: Let $S$ be a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $SS \Gamma S = \{ x \}$ for some $x \in S$. If $a \in S$ and $a \neq < x >^w$ then $a \in (x \Gamma)^{r-1} x$ for some natural number $r$ or $a \in (x \Gamma)^{n-1} x \Gamma S_n$, $s_n \in < x >^w$ for all natural numbers $n$.

Proof: Since $S$ is a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $x \in S S \Gamma S$, by theorem 5.1.12, $S \Gamma S = x \Gamma S = S \Gamma x = S \setminus \{ x \}$. Let $a \in S$. Suppose that $a \neq < x >^w$. Then $a \in \bigcap_{n=1}^{\infty} (x \Gamma)^{n-1} x \Gamma S$. Therefore $a \in (x \Gamma)^{n-1} x \Gamma S$ for all natural numbers $n$. Hence $a \in (x \Gamma)^{n-1} x$ or $a \in (x \Gamma)^{n-1} x \Gamma S_n$ for some $s_n \in S$. If $s_n \notin < x >^w$ then by theorem 5.1.13,
$s_n \in (x\Gamma)^{-1} x$ for some natural number $r$ and hence $a \in (x\Gamma)^{n-1} x = (x\Gamma)^{n-r} x$.

If $s_n \not\in \{ x > w \}$ then $a \in (x\Gamma)^{n-1} xS_n$.

**THEOREM 5.1.15 :** Let $S$ be a duo chained $\Gamma$-semigroup with $\emptyset \not\subseteq S$ for some $x \in S$. If $S$ contains strongly $\Gamma$-cancelable elements then $x$ is a strongly $\Gamma$-cancelable element and $< x >^w$ is either empty or a prime $\Gamma$-ideal of $S$.

**Proof :** Suppose if possible $x$ is not strongly $\Gamma$-cancelable element in $S$. Let $Z$ be the set of all strongly $\Gamma$-cancelable elements of $S$. Clearly $x \in Z$. So $Z$ is nonempty subset of $S$. Let $a \in Z$ and $s \in S$. Since $a \in Z$, $a$ is not strongly $\Gamma$-cancelable in $S$. So there exists $b \in S$ such that $a \Gamma b = a \Gamma c$ and $b \not= c$. Now $a \Gamma b = a \Gamma c$ implies $s \Gamma (a \Gamma b) = s \Gamma (a \Gamma c)$ and hence $(s \Gamma a) \Gamma b = (s \Gamma a) \Gamma c$ and $b \not= c$.

Therefore $s \Gamma a$ is a set of nonstrongly $\Gamma$-cancelable elements of $S$.

Therefore $s \Gamma a \subseteq Z$ and hence $Z$ is left $\Gamma$-ideal of $S$. Since $S$ is a duo $\Gamma$-semigroup, $Z$ is a $\Gamma$-ideal of $S$. Since $S \not\subseteq S = \{x\}$, by theorem 5.1.12, $S = x \Gamma S^l$. Since $x \in Z$ and $Z$ is a $\Gamma$-ideal of $S$, $Z = S$. It is a contradiction. Therefore $x$ is a strongly $\Gamma$-cancelable element in $S$. Suppose that $< x >^w \not= \emptyset$. Let $a, b \in S$ and $a \Gamma b \subseteq < x >^w$.

Suppose if possible $a \not\in< x >^w$ and $b \not\in< x >^w$. Now $a, b \not\in< x >^w$, by theorem 5.1.14, $a \in (x\Gamma)^{n-1} x, b \in (x\Gamma)^{m-1} x$ for some natural numbers $n, m$.

Therefore $(x\Gamma)^{n+m-1} x = \Gamma [ (x\Gamma)^{n-1} x ] (x\Gamma)^{m-1} x = a \Gamma b \subseteq< x >^w \subseteq (x\Gamma)^{n+m} S$.

Therefore $x \in x \Gamma S \subseteq S \Gamma S$. It is a contradiction.

Therefore either $a \in< x >^w$ or $b \in< x >^w$ and hence $< x >^w$ is a prime $\Gamma$-ideal of $S$.

**THEOREM 5.1.16 :** Let $S$ be a duo chained $\Gamma$-semigroup. Then $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents if and only if $< a >^w = \emptyset$ for every $a \in S$.

**Proof :** Suppose that $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents. If possible suppose that $< a >^w \not= \emptyset$ for some $a \in S$. By theorem 5.1.8, $< a >^w$ is a prime $\Gamma$-ideal of $S$. Since $S$ is an archimedean duo $\Gamma$-semigroup, by theorem 3.4.4, $S$ has no proper prime $\Gamma$-ideals. Therefore $< a >^w = S$. Now $a \subseteq< a >^w \subseteq< a > \subseteq< a >$. Thus $a$ is semisimple element. By theorem 3.5.8, $a$ is regular element. By theorem 1.3.30, $S$ has $\Gamma$-idempotent elements. It is a contradiction. Hence $< a >^w = \emptyset$ for every $a \in S$. Conversely suppose that $< a >^w = \emptyset$ for every $a \in S$. Since $< a >^w = \emptyset$ for every $a \in S$, by corollary 5.1.6, $S$
has no semi simple elements. By theorem 3.5.8, S has no regular elements. By theorem 1.3.30, S has no \( \Gamma \)-idempotent elements. If possible, suppose that P is proper prime \( \Gamma \)-ideal of S. Let \( x \in S \) such that \( x \not\in P \). Since \( x \not\in P \) by theorem 5.1.2, \( P = \bigcap_{n=1}^{\infty}(x\Gamma)^n P \).

Therefore \( P \subseteq <a>^w = \emptyset \). It is a contradiction. Hence S has no proper prime \( \Gamma \)-ideals. By theorem 3.4.4, S is an archimedean \( \Gamma \)-semigroup.

Now we characterize \( \Gamma \)-group.

**THEOREM 5.1.17**: Let S be a strongly cancellative archimedean duo chained \( \Gamma \)-semigroup with \( <a>^w \neq \emptyset \) for some \( a \in S \), then S is a \( \Gamma \)-group.

**Proof**: Suppose that S is a strongly \( \Gamma \)-cancellative archimedean duo chained \( \Gamma \)-semigroup with \( <a>^w \neq \emptyset \) for some \( a \in S \). Suppose if possible S has no \( \Gamma \)-idempotent elements. Since \( <a>^w \neq \emptyset \), by theorem 5.1.8, \( <a>^w \) is a prime \( \Gamma \)-ideal of S. Since S is an archimedean duo \( \Gamma \)-semigroup, by theorem 3.4.4, S has no proper prime \( \Gamma \)-ideals. It is a contradiction. Hence S has \( \Gamma \)-idempotent elements. Let \( e \) be a \( \Gamma \)-idempotent element in S. Then \( x\alpha(e\beta e) = x\alpha e \) for every \( x \in S \) and \( \alpha, \beta \in \Gamma \). Since S is strongly \( \Gamma \)-cancellative, we have \( x\alpha e = x \) for every \( x \in S \), \( \alpha \in \Gamma \). Similarly \( e\alpha x = x \) for every \( x \in S \), \( \alpha \in \Gamma \). Therefore \( e\Gamma x = x\Gamma e = x \). Hence \( e \) is the identity element in S. Let \( a \in S \). Since \( e, a \in S \) and S is archimedean \( \Gamma \)-semigroup, \( (e\Gamma)^n \subseteq S\Gamma a \Gamma S \). Therefore \( e \in S\Gamma a \Gamma S \). Since S is duo \( \Gamma \)-semigroup \( S\Gamma a \Gamma S = (S\Gamma S)\Gamma a = a\Gamma(S\Gamma S) \). Therefore \( e \in (S\Gamma S)\Gamma a \) and hence \( e = x\alpha a \) for some \( x \in S, \Gamma S \subseteq S \) and \( \alpha \in \Gamma \). Now \( e = e\alpha e = (x\alpha a)\alpha(x\alpha a) \) implies that \( x\alpha(a\alpha a)x\alpha = e = x\alpha a = x\alpha(e\alpha a) \). Since S is strongly cancellative, we have \( a\alpha x = e \). Similarly \( x\alpha a = e \). Therefore \( x\alpha a = a\alpha x = e \) and hence \( x \) is the \( a \)-inverse of \( a \) in S. Therefore S is a \( \Gamma \)-group.

**THEOREM 5.1.19**: Let S be a duo chained \( \Gamma \)-semigroup containing strongly \( \Gamma \)-cancellative elements and \( <a>^w = \emptyset \) for every \( a \in S \), then S is a strongly cancellative \( \Gamma \)-semigroup.

**Proof**: Let S be a duo chained \( \Gamma \)-semigroup containing strongly \( \Gamma \)-cancellative elements. Suppose that \( <a>^w = \emptyset \) for every \( a \in S \). Let \( Z \) be the set of all non strongly \( \Gamma \)-cancellative elements in S. Suppose if possible \( Z \) is a nonempty subset of S. If
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$x \in Z$, then there exists $y, z \in S, \alpha, \beta \in \Gamma$ such that $x\alpha y = x\beta z$ and $y \neq z$. Therefore for any $s \in S, \gamma \in \Gamma$ $s\gamma(x\alpha y) = s\gamma(x\beta z) \Rightarrow (s\gamma x)\alpha y = (s\gamma x)\beta z$ and $y \neq z$. Hence $s\gamma x \in Z$. Therefore $Z$ is a left $\Gamma$-ideal of $S$ and hence $Z$ is a $\Gamma$-ideal of $S$. If possible, suppose that $Z$ is not prime. Then there exists $a, b \in S$ such that $ab \in Z$ and $a, b \notin Z$. Since $a \notin Z, b = c = d$. It is a contradiction. Therefore $Z$ is a prime $\Gamma$-ideal of $S$. Since $< a > = \emptyset$ for every $a \in S$, by theorem 5.1.16, we have $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents. Therefore by theorem 3.4.4, $S$ has no prime $\Gamma$-ideals and hence $Z = S$. It is a contradiction to $S$ contains strongly $\Gamma$-cancellable elements. Hence $Z = \emptyset$. Thus $S$ is a strongly $\Gamma$-cancellative $\Gamma$-semigroup.

5.2. DUO NOETHERIAN $\Gamma$-SEMIGROUPS

In this section, the terms; noetherian $\Gamma$-semigroup, $\Gamma$-closed $\Gamma$-semigroup and centre of a $\Gamma$-semigroup are introduced. It is proved that if $S$ is a noetherian $\Gamma$-semigroup containing proper $\Gamma$-ideals, then $S$ has a maximal $\Gamma$-ideal. It is proved that if $H$ is the collection of all $\Gamma$-ideals in a $\Gamma$-closed duo $\Gamma$-semigroup $S$ which are not principal and $H \neq \emptyset$, then there exists a prime $\Gamma$-ideal of $S$ which is not a principal $\Gamma$-ideal. It is proved that if every prime $\Gamma$-ideal including $S$ is principal in a $\Gamma$-closed duo $\Gamma$-semigroup $S$, then every $\Gamma$-ideal in $S$ is principal. It is proved that if $S$ is a $\Gamma$-closed duo $\Gamma$-semigroup, which is a union of finite number of principal $\Gamma$-ideals and every proper prime $\Gamma$-ideal is principal, then every $\Gamma$-ideal is an intersection of a principal $\Gamma$-ideal and an $S$-primary $\Gamma$-ideal. Also it is proved that if $S$ is a $\Gamma$-closed duo $\Gamma$-semigroup, which is a union of finite number of principal $\Gamma$-ideals and every proper prime $\Gamma$-ideal of $S$ is principal and $S = S$ then every proper $\Gamma$-ideal is principal. If $S$ is a $\Gamma$-semigroup such that $S \neq S$ and every maximal $\Gamma$-ideal is principal then it is proved that (1) $S$ has at most two maximal $\Gamma$-ideals and (2) if $P$ is a proper prime $\Gamma$-ideal of $S$ then either $P$ is a principal $\Gamma$-ideal or $P = x\Gamma P$ for some $x \in S$. If every maximal $\Gamma$-ideal in a $\Gamma$-closed duo $\Gamma$-semigroup $S$ is principal and $S \neq S$, $< x > = \emptyset$ for every $x \in S$, then it is proved that $S$ is a union of two principal $\Gamma$-ideals and every $\Gamma$-ideal is an intersection of a prime $\Gamma$-ideal and an $S$-primary $\Gamma$-ideal. If $S$ is a noetherian or Archimedean duo $\Gamma$-semigroup such that $S = \bigcup_{i=1}^{n} < x_i >$ and suppose $a \notin < x_i \Gamma a >$ for all $a \in S$, which is not a product of power of $x_i$, then it is proved that $S$ is finitely generated and in particular if $S$ is noetherian strongly $\Gamma$-cancellative $\Gamma$-semigroup without identity then $S$ is finitely generated.
generated. If $S$ is a duo $\Gamma$-semigroup which is a union of finite number of principal $\Gamma$-ideals and if $S = \Gamma S$, then it is proved that $S$ contains $\Gamma$-idempotent elements. If $S$ is a strongly $\Gamma$-cancellable duo $\Gamma$-semigroup which is a union of finite number of principal $\Gamma$-ideals, then it is proved that $S$ contains identity if and only if $S = \Gamma S$. In an Archimedean duo $\Gamma$-semigroup $S$, if $S$ is a union of finite number of principal $\Gamma$-ideals or $S$ contains a maximal $\Gamma$-ideal which is finitely generated, then it is proved that every proper $\Gamma$-ideal is principal and $S$ is a union of at most two principal $\Gamma$-ideals. It is proved that if $A$ is a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$, $A = A \Gamma B$ for some $\Gamma$-ideal $B$ and $a \in A$ then $a \in a \Gamma b$ for some $b \in B$. If $S$ is a duo $\Gamma$-semigroup containing no $\Gamma$-idempotents except perhaps the identity 1 and $P$ is a finitely generated prime $\Gamma$-ideal contained properly in $x \Gamma S$ for some $x \in S$ and $x \Gamma S \neq S$, then it is proved that (1) $P$ does not contain any strongly $\Gamma$-cancellable element and (2) if $A$ is finitely generated $\Gamma$-ideal containing a strongly $\Gamma$-cancellable element then $A \neq A \Gamma B$ for any proper $\Gamma$-ideal $B$. It is proved that if $A$ is a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$ and $A^w = B$ such that $A \Gamma B = \bigcap Q_\alpha$ where $Q_\alpha$'s are primary $\Gamma$-ideals, then $A \Gamma B = B$. If $S$ is a noetherian duo $\Gamma$-semigroup without $\Gamma$-idempotents except perhaps identity, then it is proved that for any $\Gamma$-ideal $A$, $A^w \subseteq Z$ where $Z$ is the set of all nonstrongly $\Gamma$-cancellable elements and $A^w = \emptyset$ if $S$ is strongly $\Gamma$-cancellative. If $S$ is a noetherian $\Gamma$-closed duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = < m >$ for some $m \in S$ and if $x \in M$ then it is proved that $x = (m \Gamma)^r u$, $u$ is a unit or $x \in M^w$ with $x = m \Gamma x \Gamma s$. If $S$ is a noetherian duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = < m >$ for some $m \in S$ and if $P$ is a proper prime $\Gamma$-ideal of $S$ such that $P \neq M$, then it is proved that $P \subseteq M^w$. If $S$ is a noetherian duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = < m >$ for some $m \in S$ and if $S$ has no $\Gamma$-idempotents except 1, then it is proved that $M^w$ is a prime $\Gamma$-ideal and also if $Z \neq M$ where $Z$ is the set of all non cancellable elements of $S$, then $Z = M^w$. If $T$ is a $\Gamma$-closed duo $\Gamma$-semigroup and $S$ is a duo $\Gamma$-semigroup such that $S$ is a $\Gamma$-subsemigroup of $T$ and $T = x \Gamma S^1$ for some $x \in T$ and if $S$ is noetherian then it is proved that $T$ is noetherian. Further an analogue of HILBERT basis theorem is obtained for duo $\Gamma$-semigroups.

Now we introduce the notion of noetherian $\Gamma$-semigroup.

**DEFINITION 5.2.1**: A $\Gamma$-semigroup $S$ is said to be a noetherian $\Gamma$-semigroup if ascending chain of $\Gamma$-ideals becomes stationary. i.e., if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending chain of $\Gamma$-ideals of $S$, then there exists a natural number $m$ such that $A_m = A_n$ for all natural numbers $n \geq m$.  

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NOTE 5.2.2: A \( \Gamma \)-semigroup \( S \) is noetherian if and only if every \( \Gamma \)-ideal of \( S \) is a union of finite number of principal \( \Gamma \)-ideals of \( S \).

We characterize the noetherian \( \Gamma \)-semigroup.

THEOREM 5.2.3: If \( S \) is a noetherian \( \Gamma \)-semigroup containing proper \( \Gamma \)-ideals then \( S \) has a maximal \( \Gamma \)-ideal.

**Proof:** Let \( A_1 \) be a proper \( \Gamma \)-ideal of \( S \). If \( A_1 \) is not a maximal \( \Gamma \)-ideal of \( S \), then there exists a proper \( \Gamma \)-ideal \( A_2 \) of \( S \) such that \( A_1 \subset A_2 \). If \( A_2 \) is not a maximal \( \Gamma \)-ideal of \( S \), then there exists a proper \( \Gamma \)-ideal \( A_3 \) of \( S \) such that \( A_1 \subset A_2 \subset A_3 \). By continuing this process we get an ascending chain of proper \( \Gamma \)-ideals of \( S \). Since \( S \) is noetherian, the chain \( A_1 \subset A_2 \subset A_3 \ldots \) is stationary. It is a contradiction. Therefore there exists a maximal \( \Gamma \)-ideal of \( S \).

Now we introduce the notions of \( \Gamma \)-closed \( \Gamma \)-ideal and \( \Gamma \)-closed \( \Gamma \)-semigroup.

**DEFINITION 5.2.4:** A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be \( \Gamma \)-**closed** if \( a, b \in S, \alpha \in \Gamma, a\alpha b \in A \Rightarrow a\Gamma b \subseteq A \).

**DEFINITION 5.2.5:** A \( \Gamma \)-semigroup \( S \) is said to be \( \Gamma \)-**closed** if every \( \Gamma \)-ideal of \( S \) is \( \Gamma \)-closed.

**NOTE 5.2.6:** In a \( \Gamma \)-closed \( \Gamma \)-semigroup \( S \), \( < a\alpha b > = < a\Gamma b > \) where \( a, b \in S \) and \( \alpha \in \Gamma \).

**THEOREM 5.2.7:** Let \( H \) be the collection of all \( \Gamma \)-ideals in a \( \Gamma \)-closed duo \( \Gamma \)-semigroup \( S \) which are not principal. If \( H \neq \emptyset \) then there exists a prime \( \Gamma \)-ideal which is not a principal \( \Gamma \)-ideal.

**Proof:** Let \( H = \{ A_\alpha : \alpha \in \Delta \} \) be the collection of all \( \Gamma \)-ideals in a duo \( \Gamma \)-semigroup \( S \), which are not principal. If \( \bigcup_{\alpha \in \Delta} A_\alpha = \langle x \rangle \) for some \( x \in S \), then \( x \in A_\beta \) for some \( \beta \in \Delta \).

Therefore \( \langle x \rangle \subseteq A_\beta \subseteq \bigcup_{\alpha \in \Delta} A_\alpha = \langle x \rangle \) and hence \( A_\beta = \langle x \rangle \). Then \( A_\beta \in H \). It is a contradiction. Hence \( \bigcup_{\alpha \in \Delta} A_\alpha \) is not principal. So \( \bigcup_{x \in \Delta} A_\alpha \in H \). Thus \( H \) satisfies all the conditions of Zorn’s lemma. By Zorn’s lemma, \( H \) has a maximal element say \( P \). Suppose if possible \( P \) is not a prime \( \Gamma \)-ideal. Then there exists \( a, b \in S \) such that \( a\Gamma b \subseteq P \) and \( a \notin P \) and \( b \notin P \). Since \( P \) is maximal in \( H \), \( P \cup \langle b \rangle \notin H \). Therefore \( P \cup \langle b \rangle \) is a principal \( \Gamma \)-ideal. Then \( P \cup \langle b \rangle = \langle x \rangle \) for some \( x \in S \). If \( x \in P \)
then we get \( P = \langle x \rangle \) and hence \( b \in P \). It is not true. Hence \( x \notin P \). Therefore \( x \in \langle b \rangle \) and hence \( \langle b \rangle = \langle x \rangle \). Hence \( P \subseteq \langle b \rangle \). Now \( P' = \{ s \in S : s\Gamma b \subseteq P \} \) is a \( \Gamma \)-ideal of \( S \). Then clearly \( a \in P' \) and \( a \notin P \). Therefore \( P \subseteq P' \) and \( P \neq P' \). By the maximality of \( P \) in \( H \), we get \( P' = \langle y \rangle \) for some \( y \in S \). Now \( y \in P' \Rightarrow y\Gamma b \subseteq P \Rightarrow \langle y\Gamma b \rangle \subseteq P \). Let \( t \in P \). Since \( P \subseteq \langle b \rangle \), we have \( t = syb \) for some \( s \in S, y \in T \). Now \( s \in S \). \( \gamma \in T \). Now \( syb \in P \). Since \( S \) is \( \Gamma \)-closed, \( s\Gamma b \subseteq P \). Hence \( s \in P' = \langle y \rangle \).
Therefore \( s = r\beta y \) for some \( r \in S, \beta \in \Gamma \).
Now \( t = syb = (r\beta y)\gamma b = r\beta (y\gamma b) \in \langle y\gamma b \rangle \Rightarrow t \in \langle y\gamma b \rangle = \langle y\Gamma b \rangle \).
Therefore we have \( P \subseteq \langle y\Gamma b \rangle \). Hence \( P = \langle y\Gamma b \rangle \). Thus \( P \notin H \). It is a contradiction.
Therefore \( P \) is a prime \( \Gamma \)-ideal.

**COROLLARY 5.2.8**: If \( H \) is the collection of all \( \Gamma \)-ideals in a \( \Gamma \)-closed duo \( \Gamma \)-semigroup \( S \), which are not finitely generated and \( H \neq \emptyset \), then there exists a prime \( \Gamma \)-ideal which is not finitely generated.

**THEOREM 5.2.9**: If every prime \( \Gamma \)-ideal including \( S \) is principal in a \( \Gamma \)-closed duo \( \Gamma \)-semigroup \( S \), then every \( \Gamma \)-ideal in \( S \) is principal.

**Proof**: Let \( H \) be the collection of all \( \Gamma \)-ideals in \( S \) which are not principal. If \( H \neq \emptyset \) then by theorem 5.2.7, \( H \) contains a proper prime \( \Gamma \)-ideal which is not principal. It is a contradiction. Hence \( H = \emptyset \). Therefore every \( \Gamma \)-ideal in \( S \) is principal.

**COROLLARY 5.2.10**: If every prime \( \Gamma \)-ideal including \( S \) is finitely generated in a \( \Gamma \)-closed duo \( \Gamma \)-semigroup \( S \), then every \( \Gamma \)-ideal in \( S \) is finitely generated.

**THEOREM 5.2.11**: If \( S \) is a \( \Gamma \)-closed duo \( \Gamma \)-semigroup, which is a union of finite number of principal \( \Gamma \)-ideals and every proper prime \( \Gamma \)-ideal is principal, then every \( \Gamma \)-ideal is an intersection of a principal \( \Gamma \)-ideal and an \( S \)-Primary \( \Gamma \)-ideal.

**Proof**: First we prove that every primary \( \Gamma \)-ideal \( Q \) such that \( \sqrt{Q} \neq S \) is a principal \( \Gamma \)-ideal. Now \( \sqrt{Q} = P \) is a proper prime \( \Gamma \)-ideal of \( S \). Since every prime \( \Gamma \)-ideal is principal, we have \( P = \langle a \rangle \) for some \( a \in S \). Therefore by theorem 3.3.13, there exists \( n \in \mathbb{N} \) such that \( (a\Gamma)^{n-1}a \subseteq Q \).
Hence \((PG)^{n-1}P = (a\Gamma)^{n-1}a > \subseteq Q\). Now in the case, when \(Q\) is contained in every power of \(P\), we have \(Q = (PG)^{m-1}P = (a\Gamma)^{m-1}a > \subseteq Q\). On the other hand, there exists \(m \in N\) such that \(Q \subseteq (PG)^{m-1}P\) and \(Q \notin (PG)^{m}P\). Now \(A = \{x \in S : x\Gamma(a\Gamma)^{m-1}a \subseteq Q\}\) is a \(\Gamma\)-ideal of \(S\) and \(Q = A\Gamma(PG)^{m-1}P\). Since \(Q \not\subseteq (PG)^{m}P\), we get \(A \not\subseteq P\). Since \(Q\) is a primary \(\Gamma\)-ideal, \((PG)^{m-1}P \subseteq Q\) and hence \(Q = (PG)^{m-1}P = (a\Gamma)^{m-1}a > \subseteq Q\). Therefore \(Q\) is a principal \(\Gamma\)-ideal. By note 5.2.2, \(S\) is a noetherian \(\Gamma\)-semigroup. Thus by theorem 4.3.4, every \(\Gamma\)-ideal \(A\) is of the form \(Q_{1} \cap Q_{2} \cap ... \cap Q_{n}\) where each \(Q_{i}\) is a primary \(\Gamma\)-ideal such that \(P_{i} = \sqrt{Q_{i}} = \sqrt{Q_{j}} = P_{j}\) for \(i \neq j\). We may assume that \(P_{i} \not\subseteq S\) for \(i = 1, 2, ..., m\) and \(P_{i} = S\) for \(m + 1 \leq i \leq n\). Clearly \(\sqrt{Q_{m+1}} \cap \sqrt{Q_{m+2}} \cap ... \cap Q_{n} = S\). Therefore by theorem 4.3.2, \(Q_{m+1} \cap Q_{m+2} \cap ... \cap Q_{n}\) is an \(S\)-primary \(\Gamma\)-ideal. Now we claim that \(Q_{1} \cap Q_{2} \cap ... \cap Q_{m} = Q_{1}\Gamma Q_{2}\Gamma ... \Gamma Q_{m}\), which proves that \(Q_{1} \cap Q_{2} \cap ... \cap Q_{m}\) is principal. Without loss of generality, we may assume that \(P_{1}\) is maximal in \(\{P_{i}\}_{i=1}^{m}\). \(P_{2}\) is maximal in \(\{P_{i}\}_{i=2}^{m}\) and so on. This means that \(P_{i} \subseteq P_{j}\) for all \(i < j\).

Now assume for \(r < m\), \(Q_{1} \cap Q_{2} \cap ... \cap Q_{r} = Q_{1}\Gamma Q_{2}\Gamma ... \Gamma Q_{r}\).

Therefore \(Q_{1} \cap Q_{2} \cap ... \cap Q_{r} \cap Q_{r+1} = (Q_{1}\Gamma Q_{2}\Gamma ... \Gamma Q_{r}) \cap Q_{r+1} = <a > \cap Q_{r+1}\) for some \(a \in S\). Let \(x \in <a > \cap Q_{r+1}\). Now \(x \in a\Gamma y \subseteq Q_{r+1}\).

If \(a \in P_{r+1}\), then \(\sqrt{<a >} = \sqrt{Q_{1} \cap Q_{2} \cap ... \cap Q_{r}} = P_{1} \cap P_{2} \cap ... \cap P_{r} \subseteq P_{r+1}\) and thus since \(P_{r+1}\) is prime, \(P_{i} \subseteq P_{r+1}\) for some \(i \leq r\). It is a contradiction. So \(a \notin P_{r+1}\). Since \(Q_{r+1}\) is a primary \(\Gamma\)-ideal, we have \(y \in Q_{r+1}\) and hence \(x \in <a > \Gamma Q_{r+1}\).

So \(<a > \cap Q_{r+1} = <a > \Gamma Q_{r+1}\).

Therefore by induction \(Q_{1} \cap Q_{2} \cap ... \cap Q_{m} = Q_{1}\Gamma Q_{2}\Gamma ... \Gamma Q_{m}\).

Thus every \(\Gamma\)-ideal is an intersection of a principal \(\Gamma\)-ideal and an \(S\)-primary \(\Gamma\)-ideal.

**THEOREM 5.2.12**: Let \(S\) be a \(\Gamma\)-closed duo \(\Gamma\)-semigroup, which is a union of finite number of principal \(\Gamma\)-ideals. If every proper prime \(\Gamma\)-ideal of \(S\) is principal and \(S = S\Gamma S\) then every proper \(\Gamma\)-ideal is principal.

**Proof**: Since \(S\) is a \(\Gamma\)-closed duo \(\Gamma\)-semigroup which is a union of finite number of principal \(\Gamma\)-ideals, \(S = \bigcup_{i=1}^{n} <x_{i}>\) where \(x_{i} \not\subseteq x_{j}\) for all \(i \neq j\). Since \(S = S\Gamma S\),
$x_i \in < x_i \Gamma x_i >$ for $i = 1, 2, \ldots, n$. Thus $x_i$ is semi simple and hence by theorem 3.5.8, $x_i$ is regular. By theorem 1.3.30, $< x_i > = < e_i >$ for some $\Gamma -$ idempotent $e_i$ in $S$. Let $A$ be any proper $\Gamma$-ideal such that $\sqrt{A} = S$. Therefore $e_i \in (e_i \Gamma)^{n-1}e_i \subseteq A$ for all $i = 1, 2, \ldots, n$. Therefore $x_1, x_2, ..., x_n \in A$ and hence $S = A$. It is a contradiction. Therefore there exists no $\Gamma$-ideal of $A$ of $S$ such that $\sqrt{A} = S$. By theorem 5.2.9, every proper $\Gamma$-ideal is principal.

**THEOREM 5.2.13**: If $S$ is a duo $\Gamma$-semigroup such that $S \neq 5\Gamma S$ and every maximal $\Gamma$-ideal is principal then $S$ has at most two maximal $\Gamma$-ideals.

**Proof**: Let $S$ be a duo $\Gamma$-semigroup such that $S \neq 5\Gamma S$. Suppose that every maximal $\Gamma$-ideal is principal. Let $a \in S \setminus 5\Gamma S$. Then $S \setminus \{a\}$ is a maximal $\Gamma$-ideal. Therefore $S \setminus \{a\} = < b >$ for some $b \in S$. Clearly $a \neq b$. Let $b \in S \setminus 5\Gamma S$. Then $S \setminus \{a\} = < b > \subseteq 5\Gamma S$ and hence $S \setminus \{a\} = 5\Gamma S$. Let $M$ be a maximal $\Gamma$-ideal of $S$. Then $M = < c >$ for some $c \in S$. If $c \in 5\Gamma S$ then $M \subseteq 5\Gamma S$. Since $M$ is maximal, $M = 5\Gamma S = S \setminus \{a\}$. If $c \notin 5\Gamma S$ then $c \notin S \setminus \{a\}$ and hence $c = a$. Thus $M = < a >$. So if $b \in S \setminus 5\Gamma S$, $S$ can have at most two maximal $\Gamma$-ideals, namely $S \setminus \{a\}$ and $< a >$. Let $b \notin 5\Gamma S$. Then $S = < b > \cup \{a\} = \{a\} \cup \{b\} \cup 5\Gamma S$. Let $M = < c >$ be a maximal $\Gamma$-ideal. If $c \notin 5\Gamma S$ then $c = a$ or $c = b$. Then $M = S \setminus \{a\}$ or $M = S \setminus \{b\}$. If $c \in 5\Gamma S$ then $M = 5\Gamma S$ and hence $M$ is properly contained in a proper $\Gamma$-ideal $S \setminus \{a\}$. It is a contradiction. Hence $S$ has at most two maximal $\Gamma$-ideals.

**THEOREM 5.2.14**: Let $S$ be a duo $\Gamma$-semigroup such that $S \neq 5\Gamma S$ and every maximal $\Gamma$-ideal is principal. If $P$ is a proper prime $\Gamma$-ideal of $S$ then either $P$ is a principal $\Gamma$-ideal or $P = x \Gamma P$ for some $x \in S$.

**Proof**: Let $P$ be any proper prime $\Gamma$-ideal and $a \in S \setminus 5\Gamma S$. Now $S \setminus \{a\}$ is a maximal $\Gamma$-ideal. Therefore $S \setminus \{a\} = < b >$ for some $b \in S$. If $a \notin P$ then $P \subseteq S \setminus \{a\} = < b >$. If $b \notin P$ then $P = < b >$. If $b \notin P$ then $P = b \Gamma P$, since $P$ is a prime $\Gamma$-ideal. Let $a \in P$. If $b \notin P$ then $P = S$. If $b \notin P$ then $P \subseteq S \setminus \{b\}$. Since $S \setminus \{b\}$ is maximal $\Gamma$-ideal, we have $P \subseteq S \setminus \{b\} = < x >$ for some $x \in S$. If $x \notin P$ then $P = < x >$. If $x \in P$, let $y \in P$. Then $y \in < x >$. So $y \in x\Gamma S \subseteq P$ for some $s \in S$. Since $P$ is prime, $s \in P$. Hence $y \in P \subseteq x\Gamma P$. Clearly $x\Gamma P \subseteq P$. Hence $P = < x >$ or $P = x \Gamma P$ for some $x \in S$. 

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THEOREM 5.2.15: If every maximal \( \Gamma \)-ideal in a \( \Gamma \)-closed duo \( \Gamma \)-semigroup \( S \) is principal and \( S \neq STS \), \( <x>^w = \emptyset \) for every \( x \in S \), then \( S \) is a union of two principal \( \Gamma \)-ideals and every \( \Gamma \)-ideal is an intersection of a prime \( \Gamma \)-ideal and an \( S \)-primary \( \Gamma \)-ideal.

Proof: Let \( P \) be any proper prime \( \Gamma \)-ideal of \( S \). By theorem 5.2.14, either \( P \) is a principal \( \Gamma \)-ideal or \( P = x \Gamma P \) for some \( x \). If \( P = x \Gamma P \) for some \( x \), then for all natural numbers \( n \). Thus \( P = \bigcap_{n=1}^{\infty} (x \Gamma)^n = \bigcap_{n=1}^{\infty} (x \Gamma)^{n-1} x = <x>^w = \emptyset \).

It is a contradiction. Therefore \( P = <x> \) for some \( x \in S \). Thus every proper prime \( \Gamma \)-ideal is a principal \( \Gamma \)-ideal. If \( a \in S \setminus STS \) then by hypothesis, the maximal \( \Gamma \)-ideal \( S \setminus \{a\} \) is of the form \( <b> \) for some \( b \in S \). Therefore \( S = \{a\} \cup <b> = <a> \cup <b> \). By theorem 5.2.11, every \( \Gamma \)-ideal of \( S \) is an intersection of a prime \( \Gamma \)-ideal and an \( S \)-primary \( \Gamma \)-ideal of \( S \).

THEOREM 5.2.16: Let \( S \) be a duo noetherian \( \Gamma \)-semigroup such that \( S = \bigcup_{i=1}^{n} <x_i> \).

Suppose \( a \in <x_i \Gamma a> \) for all \( a \in S \), which is not a product of power of \( x_i \)'s. Then \( S \) is finitely generated. In particular if \( S \) is noetherian strongly \( \Gamma \)-cancellative \( \Gamma \)-semigroup without identity then \( S \) is finitely generated.

Proof: Suppose that there exists an element \( a \) such that \( a \) is not a product of \( x_i \)'s. If \( a = x_i \alpha_1 s_1 \) for \( \alpha_1 \in \Gamma \), where \( a \neq s_1 \) is not a product of powers of \( x_i \)'s. Hence \( s_1 = x_j \alpha_2 s_2 \) for \( \alpha_2 \in \Gamma \), where \( s_2 \) is not product of powers of \( x_i \)'s. If \( s_2 \in <s_1> \) then \( s_2 = s_3 \alpha_3 \) for some \( \alpha_3 \in \Gamma \) and hence \( s_1 = x_j \alpha_2 (s_1 \alpha_3 \alpha_2 \alpha_3) \in <x_j \Gamma s_1> \), which is not true. Hence \( <s_1> \in <s_2> \). By continuing this process, we get a nonterminating chain of \( \Gamma \)-ideals \( <s_1> \subset <s_2> \subset <s_3> \subset \cdots \). Since \( S \) is noetherian, it is a contradiction. So \( S \) is finitely generated. If \( S \) is a strongly \( \Gamma \)-cancellative \( \Gamma \)-semigroup and if \( a = a \beta_1 (b \beta_2 \alpha) \) for \( \beta_1, \beta_2 \in \Gamma \), then \( b \beta_2 \alpha \) is an identity in \( S \). It is a contradiction. So \( a \notin <x_i \Gamma a> \) for all \( a \in S \). As above, we have \( S \) is finitely generated.

THEOREM 5.2.17: Let \( S \) be a duo \( \Gamma \)-semigroup which is a union of finite number of principal \( \Gamma \)-ideals. If \( S = STS \), then \( S \) contains \( \Gamma \)-idempotent elements.

Proof: Suppose that \( S = \bigcup_{i=1}^{n} <x_i> \) and \( x_i \notin <x_j> \) for \( i \neq j \) and \( S = STS \). Since \( S = STS \), we have \( x_i \notin <x_i> \Gamma <x_i> \) for each \( i = 1, 2, 3, \ldots, n \). Therefore each \( x_i \) is semi
THEOREM 5.2.18: Let $S$ be a strongly $\Gamma$-cancellable duo $\Gamma$-semigroup which is a union of finite number of principal $\Gamma$-ideals. Then $S$ contains identity if and only if $S = \mathbb{S}^\Gamma S$.

*Proof:* Suppose that $S$ is a strongly $\Gamma$-cancellable duo $\Gamma$-semigroup and $S = \mathbb{S}^\Gamma S$. By theorem 5.2.17, $S$ contains $\Gamma$-idempotent element say $e$. Let $a \in S$. Then $a\alpha(e\beta e) = a\alpha e$. Since $S$ is strongly $\Gamma$-cancellative, $a\alpha e = a$. Similarly $e\alpha a = a$. Then $e$ is the identity in $S$. Therefore $S$ contains the identity. Conversely suppose that $S$ contains the identity. Then clearly $S = \mathbb{S}^\Gamma S$.

THEOREM 5.2.19: Let $S$ be a duo archimedean $\Gamma$-semigroup. If $S$ is a union of finite number of principal $\Gamma$-ideals, then every proper $\Gamma$-ideal is principal and $S$ is a union of at most two principal $\Gamma$-ideals.

*Proof:* Suppose that $S = \bigcup_{j=1}^{n} < x_i >$. Let $H$ be the collection of all proper $\Gamma$-ideals which are not principal. If $H \neq \emptyset$ then clearly $H$ is a partially ordered set under set inclusion. Let $\{ A_\alpha \}$ be a chain of $\Gamma$-ideals in $H$. If $S = \bigcup A_\alpha$ then $x_i \in A_\alpha$ for some natural number $i$. If we take $j = \max \{ 1, 2, 3, \ldots, n \}$ then $x_i \in A_j$ for $i = 1, 2, 3, \ldots, n$. So $S = \bigcup_{i=1}^{n} < x_i > \subseteq A_j \subseteq S$, and hence $A_j = S$. It is a contradiction. Hence $S \neq \bigcup A_\alpha$. If $\bigcup A_\alpha = < a >$ for some $a \in S$, then $a \in A_i$ for some $i$ and hence $A_i = < a >$, which is not true. Thus $\bigcup A_\alpha \in H$. Therefore $H$ satisfies the hypothesis of Zorn’s lemma. By Zorn’s lemma, there exists a maximal element $P$ in $H$. By corollary 5.2.8, $P$ is a prime $\Gamma$-ideal of $S$. Since $S$ is a duo archimedean $\Gamma$-semigroup, by theorem 3.4.4, $S$ has no proper prime $\Gamma$-ideals. It is a contradiction. Hence $H = \emptyset$. Therefore every proper $\Gamma$-ideal of $S$ is a principal $\Gamma$-ideal. Let $S = \bigcup_{i=1}^{n} < x_i >$ with $x_i \in < x_j >$ for $i \neq j$. If $n > 2$, then $S \neq < x_1 > \cup < x_2 >$. Since $< x_1 > \cup < x_2 >$ is a proper $\Gamma$-ideal, $< x_1 > \cup < x_2 >$ is a principal $\Gamma$-ideal. Thus either $< x_1 > \subseteq < x_2 >$ or $< x_2 > \subseteq < x_1 >$. This contradicts the choice of $x_i$’s. Thus $n \leq 2$. 

simple in $S$. By theorem 3.5.8, $x_i$ is regular in $S$ and hence by theorem 1.3.30, $S$ contains $\Gamma$-idempotents.
THEOREM 5.2.20: Let S be an archimedean duo $\Gamma$-semigroup. If S contains a maximal $\Gamma$-ideal which is finitely generated, then every proper $\Gamma$-ideal is principal and S is a union of at most two principal $\Gamma$-ideals.

Proof: Suppose that S contains a maximal $\Gamma$-ideal M which is finitely generated. Let $\alpha \in S/M$. Since M is maximal, $S = M \cup <\alpha>$. So S is a union of finite number of principal $\Gamma$-ideals. Therefore by theorem 5.2.19, every $\Gamma$-ideal is principal and S is a union of at most two principal $\Gamma$-ideals.

THEOREM 5.2.21: Let S be an archimedean duo $\Gamma$-semigroup with $S = \bigcup_{i=1}^{n} <x_i>$. If $\alpha \in <x_i \Gamma a>$ for all $\alpha \in S$, which is not a product of powers of $x_i$'s, then S is finitely generated.

Proof: Let S be an archimedean duo $\Gamma$-semigroup with $S = \bigcup_{i=1}^{n} <x_i>$. By theorem 5.2.19, S is a union of at most two principal $\Gamma$-ideals. By theorem 5.2.16, S is finitely generated.

THEOREM 5.2.22: Let A be a finitely generated $\Gamma$-ideals of a duo $\Gamma$-semigroup S. If $A = A \Gamma B$ for some $\Gamma$-ideal B and if $\alpha \in A$ then $\alpha \in a\Gamma b$ for some $b \in B$.

Proof: Suppose that A is a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup S. Without loss of generality, assume that $A = \bigcup_{i=1}^{n} <x_i>$ with $x_i \notin <x_j>$ for $i \neq j$. Suppose that $A = A \Gamma B$ for some $\Gamma$-ideal B of S. Then $A = A \Gamma B = (\bigcup_{i=1}^{n} <x_i>) \Gamma B = \bigcup_{i=1}^{n} (x_i \Gamma B)$. Let $\alpha \in A$. If $a = x_i$ for some $i$ where $1 \leq i \leq n$, then $x_i \notin <x_j>$ for $i \neq j$. So $x_i \notin x_i \Gamma B$. Thus $\alpha \in a\Gamma b$. If $a \neq x_i$ for all $1 \leq i \leq n$, then $\alpha \in <x_i>$ for some $1 \leq i \leq n$. Therefore $\alpha = x_i a \alpha s$ for some $s \in S$, $\alpha \in \Gamma$. So $\alpha = x_i a \alpha s \in (x_i a \alpha s) \Gamma B = a\Gamma B$. Therefore $\alpha \in a\Gamma b$ for some $b \in B$.

THEOREM 5.2.23: Let S be a duo $\Gamma$-semigroup containing no $\Gamma$-idempotents except perhaps the identity 1. If P is a finitely generated prime $\Gamma$-ideal contained properly in $x\Gamma S$ for some $x \in S$ and $x\Gamma S \neq S$, then P does not contain any strongly $\Gamma$-cancellable element.
**Proof**: Suppose that $S$ is a duo $\Gamma$-semigroup containing no $\Gamma$-idempotents except the identity $1$ and $P$ is a finitely generated prime $\Gamma$-ideal such that $P \subseteq x \Gamma S$ for some $x \in S$ and $x \Gamma S \neq S$. Since $P \subseteq x \Gamma S$, $x \notin P$. Clearly $P \subseteq x \Gamma P$. Let $p \in P$. Since $P \subseteq x \Gamma S$, $p \notin x \Gamma S$. So $p = x \alpha s$ for some $\alpha \in \Gamma, s \in S$. Now $x \alpha s \in P$, $P$ is prime $\Rightarrow s \in P$. Therefore $p = x \alpha s \in x \Gamma P$ and hence $P \subseteq x \Gamma P$. Therefore $P = x \Gamma P$. Assume that $a$ is a strongly cancellable element in $P$. By theorem 5.2.22, $a = a \beta b, \ b \in x \Gamma s$. Therefore $a \beta b = (a \beta b) \beta b = a \beta (b \beta b)$. Since $a$ is strongly $\Gamma$-cancellative, we have $b = b \beta b$. It is a contradiction. Hence $P$ does not contain strongly cancellable elements.

**THEOREM 5.2.24**: Let $S$ be a duo $\Gamma$-semigroup containing no $\Gamma$-idempotents except perhaps the identity $1$ and $P$ be a finitely generated prime $\Gamma$-ideal contained properly in $x \Gamma S$ for some $x \in S$ and $x \Gamma S \neq S$. If $A$ is finitely generated $\Gamma$-ideal containing a strongly $\Gamma$-cancellable element then $A \neq A \Gamma B$ for any proper $\Gamma$-ideal $B$.

**Proof**: Suppose that $A$ is a finitely generated $\Gamma$-ideal containing a strongly $\Gamma$-cancellable element say $a$. Suppose if possible $A = A \Gamma B$ for some proper $\Gamma$-ideal $B$. Now $\alpha \in A = A \Gamma B$ implies that $a = a \alpha b$ for some $b \in B, \alpha \in \Gamma$.

Therefore $a \alpha b = (a \alpha b) \alpha b = a \alpha (b \alpha b)$. Since $a$ is strongly $\Gamma$-cancellative, $b \alpha b = b$. Therefore $B$ contains $\Gamma$-idempotent elements. It is a contradiction. Hence $A \neq A \Gamma B$.

**THEOREM 5.2.25**: Let $A$ be a finitely generated $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$ and $A^w = B$ such that $A \Gamma B = \bigcap Q_\alpha$ where $Q_\alpha$'s are primary $\Gamma$-ideals. Then $A \Gamma B = B$.

**Proof**: Since $A, B$ are two $\Gamma$-ideals of a duo $\Gamma$-semigroup $S$, clearly we have $A \Gamma B \subseteq B$.

Let $\sqrt{Q_\alpha} = P_\alpha$ for each $\alpha$. Since each $Q_\alpha$ is a primary $\Gamma$-ideal of $S$, $\sqrt{Q_\alpha} = P_\alpha$ is a prime $\Gamma$-ideal of $S$ for each $\alpha$. Now $A \Gamma B = \bigcap Q_\alpha \subseteq Q_\alpha$ for each $\alpha$. Let $A \not\subseteq P_\alpha$. Since $A \Gamma B \subseteq Q_\alpha$ for each $\alpha$. $A \not\subseteq P_\alpha = \sqrt{Q_\alpha} \Rightarrow B \not\subseteq Q_\alpha$. Let $A \not\subseteq P_\alpha = \sqrt{Q_\alpha}$. Since $A = \bigcup_{i=1}^{n} x_i > x_i \in P_\alpha = \sqrt{Q_\alpha}$ for $i = 1, 2, 3, \ldots, n$. Then $(x_i \Gamma)^{i-1} x_i \subseteq Q_\alpha$ for $i = 1, 2, 3, \ldots, n$. Let $m = \text{Max} \{r_1, r_2, \ldots, r_n\}$. Then $(A \Gamma)^{m-1} A \subseteq Q_\alpha$.

Since $B = A^w = \bigcap_{m=1}^{\infty} (A \Gamma)^{m-1} A \subseteq (A \Gamma)^{m-1} A \subseteq Q_\alpha$. Thus $B \subseteq \bigcap Q_\alpha = A \Gamma B \Rightarrow A \Gamma B = B$. 

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Theorem 5.2.26: Let $S$ be a noetherian duo $\Gamma$-semigroup without $\Gamma$-idempotents except perhaps identity. Then for any $\Gamma$-ideal $A$, $A^w \subseteq Z$ where $Z$ is the set of all non-strongly $\Gamma$-cancellable elements and $A^w = \emptyset$ if $S$ is strongly cancellative.

Proof: If $A^w = \emptyset$, then clearly $A^w \subseteq Z$. If $A^w \neq \emptyset$, by theorem 5.2.25, $A\Gamma A^w = A^w$. By theorem 5.2.24, we have $A^w$ does not contain strongly $\Gamma$-cancellable elements. Thus $A^w \subseteq Z$. If $S$ is strongly $\Gamma$-cancellable then $Z = \emptyset$ and hence $A^w \subseteq Z \Rightarrow A^w = \emptyset$.

Theorem 5.2.27: Let $S$ be a noetherian $\Gamma$-closed duo monoid with a unique maximal $\Gamma$-ideal $M = \langle m \rangle$ for some $m \in S$. If $x \in M$ then $x = (m\Gamma)^n u$ where $u$ is a unit or $x \in M^w$ with $x = m \Gamma x \Gamma s$.

Proof: Let $x \in M$. Now $x \in M = \langle m \rangle$ implies that $x = m\Gamma t_1$ for some $t_1 \in S$. If $t_1$ is not a unit, then $t_1 = m\Gamma t_2$ for some $t_2 \in S$ and hence $x = m\Gamma (m\Gamma t_2) = (m\Gamma)^2 t_2$. By preceding the same process, we get $x = (m\Gamma)^n t_n$ for some natural number $n$, for some $t_n \in S$. If $t_n$ is a unit for some natural number $n$, then $x = (m\Gamma)^n u$ where $u = t_n$, is a unit. If $t_n$ is not a unit, then $x = (m\Gamma)^n t_n$ for $n = 1, 2, 3, \ldots$ and hence $x = (m\Gamma)^n t_n \subseteq \bigcap_{n=1}^{\infty} \langle (m\Gamma)^{n-1} m \rangle \bigcap_{n=1}^{\infty} \langle (m > \Gamma)^{n-1} m \rangle = \langle m >^w = M^w$.

Therefore $x \in M^w$. Since $S$ is noetherian, the chain $\langle t_1 \rangle \subseteq \langle t_2 \rangle \subseteq \langle t_3 \rangle \ldots$ is stationary. Therefore there exists a natural number $n$ such that $\langle t_n \rangle = \langle t_{n+1} \rangle$ for all natural numbers $n$. Therefore $t_{n+1} = t_n \Gamma s$ for some $s \in S$.

Now $x \in (m\Gamma)^n t_n = (m\Gamma)^n t_{n+1} = (m\Gamma)^n t_n \Gamma s = x \Gamma s$

$\therefore x = (m\Gamma)^n t_n = m\Gamma (m\Gamma)^{n-1} t_n = m\Gamma x \Gamma s$

Theorem 5.2.28: Let $S$ be a noetherian duo monoid with a unique maximal $\Gamma$-ideal $M = \langle m \rangle$ for some $m \in S$. If $P$ is a proper prime $\Gamma$-ideal of $S$ such that $P \neq M$, then $P \subseteq M^w$.

Proof: Let $P$ be any proper prime $\Gamma$-ideal of $S$ such that $P \neq M$. Then there exists $x \in M$ such that $x \notin P$. By theorem 5.2.14, $P = x \Gamma P$. Thus $P = (x\Gamma)^n P$ for all natural numbers $n$. Therefore $P = \bigcap_{n=1}^{\infty} (x\Gamma)^n P \subseteq \bigcap_{n=1}^{\infty} \langle (x\Gamma)^{n-1} x \rangle \subseteq M^w$ and hence $P \subseteq M^w$.

Theorem 5.2.29: Let $S$ be a noetherian duo $\Gamma$-monoid with a unique maximal $\Gamma$-ideal $M = \langle m \rangle$ for some $m \in S$. If $S$ has no $\Gamma$-idempotents except 1, then $M^w$ is a prime $\Gamma$-ideal and also if $Z \neq M$ where $Z$ is the set of all non strongly $\Gamma$-cancellable elements of $S$, then $Z = M^w$. 

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\textbf{Proof} : Suppose that \(S\) has no \(\Gamma\)-idempotent elements except 1 and \(Z\) is the set of all non strongly \(\Gamma\)-cancellable elements of \(S\). Therefore \(Z\) is a proper prime \(\Gamma\)-ideal of \(S\). If \(Z \neq M\), then by theorem 5.2.28, \(Z \subseteq M^w\). Since \(Z\) is a prime \(\Gamma\)-ideal of \(S\) and \(M\) is \(\Gamma\)-ideal of \(S\), by theorem 5.2.26, \(M^w \subseteq Z\) and hence \(Z = M^w\).

\textbf{THEOREM 5.2.30} : Let \(T\) be a \(\Gamma\)-closed duo \(\Gamma\)-semigroup and \(S\) be a duo \(\Gamma\)-semigroup such that \(S\) is a \(\Gamma\)-subsemigroup of \(T\) and \(T = x \Gamma S^l\) for some \(x \in T\). If \(S\) is noetherian then \(T\) is noetherian.

\textbf{Proof} : Suppose that \(S\) is noetherian duo \(\Gamma\)-semigroup. Let \(A^1\) be a proper \(\Gamma\)-ideal of \(T\). Let \(A = \{ a \in S : x \Gamma a \subseteq A^1 \}\). Let \(a \in A\) and \(s \in S\). Now \(a \in A \Rightarrow x \Gamma a \subseteq A^1\). Now \(s \in S \subseteq T \Rightarrow s \in T\). Since \(A^1\) is a \(\Gamma\)-ideal of \(T\), \((x \Gamma a) \Gamma s \subseteq A^1\). So \(x \Gamma (a \Gamma s) \subseteq A^1\).

Thus \(a \Gamma s \subseteq A\). Therefore \(A\) is a right \(\Gamma\)-ideal of \(S\). Since \(S\) is a duo \(\Gamma\)-semigroup, \(A\) is a \(\Gamma\)-ideal of \(S\). Since \(S\) is noetherian, \(A = \bigcup_{i=1}^{n} a_i \Gamma S^l = \bigcup_{i=1}^{n} a_i \Gamma S^l\). Write \(B = \bigcup_{i=1}^{n} x \Gamma a_i \Gamma S^l\). Let \(y \in A^1\). Clearly \(y \neq x\) and \(y \in T = x \Gamma S^l\). So \(y = x \alpha s\) for some \(s \in S\), \(\alpha \in \Gamma\). Since \(x \alpha s = y \in A^1\) and \(A^1\) is \(\Gamma\)-closed we get \(x \Gamma s \subseteq A^1\) and hence \(s \in A\) and so \(s < a_i \geq a_i \Gamma S^l\) for some \(i\). Therefore \(y = x \alpha s \in x \Gamma a_i \Gamma S^l \subseteq B\). Therefore \(A^1 \subseteq B\). Let \(z \in B\).

Then \(z \in x \Gamma a_i \Gamma S^l\). Since \(a_i \Gamma S^l \subseteq A\), we have \(x \Gamma a_i \Gamma S^l \subseteq A^1\) and hence \(z \in A^1\). Therefore \(B \subseteq A^1\) and hence \(A^1 = B\). Now \(\bigcup_{i=1}^{n} x \Gamma a_i \Gamma S^l \subseteq A^1 = B = \bigcup_{i=1}^{n} x \Gamma a_i \Gamma S^l\).

Hence \(A^1 = \bigcup_{i=1}^{n} x \Gamma a_i \Gamma S^l\). \(\therefore T\) is noetherian.

Now we introduce the notion of center of \(S\).

\textbf{DEFINITION 5.2.31} : Let \(S\) be a \(\Gamma\)-semigroup. The set \(\mathcal{C}(S) = \{ a \in S : a \Gamma s = s \Gamma a \text{ for all } s \in S \}\) is called the \textit{center} of \(S\).

A close analogue of HILBERT’s basis theorem for commutative semigroups has been proved by SATYANARAYANA[44]. ANJANEYULU[4] extended the HILBERT’s basis theorem for duo semigroups. Here we extend it for duo \(\Gamma\)-semigroups.
THEOREM 5.2.32: (ANALOGUE OF HILBERT BASIS THEOREM): 

Let T be a $\Gamma$-closed duo $\Gamma$-semigroup and S be a duo $\Gamma$-semigroup such that S is a $\Gamma$-subsemigroup of T. Suppose $T = \bigcup_{i=1}^{n} x_{i} \Gamma S$, $x_{i}$'s are in the centre of T, $x_{i} \alpha x_{j} \in x_{i} \Gamma S$ or $x_{j} \Gamma S$ for some $\alpha \in \Gamma$, $i \neq j$ and $S \subseteq x_{i} \Gamma S$ for every i. Then T is noetherian if S is noetherian.

Proof: We prove the theorem by using induction on the number $n$ of generators of T. i.e. the number of $x_{i}$'s. By theorem 5.2.30, the result is evident for all T with one generator. Suppose that the theorem is true for all T with the number of generators $\leq n - 1$. Let $A^{1}$ be a $\Gamma$-ideal of T. Let $A = \{a \in S : x_{i} \Gamma a \subseteq A^{1}\}$. Clearly A is a $\Gamma$-ideal of S and hence $A = \bigcup_{i=1}^{n} a_{i} \Gamma S^{i}$, $a_{i} \in S$. If $T_{1} = \bigcup_{i=2}^{n} x_{i} \Gamma S$ and $L_{1} = A^{1} \cap T_{1}$, then $T_{1}$ is a $\Gamma$-subsemigroup of T containing S and $L_{1}$ is a $\Gamma$-ideal in $T_{1}$. Then by induction, $L_{1} = \bigcup_{i=1}^{r} \beta_{i} \Gamma T^{i}$. We first claim that $A^{1} = B \cup L_{1}$ where $B = \bigcup_{i=1}^{m} x_{i} \Gamma a_{i} \Gamma S^{i}$. We claim that $A^{1} = B \cup L_{1}$. If $x \in B$ then $x \in x_{i} \Gamma a_{i} \subseteq A^{1}$ or $x \in x_{i} \Gamma a_{i} \Gamma S^{i} \subseteq A^{1}$ and hence $x \in A^{1}$. Clearly $L_{1} \subseteq A^{1}$. Therefore $B \cup L_{1} \subseteq A^{1}$. Let $x \in A^{1}$. If $x \in T_{1}$ then $x \in A^{1} \cap T_{1} = L_{1}$ and hence $x \in B \cup L_{1}$. If $x \notin T_{1}$ then $x \in x_{i} \Gamma S$ and hence $x = x_{i} \gamma s$ for some $\gamma \in \Gamma$, $s \in S$. Since T is $\Gamma$-closed, $A^{1}$ is a $\Gamma$-closed $\Gamma$-ideal of T. So $x_{i} \gamma s = x \in A^{1} \Rightarrow x_{i} \Gamma S \subseteq A^{1} \Rightarrow s \in A^{1}$. $x \in x_{i} \Gamma S \subseteq x_{i} \Gamma a_{i} \Gamma S^{i}$ for some $i \Rightarrow x \in B$. Therefore $A^{1} \subseteq B \cup L_{1}$. Hence $A^{1} = B \cup L_{1}$. Now $A^{1} = B \cup L_{1} = \bigcup_{i=1}^{m} x_{i} \Gamma a_{i} \Gamma S^{i} \cup \bigcup_{i=1}^{r} \beta_{i} \Gamma T^{i}$. Therefore $A^{1}$ is finitely generated $\Gamma$-ideal. Hence T is noetherian.

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