Chapter 3

IDEALS IN DUO Γ-SEMIGROUPS

KRULL [29] proved that the nil-radical of an ideal A in a commutative ring is equal to the intersection of all minimal prime ideals containing A. SATYANARAYANA [43] obtained KRULL’s theorem [29] for commutative semigroups. ANJANEYULU [4] introduced the notions of ideals in duo semigroups and exhibit some examples and some classes of duo semigroups. He obtained KRULL’s theorem [29] for pseudo symmetric semigroups which includes duo semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [32], [33], [34] and [35] introduced the notions of duo Γ-semigroups and obtained KRULL’s theorem for pseudo and semipseudo symmetric Γ-semigroups. In this thesis we introduce and made a study on ideals in duo Γ-semigroups and obtained an analogue of KRULL’s theorem [29] in duo Γ-semigroups.

This chapter is divided into 5 sections. In section 1, the terms; left duo Γ-semigroup, right duo Γ-semigroup, duo Γ-semigroup are introduced. It is proved that a Γ-semigroup S is a duo Γ-semigroup if and only if
\[ x \Gamma S^1 = S^1 \Gamma x \] for all \( x \in S \). Further it is proved that (1) every commutative Γ-semigroup is a duo Γ-semigroup (2) every normal Γ-semigroup is a duo Γ-semigroup (3) every quasi commutative Γ-semigroup is a duo Γ-semigroup (4) every generalized Γ-semigroup is a left duo Γ-semigroup.

In section 2, it is proved that (1) if A is a Γ-ideal in a left duo Γ-semigroup S, then 
\[ A_\Gamma(a) = \{ x \in S : x \Gamma a \subseteq A \} \] is a Γ-ideal of S for all \( a \in S \), (2) if A is a Γ-ideal in a right duo Γ-semigroup S, then 
\[ A_\Gamma(a) = \{ x \in S : a \Gamma x \subseteq A \} \] is a Γ-ideal of S for all \( a \in S \), (3) if A is a Γ-ideal in a duo Γ-semigroup S, then 
\[ A_\Gamma(a) = \{ x \in S : x \Gamma a \subseteq A \} \] and 
\[ A_\Gamma(a) = \{ x \in S : a \Gamma x \subseteq A \} \] are Γ-ideals of S for all \( a \in S \). Further it is proved that (1) if A is a Γ-ideal in a left duo Γ-semigroup S and \( x, y \in S \), then \( x \Gamma y \subseteq A \) implies \( x \Gamma s \Gamma y \subseteq A \) for all \( s \in S \), (2) if A is a Γ-ideal in a right duo Γ-semigroup S and \( x, y \in S \), then \( x \Gamma y \subseteq A \) implies \( x \Gamma s \Gamma y \subseteq A \) for all \( s \in S \), (3) if A is a Γ-ideal in a duo Γ-semigroup S and \( x, y \in S \), then \( x \Gamma y \subseteq A \) implies \( x \Gamma s \Gamma y \subseteq A \). It is proved that if A is a Γ-ideal in a duo Γ-semigroup S and \( a, b \in S \), then (1) \( a \Gamma b \in A \) iff \( < a > \Gamma < b > \subseteq A \), (2) \( a_1 \Gamma a_2 \Gamma \ldots \Gamma a_n \Gamma a_n \subseteq A \) iff \( < a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A \), (3) for any natural number \( n \), \( (a \Gamma)^{n-1} a \subseteq A \) iff \( < a > \Gamma (a \Gamma)^{n-1} \subseteq A \). It is also proved that in a duo Γ-semigroup S, a Γ-ideal P is prime Γ-ideal if and only if P is a completely prime Γ-ideal. Further it is proved that a
Γ-ideal A of a duo Γ-semigroup S is a completely semiprime Γ-ideal of S if and only if A is a semiprime Γ-ideal.

In section 3, it is proved that, if \( A_1 = \) the intersection of all completely prime Γ-ideals of S containing A, \( A_2 = \{ x \in S : (x \Gamma)^{n-1} x \subseteq A \) for some natural number \( n \} \), \( A_3 = \) the intersection of all prime Γ-ideals of S containing A, \( A_4 = \{ x \in S : (< x > \Gamma)^{n-1} < x > \subseteq A \) for some natural number \( n \} \) for a Γ-ideal A of a Γ-semigroup S, then \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \). If A is a Γ-ideal of a commutative/duo Γ-semigroup then it is proved that \( A_1 = A_2 = A_3 = A_4 \). It is proved that if A is a Γ-ideal in a duo Γ-semigroup S, then (1) \( A_2 \) is the minimal completely semiprime Γ-ideal of S containing A, (2) \( A_4 \) is the minimal semiprime Γ-ideal of S containing A. It is proved that if \( a \in \sqrt{A} \), then there exist a positive integer \( n \) such that \( (a \Gamma)^{n-1} a \subseteq A \). Further if A is a Γ-ideal of a duo Γ-semigroup S then it is proved that (1) \( A_1 = \) the intersection of all completely prime Γ-ideals of S containing A, (2) \( A'_1 \) = the intersection of all minimal completely prime Γ-ideals of S containing A, (3) \( A''_1 \) = the minimal completely semiprime Γ-ideal of S containing A, (4) \( A_2 = \{ x \in S : (x \Gamma)^{n-1} x \subseteq A \) for some natural number \( n \} \), (5) \( A_3 = \) the intersection of all prime Γ-ideals of S containing A, (6) \( A'_3 \) = the intersection of all minimal prime Γ-ideals of S containing A, (7) \( A''_3 \) = the minimal semiprime Γ-ideal of S containing A, (8) \( A_4 = \{ x \in S : (< x > \Gamma)^{n-1} < x > \subseteq A \) for some natural number \( n \} \) are equal.

In section 4, the terms; Archimedean Γ-semigroup and strongly Archimedean Γ-semigroup are introduced. It is proved that if S is a duo Γ-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime Γ-ideals and (4) S has no proper prime Γ-ideals; are equivalent.

In section 5, the terms; left simple Γ-semigroup, right simple Γ-semigroup, simple Γ-semigroup are introduced. It is proved that (1) a Γ-semigroup S is a left simple Γ-semigroup if and only if \( S \Gamma a = S \) for all \( a \in S \), (2) a Γ-semigroup S is a right simple Γ-semigroup if and only if \( a \Gamma S = S \) for all \( a \in S \), (3) a Γ-semigroup S is a simple Γ-semigroup if and only if \( S \Gamma a \Gamma S = S \) for all \( a \in S \). It is also proved that if S is a left simple Γ-semigroup or a right simple Γ-semigroup then S is a simple Γ-semigroup. Further it is proved that if S is a duo Γ-semigroup and \( a \in S \) then (1) \( a \) is regular, (2) \( a \) is
left regular, (3) $a$ is right regular, (4) $a$ is intra regular and (5) $a$ is semisimple are equivalent.

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3.1. DUO $\Gamma$-SEMIGROUPS

Duo $\Gamma$-semigroups played an important role in the theory of $\Gamma$-semigroups. In this section the terms; left duo $\Gamma$-semigroup, right duo $\Gamma$-semigroup, duo $\Gamma$-semigroup are introduced. It is proved that a $\Gamma$- semigroup $S$ is a duo $\Gamma$- semigroup if and only if $x\Gamma S^1 = S^1 \Gamma x$ for all $x \in S$. Further it is proved that (1) every commutative $\Gamma$- semigroup is a duo $\Gamma$- semigroup (2) every normal $\Gamma$-semigroup is a duo $\Gamma$- semigroup (3) every quasi commutative $\Gamma$- semigroup is a duo $\Gamma$-semigroup (4) every generalized $\Gamma$- semigroup is a left duo $\Gamma$- semigroup.

We now introduce a left duo $\Gamma$-semigroup, right duo $\Gamma$-semigroup and duo $\Gamma$-semigroup.

**DEFINITION 3.1.1 :** A $\Gamma$- semigroup $S$ is said to be a left duo $\Gamma$- semigroup provided every left $\Gamma$- ideal of $S$ is a two sided $\Gamma$- ideal of $S$.

**DEFINITION 3.1.2 :** A $\Gamma$- semigroup $S$ is said to be a right duo $\Gamma$- semigroup provided every right $\Gamma$-ideal of $S$ is a two sided $\Gamma$- ideal of $S$.

**DEFINITION 3.1.3 :** A $\Gamma$- semigroup $S$ is said to be a duo $\Gamma$- semigroup provided it is both a left duo $\Gamma$- semigroup and a right duo $\Gamma$- semigroup.

**THEOREM 3.1.4 :** A $\Gamma$-semigroup $S$ is a duo $\Gamma$- semigroup if and only if $x\Gamma S^1 = S^1 \Gamma x$ for all $x \in S$.

**Proof :** Suppose that $S$ is a duo $\Gamma$-Semigroup and $x \in S$.

Let $t \in x\Gamma S^1$. Then $t=x\gamma s$ for some $s \in S^1$, $\gamma \in \Gamma$.

Since $S^1 \Gamma x$ is a left $\Gamma$-ideal of $S$, $S^1 \Gamma x$ is a $\Gamma$-ideal of $S$.

So $x \in S^1 \Gamma x$, $\gamma \in \Gamma$, $s \in S$, $S^1 \Gamma x$ is a $\Gamma$-ideal $\Rightarrow x\gamma s \in S^1 \Gamma x \Rightarrow t \in S^1 \Gamma x$.

Therefore $x\Gamma S^1 \subseteq S^1 \Gamma x$. Similarly we can prove that $S^1 \Gamma x \subseteq x\Gamma S^1$. Therefore $S^1 \Gamma x = x\Gamma S^1$.

Conversely suppose that $S^1 \Gamma x = x\Gamma S^1$ for all $x \in S$. Let $A$ be a left $\Gamma$-ideal of $S$.

Let $x \in A$, $s \in S$ and $a \in \Gamma$. Then $xas \in x\Gamma S^1 = S^1 \Gamma x \Rightarrow xas = tfx$ for some $t \in S^1$, $\beta \in \Gamma$.
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$x \in A, t \in S, \beta \in \Gamma, A$ is a left $\Gamma$-ideal of $S \Rightarrow t\beta x \in A \Rightarrow x\alpha s \in A$.

Therefore $A$ is a right $\Gamma$-ideal of $S$ and hence $A$ is a $\Gamma$-ideal of $S$.

Therefore $S$ is left duo $\Gamma$-semigroup.

Similarly we can prove that $S$ is a right duo $\Gamma$-semigroup. Hence $S$ is duo $\Gamma$-semigroup.

THEOREM 3.1.5: Every commutative $\Gamma$-semigroup is a duo $\Gamma$-semigroup.

Proof: Suppose that $S$ is a commutative $\Gamma$-semigroup. Therefore $x\Gamma S^1 = S^1 \Gamma x$ for all $x \in S$. By theorem 3.1.4, $S$ is a duo $\Gamma$-semigroup.

THEOREM 3.1.6: Every normal $\Gamma$-semigroup is a duo $\Gamma$-semigroup.

Proof: Suppose that $S$ is normal $\Gamma$-semigroup.

Then $a\Gamma S = S\Gamma a$ for all $a \in S \Rightarrow a\Gamma S^1 = S^1 \Gamma a$ for all $a \in S$.

By theorem 3.1.4, $S$ is a duo $\Gamma$-semigroup.

THEOREM 3.1.7: Every quasi commutative $\Gamma$-semigroup is a duo $\Gamma$-semigroup.

Proof: Suppose that $S$ is a quasi commutative $\Gamma$-semigroup. Then for $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a\gamma b = (b\gamma)^n a$ for all $\gamma \in \Gamma$. Let $A$ be a left $\Gamma$-ideal of $S$.

Therefore $S\Gamma A \subseteq A$. Let $a \in A$ and $s \in S$. Since $S$ is a quasi commutative $\Gamma$-semigroup, there exists a natural number $n$ such that $a\gamma s = (s\gamma)^n a \subseteq S\Gamma A \subseteq A$.

Therefore $a\Gamma s \subseteq A$ for all $a \in A$ and $s \in S$ and hence $A\Gamma S \subseteq A$. Therefore $S$ is a left duo $\Gamma$-semigroup. Similarly we can prove that $S$ is a right duo $\Gamma$-semigroup. Therefore every quasi commutative $\Gamma$-semigroup is a duo $\Gamma$-semigroup.

THEOREM 3.1.8: Every generalized commutative $\Gamma$-semigroup is a left duo $\Gamma$-semigroup.

Proof: Let $S$ be a generalized commutative $\Gamma$-semigroup. Therefore 1 is an $r$-element.

Let $A$ be a left $\Gamma$-ideal of $S$. Let $x \in A$ and $s \in S$.

Now $x\Gamma s = I\Gamma x\Gamma s = b\Gamma s\Gamma x = (b\Gamma s)\Gamma x \subseteq A$. Therefore $A$ is a $\Gamma$-ideal of $S$.

Therefore $S$ is a left duo $\Gamma$-semigroup.

3.2. \(\Gamma\)-IDEALS IN DUO \(\Gamma\)-SEMIGROUPS

In this section, it is proved that (1) if $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$, then $A(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$, (2) if $A$ is a $\Gamma$-ideal in a
right duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

(3) If $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ and $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are $\Gamma$-ideals of $S$ for all $a \in S$. Further it is proved that (1) if $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$.

We now characterize left duo $\Gamma$-semigroups.

**Theorem 3.2.1:** If $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

**Proof:** Let $x \in A_\Gamma(a)$ and $s \in S$. $x \in A_\Gamma(a) \Rightarrow x\Gamma a \subseteq A$.

$x\Gamma a \subseteq A$, $s \in S$, $A$ is a $\Gamma$-ideal $\Rightarrow s\Gamma x\Gamma a \subseteq A \Rightarrow s\Gamma x \subseteq A_\Gamma(a)$.

Therefore $A_\Gamma(a)$ is a left $\Gamma$-ideal of $S$. Since $S$ is a left duo $\Gamma$-semigroup, $A_\Gamma(a)$ is a $\Gamma$-ideal of $S$.

**Theorem 3.2.2:** If $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$.

**Proof:** Suppose that $x\Gamma y \subseteq A$. Let $s \in S$.

$x\Gamma y \subseteq A \Rightarrow x \in A_\Gamma(y)$.

$x \in A_\Gamma(y)$, $s \in S$, $A_\Gamma(y)$ is a $\Gamma$-ideal of $S \Rightarrow x\Gamma s \subseteq A_\Gamma(y) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

We now characterize right duo $\Gamma$-semigroups.

**Theorem 3.2.3:** If $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

**Proof:** Let $x \in A_\Gamma(a)$ and $s \in S$. $x \in A_\Gamma(a) \Rightarrow a\Gamma x \subseteq A$.

$a\Gamma x \subseteq A$, $s \in S$, $A$ is a $\Gamma$-ideal $\Rightarrow a\Gamma x\Gamma s \subseteq A \Rightarrow x\Gamma s \subseteq A_\Gamma(a)$. 

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Therefore $A_t(a)$ is a right $\Gamma$-ideal of $S$.
Since $S$ is a right duo $\Gamma$-semigroup, $A_t(a)$ is a $\Gamma$-ideal of $S$.

**THEOREM 3.2.4**: If $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma_s\Gamma y \subseteq A$.

**Proof**: Suppose that $x\Gamma y \subseteq A$. Let $s \in S$. $x\Gamma y \subseteq A \Rightarrow y \in A_t(x)$.

$y \in A_t(x)$, $s \in S$, $A_t(x)$ is a $\Gamma$-ideal of $S \Rightarrow s\Gamma y \subseteq A_t(x) \Rightarrow x\Gamma_s\Gamma y \subseteq A$.

We now characterize duo $\Gamma$-semigroups.

**COROLLARY 3.2.5**: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma_s\Gamma y \subseteq A$.

**THEOREM 3.2.6**: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A_t(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are $\Gamma$-ideals of $S$ for all $a \in S$.

**Proof**: Since $S$ is a duo $\Gamma$-semigroup, $S$ is left duo $\Gamma$-semigroup and hence by theorem 3.2.1, $A_t(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a $\Gamma$-ideal of $S$. Again $S$ is right duo $\Gamma$-semigroup and hence by theorem 3.2.3, $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$.

**THEOREM 3.2.7**: Let $A$ be a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $a, b \in S$. Then $a\Gamma b \in A$ if and only if $a > \Gamma < b > \subseteq A$.

**Proof**: Suppose that $a > \Gamma < b > \subseteq A$. Then $a\Gamma b \subseteq a > \Gamma < b > \subseteq A$.

Conversely suppose that $a\Gamma b \subseteq A$. Since $S$ is a duo $\Gamma$-semigroup. By corollary 3.2.5, $a\Gamma b \subseteq A \Rightarrow a\Gamma_s\Gamma b \subseteq A$ for all $s \in S \Rightarrow a\Gamma S^1 \Gamma b \subseteq A$. Since $A$ is a $\Gamma$-ideal, $a\Gamma S^1 \Gamma b \subseteq A \Rightarrow S^1 \Gamma a \Gamma S^1 \Gamma b \Gamma S^1 \subseteq A \Rightarrow a > \Gamma < b > \subseteq A$.

**THEOREM 3.2.8**: Let $A$ be a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$.
Then $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq A$ if and only if $a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

**Proof**: Suppose that $a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Then $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Conversely suppose that $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq A$.

Then for any $t \in a_1 > \Gamma < a_2 > \ldots \Gamma < a_n >$, we have $t = s_1 a_1 \beta_1 s_2 a_2 \beta_2 \ldots a_n a_n \beta_n s_{n+1}$, where $s_i \in S$ and $a_i, \beta_i \in \Gamma$.

Since $x, y \in S$, $x\Gamma y \subseteq A \Rightarrow x\Gamma_s\Gamma y \subseteq A$, we have $t \in A$.  

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Therefore $<a_1> \Gamma <a_2> \ldots \Gamma <a_n> \subseteq A$.

COROLLARY 3.2.9: Let $A$ be a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$. Then for any natural number $n$, $(a \Gamma)^{n-1}a \subseteq A$ if and only if $(<a \Gamma>)^{n-1} <a> \subseteq A$.

Proof: The proof follows from theorem 3.2.8, by taking $a_1 = a_2 = a_3 = \ldots = a_n = a$.

THEOREM 3.2.10: Let $S$ be a duo $\Gamma$-semigroup. A $\Gamma$-ideal $P$ of $S$ is prime $\Gamma$-ideal if and only if $P$ is a completely prime $\Gamma$-ideal.

Proof: Suppose that $P$ is a prime $\Gamma$-ideal of $\Gamma$-semigroup $S$. Let $x, y \in S$ and $x\Gamma y \subseteq P$. Now $x\Gamma y \subseteq P, P$ is a $\Gamma$-ideal $\Rightarrow x\Gamma y \Gamma S^1 \subseteq P$.

Since $S$ is duo $\Gamma$-semigroup, $x\Gamma S^1 \Gamma y = x\Gamma y \Gamma S^1 \subseteq P$.

By corollary 2.2.7, either $x \in P$ or $y \in P$. Hence $P$ is a completely prime $\Gamma$-ideal.

Conversely suppose that $P$ is a completely prime $\Gamma$-ideal of $S$.

By theorem 2.2.8, $P$ is a prime $\Gamma$-ideal of $S$.

COROLLARY 3.2.11: Let $S$ be a commutative $\Gamma$-semigroup. A $\Gamma$-ideal $P$ of $S$ is prime $\Gamma$-ideal if and only if $P$ is a completely prime $\Gamma$-ideal.

THEOREM 3.2.12: Let $S$ be a duo $\Gamma$-semigroup. A $\Gamma$-ideal $A$ of $S$ is completely semiprime iff semiprime.

Proof: Suppose that $A$ is a completely semiprime $\Gamma$-ideal of $S$.

By theorem 2.3.7, $A$ is a semiprime $\Gamma$-ideal of $S$.

Conversely Suppose that $A$ is a semiprime $\Gamma$-ideal of $S$. Let $x \in S$ and $x\Gamma x \subseteq A$.

Now $x\Gamma x \subseteq A \Rightarrow s\Gamma x \Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma S \Gamma x \subseteq A$

$\Rightarrow x \in A$, since $A$ is semiprime. Therefore $A$ is a completely semiprime $\Gamma$-ideal of $S$.

COROLLARY 3.2.13: Let $S$ be a commutative $\Gamma$-semigroup. A $\Gamma$-ideal $A$ of $S$ is completely semiprime iff semiprime.

3.3. $\Gamma$-RADICALS IN DUO $\Gamma$-SEMIGROUPS

In this section, it is proved that, if $A_1 = \text{the intersection of all completely prime}$ $\Gamma$-ideals of $S$ containing $A$, $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \}$, $A_3 = \text{the intersection of all prime } \Gamma$-ideals of $S$ containing $A$, $A_4 = \{x \in S : (<x \Gamma>)^{n-1} <x> \subseteq A \text{ for some natural number } n \}$ for a $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$, then
A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1. If \( A \) is a \( \Gamma \)-ideal of a commutative/duo \( \Gamma \)-semigroup then it is proved that \( A_1 = A_2 = A_3 = A_4 \). It is proved that if \( A \) is a \( \Gamma \)-ideal in a duo \( \Gamma \)-semigroup \( S \), then (1) \( A_2 \) is the minimal completely semiprime \( \Gamma \)-ideal of \( S \) containing \( A \), (2) \( A_4 \) is the minimal semiprime \( \Gamma \)-ideal of \( S \) containing \( A \). It is proved that if \( a \in \sqrt{A} \), then there exist a positive integer \( n \) such that \( (a\Gamma)^{n-1}a \not\subseteq A \). Further if \( A \) is a \( \Gamma \)-ideal of a duo \( \Gamma \)-semigroup \( S \) then it is proved that (1) \( A_1 \) = the intersection of all completely prime \( \Gamma \)-ideals of \( S \) containing \( A \), (2) \( A_4 \) = the intersection of all minimal completely prime \( \Gamma \)-ideals of \( S \) containing \( A \), (3) \( A_3 \) = the minimal completely semiprime \( \Gamma \)-ideal of \( S \) containing \( A \), (4) \( A_2 = \{ x \in S : (x\Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \} \), (5) \( A_3 = \text{the intersection of all prime } \Gamma \)-ideals of \( S \) containing \( A \), (6) \( A_4 \) = the intersection of all minimal prime \( \Gamma \)-ideals of \( S \) containing \( A \), (7) \( A_5 \) = the minimal semiprime \( \Gamma \)-ideal of \( S \) containing \( A \), (8) \( A_4 = \{ x \in S : (\langle x > \Gamma)^{n-1} < x > \subseteq A \text{ for some natural number } n \} \) are equal.

**NOTATION 3.3.1** : If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then we associate the following four types of sets.

\[ A_1 = \text{The intersection of all completely prime } \Gamma \text{-ideals of } S \text{ containing } A. \]

\[ A_2 = \{ x \in S : (x\Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \} \]

\[ A_3 = \text{The intersection of all prime ideals of } S \text{ containing } A. \]

\[ A_4 = \{ x \in S : (\langle x > \Gamma)^{n-1} < x > \subseteq A \text{ for some natural number } n \} \]

**NOTE 3.3.2** : If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) then \( \text{rad } A = A_3 \) and \( \text{c.rad } A = A_4 \).

**THEOREM 3.3.3** : If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \).

**Proof** : (i) \( A \subsetneq A_4 \) : Let \( x \in A \). Then \( (\langle x > \Gamma)^{0} < x > \subseteq A \) and hence \( x \in A_4 \). \( A \subsetneq A_4 \).

(ii) \( A_4 \subsetneq A_3 \) : Let \( x \in A_4 \). Then \( (\langle x > \Gamma)^{n-1} < x > \subseteq A \) for some \( n \in \mathbb{N} \).

Let \( P \) be any prime \( \Gamma \)-ideal of \( S \) containing \( A \).

Then \( (\langle x > \Gamma)^{n-1} < x > \subseteq A \Rightarrow (\langle x > \Gamma)^{n-1} < x > \subseteq P. \)

Since \( P \) is prime, \( < x > \subseteq P \) and hence \( x \in P \).

Since this is true for all prime \( \Gamma \)-ideals \( P \) containing \( A \), \( x \in A_3 \). Therefore \( A_4 \subseteq A_3 \).

(iii) \( A_3 \subsetneq A_2 \) : Let \( x \in A_3 \). Suppose if possible \( x \not\in A_2 \). Then \( (x\Gamma)^{n-1}x \not\subseteq A \) for all \( n \in \mathbb{N} \).

Consider \( T = \bigcup (x\Gamma)^{r-1}x \), where \( x \in S \) and \( n \) is a natural number.

Let \( a, b \in T \). Then \( a \in (x\Gamma)^{r-1}x, b \in (x\Gamma)^{s-1}x \) for some \( r, s \in \mathbb{N} \).
Therefore \( a\Gamma b = (x\Gamma)^{s-1}x\Gamma(x\Gamma)^{s-1}x = (x\Gamma)^{s+1}x \subseteq T.\)

Therefore \( T \) is a \( \Gamma \)-subsemigroup of \( S \) and \( T \) is a \( c \)-system of \( S \) and \( x \in T.\)

By theorem 2.2.4, \( P = S\setminus T \) is a completely prime \( \Gamma \)-ideal of \( S \) and \( x \not\in P.\)

By theorem 2.2.8, \( P \) is prime \( \Gamma \)-ideal of \( S \) and \( x \not\in P.\)

Therefore \( x \not\in A_3. \) It is a contradiction. \( \therefore x \in A_2 \) and hence \( A_3 \subseteq A_2. \)

(iv) \( A_2 \subseteq A_1: \) Let \( x \in A_2. \) Now \( x \in A_2 \Rightarrow (x\Gamma)^{n-1}x \subseteq A \) for some natural number \( n.\)

Let \( P \) be any completely prime \( \Gamma \)-ideal of \( S \) containing \( A.\)

Then \( (x\Gamma)^{n-1}x \subseteq A \subseteq P \Rightarrow (x\Gamma)^{n-1}x \subseteq P \Rightarrow x \in P. \) Therefore \( x \in A_1. \) Therefore \( A_2 \subseteq A_1. \)

Hence \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1. \)

**THEOREM 3.3.4:** If \( A \) is a \( \Gamma \)-ideal of a commutative \( \Gamma \)-semigroup \( S, \) then \( A_1 = A_2 = A_3 = A_4. \)

**Proof:** By theorem 3.3.3, \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1. \) By corollary 3.2.11, in a commutative \( \Gamma \)-semigroup \( S, \) a \( \Gamma \)-ideal \( P \) is a prime \( \Gamma \)-ideal iff \( P \) is a completely prime \( \Gamma \)-ideal. \( \) So \( A_1 = A_3. \) By theorem 3.2.13, in a commutative \( \Gamma \)-semigroup \( S, \) a \( \Gamma \)-ideal \( P \) is a semiprime \( \Gamma \)-ideal iff \( P \) is a completely semiprime \( \Gamma \)-ideal. \( \) So \( A_4 = A_2. \)

Therefore \( A_1 = A_2 = A_3 = A_4. \)

**NOTE 3.3.5:** If \( A \) is a \( \Gamma \)-ideal in a arbitrary \( \Gamma \)-semigroup, then \( A_1, A_2, A_3, A_4 \) need not be equal.

**EXAMPLE 3.3.6:** Let \( S \) be the free \( \Gamma \)-semigroup generated by two alphabets \( a, b. \) It is clear that \( A = S\Gamma a\Gamma a\Gamma S \) is a \( \Gamma \)-ideal in \( S. \) Since \( (a\Gamma)^3a \subseteq S\Gamma a\Gamma a\Gamma S = A, \) We have \( a \in A_2. \)

Evidently \( (a\Gamma b\Gamma)^{n-1}a\Gamma b \not\subseteq S\Gamma a\Gamma a\Gamma S \) for all natural number \( n \) and thus \( a\Gamma b \not\subseteq A_2. \) Thus \( A_2 \) is not a \( \Gamma \)-ideal in \( S. \) Therefore \( A_1 \neq A_2 \) and \( A_2 \neq A_3. \)

**EXAMPLE 3.3.7:** Let \( S \) be the free \( \Gamma \)-semigroup over the countable infinite alphabet \( \{ x_1, x_2, \ldots \} \) and \( \Gamma \) as \( \{ \alpha_1, \alpha_2, \ldots \}. \) Consider the \( \Gamma \)-ideal \( A = \bigcup_{l(s)=1} (x > s \Gamma)^{l(s)-1}x \leq A, \) where \( l(s) \) is the length of the word \( s. \) For any \( s \in S, \)

\(< x_j\Gamma a\Gamma x_j \geq l(s)+1 \leq x_j\Gamma a\Gamma x_j \subseteq A \) and hence \( x_j\Gamma s\Gamma x_j \subseteq A_4 \) for all \( s \in S. \) If \( A_3 = A_4, \) then \( A_4 \) is a semiprime \( \Gamma \)-ideal and hence \( x_j \in A_4. \) Therefore \( (x > x_j \Gamma)^{n-1}x \leq A \) for some natural number \( n. \) Consider the word \( t = x_1\alpha_1x_2\alpha_2x_3\alpha_3x_4\alpha_4 \ldots \ldots \alpha_{n-1}x_j\alpha_nx_{n+1}. \)

Now \( t \in x > x_j \Gamma^{n-1}x \leq A. \) So \( t \in s\Gamma \geq l(s)+1 \leq s \) for some \( s \in S \) with \( l(s) > 1. \)

Thus in \( t, s \) occurs at least two times, which is a contradiction. So \( A_3 \neq A_4. \)
THEOREM 3.3.8: If A is a Γ-ideal of a duo Γ-semigroup S, then \( A_1 = A_2 = A_3 = A_4 \).

**Proof**: By theorem 3.3.3, \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \). By theorem 3.1.10, in a duo Γ-semigroup S, a Γ-ideal P is a prime Γ-ideal iff P is a completely prime Γ-ideal.

So \( A_1 = A_3 \). By theorem 3.2.12, in a duo Γ-semigroup S, a Γ-ideal P is a semiprime Γ-ideal iff P is a completely semiprime Γ-ideal. So \( A_4 = A_2 \).

Therefore \( A_1 = A_2 = A_3 = A_4 \).

THEOREM 3.3.9: If A is a Γ-ideal of a duo Γ-semigroup S, then \( \text{rad} \ A = c.\text{rad} \ A \).

**Proof**: By theorem 3.3.8, \( \text{rad} \ A = c.\text{rad} \ A \).

THEOREM 3.3.10: If A is a Γ-ideal in a duo Γ-semigroup S. Then \( A_2 = \{ x \in S : (x\Gamma)^{n-1} x \subseteq A \text{ for some } n \in \mathbb{N} \} \) is the minimal completely semiprime Γ-ideal of S containing A.

**Proof**: Clearly \( A \subseteq A_2 \) and hence \( A_2 \) is nonempty subset of S. Let \( x \in A_2 \) and \( s \in S \).

Since \( x \in A_2 \), \((x\Gamma)^{n-1} x \subseteq A \text{ for some } n \in \mathbb{N} \). Now \((x\Gamma s)^{n-1} x\Gamma s \subseteq A \) and \((s\Gamma x)^{n-1} s\Gamma x \subseteq A \) implies \( x\Gamma s, s\Gamma x \in A_2 \). Therefore \( A_2 \) is a Γ-ideal of S containing \( A \). Let \( x \in S \) such that \( x\Gamma x \subseteq A_2 \). Then \((x\Gamma x\Gamma)^{n-1} x \Gamma x \subseteq A \) implies \((x\Gamma)^{2n-1} x \subseteq A \Rightarrow x \in A_2 \). Thus \( A_2 \) is a completely semiprime Γ-ideal of S containing \( A \). Let \( P \) be a completely semi prime Γ-ideal of S containing \( A \). Let \( x \in A_2 \). Then \((x\Gamma)^{n-1} x \subseteq A \text{ for some } n \in \mathbb{N} \). Since \( A \subseteq P \), then \((x\Gamma)^{n-1} x \subseteq P \), for some \( n \in \mathbb{N} \). Since \( P \) is completely semiprime Γ-ideal of S, \((x\Gamma)^{n-1} x \subseteq P \Rightarrow x \in P \). Therefore \( A_2 \subseteq P \) and hence \( A_2 \) is the minimal completely semiprime Γ-ideal of S containing \( A \).

THEOREM 3.3.11: If A is a Γ-ideal in a duo Γ-semigroup S, then \( A_4 = \{ x \in S : (< x >\Gamma)^{n-1} < x > \subseteq A \text{ for some } n \in \mathbb{N} \} \) is the minimal semiprime Γ-ideal of S containing A.

**Proof**: Clearly \( A \subseteq A_4 \) and hence \( A_4 \) is nonempty subset of S. Let \( x \in A_4 \) and \( s \in S \).

Since \( x \in A_4 \), \((< x >\Gamma)^{n-1} < x > \subseteq A \text{ for some } n \in \mathbb{N} \).

Now \((< x \Gamma s >\Gamma)^{n-1} < x \Gamma s > \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A \) and \((s\Gamma x >\Gamma)^{n-1} s\Gamma x \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A \) implies \( x\Gamma s, s\Gamma x \in A_4 \).

Therefore \( A_4 \) is a Γ-ideal of S containing \( A \). Let \( x \in S \) such that \((< x > \Gamma) < x > \subseteq A_4 \).
Then $<(x >\Gamma <x >\Gamma)^{n-1} <x >\subseteq A$ implies $(<x >\Gamma)^{2n-1} <x >\subseteq A \Rightarrow x \in A_{4}$

Thus $A_{4}$ is semiprime $\Gamma$-ideal of $S$ containing $A$.

Let $Q$ be a semiprime $\Gamma$-ideal of $S$ containing $A$. Let $x \in A_{4}$. Then $(<x >\Gamma)^{n-1} <x >\subseteq A$ for some $n \in \mathbb{N}$. Since $A \subseteq Q$, then $(<x >\Gamma)^{n-1} <x >\subseteq Q$ for some $n \in \mathbb{N}$.

Since $Q$ is a semiprime $\Gamma$-ideal of $S$, $(<x >\Gamma)^{n-1} <x >\subseteq Q \Rightarrow x \in Q$.

Therefore $A_{4} \subseteq Q$ and hence $A_{4}$ is the minimal semiprime $\Gamma$-ideal of $S$ containing $A$.

COROLLARY 3.3.12 : If $A$ is a $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$ then

1. $A_{1}$ = the intersection of all completely prime $\Gamma$-ideals of $S$ containing $A$,
2. $A_{1}'$ = the intersection of all minimal completely prime $\Gamma$-ideals of $S$ containing $A$,
3. $A_{1}''$ = the minimal completely semiprime $\Gamma$-ideal of $S$ containing $A$,
4. $A_{2} = \{ x \in S : (x\Gamma)^{n-1} x \subseteq A \text{ for some natural number } n \}$,
5. $A_{3}$ = the intersection of all prime $\Gamma$-ideals of $S$ containing $A$,
6. $A_{3}'$ = the intersection of all minimal prime $\Gamma$-ideals of $S$ containing $A$,
7. $A_{3}''$ = the minimal semiprime $\Gamma$-ideal of $S$ containing $A$,
8. $A_{4} = \{ x \in S : (<x >\Gamma)^{n-1} <x >\subseteq A \text{ for some natural number } n \}$ are equal.

THEOREM 3.3.13 : If $a \in \sqrt{A}$, then there exist a positive integer $n$ such that $(a\Gamma)^{n-1} a \subseteq A$.

Proof : By theorem 3.3.3, $A_{3} \subseteq A_{2}$ and hence $a \in \sqrt{A} = A_{3} \subseteq A_{2}$.

Therefore $(a\Gamma)^{n-1} a \subseteq A$ for some $n \in \mathbb{N}$.

3.4. ARCHIMEDEAN $\Gamma$-SEMIGROUPS

In this section, the terms; Archimedean $\Gamma$-semigroup and strongly Archimedean $\Gamma$-semigroup are introduced. It is proved that if $S$ is a duo $\Gamma$-semigroup, then the conditions (1) $S$ is strongly Archimedean, (2) $S$ is Archimedean, (3) $S$ has no proper completely prime $\Gamma$-ideals and (4) $S$ has no proper prime $\Gamma$-ideals; are equivalent.

We now introduce the notions of archimedean $\Gamma$-semigroup and strongly archimedean $\Gamma$-semigroup.

DEFINITION 3.4.1 : A $\Gamma$-semigroup $S$ is said to be an archimedean $\Gamma$-semigroup provided for any $a,b \in S$, there exists a natural number $n$ such that $(a\Gamma)^{n-1} a \subseteq <b >$. 
DEFINITION 3.4.2: A \( \Gamma \)-semigroup \( S \) is said to be a strongly archimedean \( \Gamma \)-semigroup provided for any \( a, b \in S \), there is a natural number \( n \) such that \( (a > \Gamma)^{n-1} < a \subseteq < b > \).

We now characterize archimedean \( \Gamma \)-semigroups.

THEOREM 3.4.3: If \( S \) is a duo \( \Gamma \)-semigroup, then \( S \) is strongly archimedean if and only if archimedean.

Proof: Suppose that \( S \) is strongly Archimedean. Then for any \( a, b \in S \), there is a natural number \( n \) such that \( (a > \Gamma)^{n-1} < a \subseteq < b > \). Therefore \( (a\Gamma)^{n-1}a \subseteq (a > \Gamma)^{n-1} < a \subseteq < b > \) and hence \( S \) is Archimedean.

Conversely suppose that \( S \) is archimedean. Let \( a, b \in S \). Since \( S \) is archimedean, there exists a natural number \( n \) such that \( (a > \Gamma)^{n-1} < a \subseteq < b > \subseteq S \Gamma b \Gamma S \). Since \( S \Gamma b \Gamma S \) is a \( \Gamma \)-ideal of a duo \( \Gamma \)-semigroup \( S \), by corollary 3.2.5, \( (a\Gamma)^{n-1}a \subseteq S \Gamma b \Gamma S \Rightarrow (a > \Gamma)^{n-1} < a \subseteq S \Gamma b \Gamma S \). Therefore \( S \) is a strongly Archimedean duo \( \Gamma \)-semigroup.

THEOREM 3.4.4: If \( S \) is a duo \( \Gamma \)-semigroup, then \( S \) is archimedean if and only if \( S \) has no proper prime \( \Gamma \)-ideals.

Proof: Suppose that \( S \) is archimedean \( \Gamma \)-semigroup. Let \( P \) be prime \( \Gamma \)-ideal of \( S \). Let \( a, b \in S \). Since \( P \) is \( \Gamma \)-ideal, \( S \Gamma a \Gamma S \subseteq P \). Since \( S \) is archimedean, \( (b\Gamma)^{n-1} \subseteq S \Gamma a \Gamma S \) for some natural number \( n \). Thus \( (b\Gamma)^{n-1} \subseteq S \Gamma a \Gamma S \subseteq P \). Since \( S \) is a duo \( \Gamma \)-semigroup, by theorem 3.2.10, \( P \) is completely prime. Thus \( (b\Gamma)^{n-1}b \subseteq P \Rightarrow b \in P \). Hence \( S = P \). Therefore \( S \) has no proper prime \( \Gamma \)-ideals.

Conversely suppose that \( S \) has no proper prime \( \Gamma \)-ideals. Then for any \( b \in S \), the intersection of all prime \( \Gamma \)-ideals of \( S \) containing \( B = < b > \) is \( S \) itself. Therefore \( B = S \). We have \( B = \{ x \in S : (x > \Gamma)^{n-1} < x \subseteq < b > \text{ for some } n \in N \} = S \).

Therefore for any \( a \in S \), \( (a > \Gamma)^{n-1} < a \subseteq < b > \) for some natural number \( n \).

So \( (a > \Gamma)^{n-1} < a \subseteq S \Gamma b \Gamma S \). Thus \( S \) is strongly archimedean.

Hence by theorem 3.4.3, \( S \) is archimedean.

COROLLARY 3.4.5: If \( S \) is a duo \( \Gamma \)-semigroup, then the conditions (1) \( S \) is strongly Archimedean, (2) \( S \) is Archimedean, (3) \( S \) has no proper completely prime \( \Gamma \)-ideals and (4) \( S \) has no proper prime \( \Gamma \)-ideals are equivalent.
3.5. SIMPLE Γ-SEMIGROUPS

In this section, the terms; left simple Γ-semigroup, right simple Γ-semigroup, simple Γ-semigroup are introduced. It is proved that (1) a Γ-semigroup S is a left simple Γ-semigroup if and only if $S\Gamma a = S$ for all $a \in S$, (2) a Γ-semigroup S is a right simple Γ-semigroup if and only if $aS = S$ for all $a \in S$, (3) a Γ-semigroup S is a simple Γ-semigroup if and only if $S\Gamma a\Gamma S = S$ for all $a \in S$. It is also proved that if S is a left simple Γ-semigroup or a right simple Γ-semigroup then S is a simple Γ-semigroup. Further it is proved that if S is a duo Γ-semigroup and $a \in S$ then (1) a is regular, (2) a is left regular, (3) a is right regular, (4) a is intra regular and (5) a is semisimple are equivalent.

We now introduce a left simple Γ-semigroup.

DEFINITION 3.5.1: A Γ-semigroup S is said to be a left simple Γ-semigroup if S is its only left Γ-ideal.

We now characterize left simple Γ-semigroups.

THEOREM 3.5.2: A Γ-semigroup S is a left simple Γ-semigroup if and only if $S\Gamma a = S$ for all $a \in S$.

Proof: Suppose that S is a left simple Γ-semigroup and $a \in S$.

Let $t \in S\Gamma a$, $s \in S$, $\gamma \in \Gamma$.

$t \in S\Gamma a \Rightarrow t = s_1\alpha a$ where $s_i \in S$ and $\alpha \in \Gamma$.

Now $s\gamma t = s\gamma(s_1\alpha a) = (s\gamma s_1)\alpha a \in S\Gamma a \Rightarrow S\Gamma a$ is a left Γ-ideal of S.

Since S is a left simple Γ-semigroup, $S\Gamma a = S$.

Therefore $S\Gamma a = S$ for all $a \in S$.

Conversely suppose that $S\Gamma a = S$ for all $a \in S$. Let L be a left Γ-ideal of S.

Let $l \in L$. Then $l \in S$. By assumption $S\Gamma l = S$.

Let $s \in S$. Then $s \in S\Gamma l \Rightarrow s = tal$ for some $t \in S$, $\alpha \in \Gamma$.

$l \in L$, $t \in S$, $a \in \Gamma$ and L is a left Γ-ideal $\Rightarrow tal \in L \Rightarrow s \in L$.

Therefore S $\subseteq L$. Clearly L $\subseteq S$ and hence S = L.

Therefore S is the only left Γ-ideal of S. Hence S is left simple Γ-semigroup.
We now introduce a right simple $\Gamma$-semigroup.

**DEFINITION 3.5.3:** A $\Gamma$-semigroup $S$ is said to be a *right simple $\Gamma$-semigroup* if $S$ is its only right $\Gamma$-ideal.

We now characterize right simple $\Gamma$-semigroups.

**THEOREM 3.5.4:** A $\Gamma$-semigroup $S$ is a right simple $\Gamma$-semigroup if and only if $a\Gamma S = S$ for all $a \in S$.

*Proof:* Suppose that $S$ is a right simple $\Gamma$-semigroup and $a \in S$. Let $t \in a\Gamma S, s \in S, \gamma \in \Gamma$. 
$t \in a\Gamma S \Rightarrow t = a\alpha s_i$ where $s_i \in S$ and $\alpha \in \Gamma$.

Now $t\gamma s = (a\alpha s_i)\gamma s = a\alpha(s_i\gamma s) \in a\Gamma S \Rightarrow a\Gamma S$ is a right $\Gamma$-ideal of $S$.

Since $S$ is a right simple $\Gamma$-semigroup, $a\Gamma S = S$.

Therefore $a\Gamma S = S$ for all $a \in S$.

Conversely suppose that $a\Gamma S = S$ for all $a \in S$.

Let $R$ be a right $\Gamma$-ideal of a $\Gamma$-semigroup $S$.

Let $r \in R$. Then $r \in S$. By assumption $r\Gamma S = S$.

Let $s \in S$. Then $s \in r\Gamma S \Rightarrow s = rat$ for some $t \in S, \alpha \in \Gamma$.

$r \in R, t \in S, \alpha \in \Gamma$ and $R$ is a right $\Gamma$-ideal $\Rightarrow rat \in R \Rightarrow s \in R$.

Therefore $S \subseteq R$. Clearly $R \subseteq S$ and hence $S = R$.

Therefore $S$ is the only right $\Gamma$-ideal of $S$. Hence $S$ is right simple $\Gamma$-semigroup.

We now introduce a simple $\Gamma$-semigroup.

**DEFINITION 3.5.5:** A $\Gamma$-semigroup $S$ is said to be *simple $\Gamma$-semigroup* if $S$ is its only two-sided $\Gamma$-ideal.

We now characterize simple $\Gamma$-semigroups.

**THEOREM 3.5.6:** If $S$ is a left simple $\Gamma$-semigroup or a right simple $\Gamma$-semigroup then $S$ is a simple $\Gamma$-semigroup.

*Proof:* Suppose that $S$ is a left simple $\Gamma$-semigroup. Then $S$ is the only left $\Gamma$-ideal of $S$.

If $A$ is a $\Gamma$-ideal of $S$, then $A$ is a left $\Gamma$-ideal of $S$ and hence $A = S$.

Therefore $S$ itself is the only $\Gamma$-ideal of $S$ and hence $S$ is a simple $\Gamma$-semigroup.

Suppose that $S$ is a right simple $\Gamma$-semigroup. Then $S$ is the only right $\Gamma$-ideal of $S$.

If $A$ is a $\Gamma$-ideal of $S$, then $A$ is a right $\Gamma$-ideal of $S$ and hence $A = S$.

Therefore $S$ itself is the only $\Gamma$-ideal of $S$ and hence $S$ is a simple $\Gamma$-semigroup.
THEOREM 3.5.7 : A Γ-semigroup S is simple Γ-semigroup if and only if SΓaΓS = S for all a ∈ S.

Proof : Suppose that S is a simple Γ-semigroup and a ∈ S. Let t ∈ SΓaΓS, s ∈ S and γ ∈ Γ.

\( t ∈ SΓaΓS \Rightarrow t = s_1ααβs_2 \) \( s_1, s_2 ∈ S \) and α, β ∈ Γ.

Now 1γs = (s_1ααβs_2)γ s = s_1ααβ(s_1γs) ∈ SΓaΓS

and sγt = sγ(s_1ααβs_2) = (sγs_1)ααβs_2 ∈ SΓaΓS. Therefore SΓaΓS is a Γ-ideal of S.

Since S is a simple Γ-semigroup, S itself is the only Γ-ideal of S and hence SΓaΓS = S.

Conversely suppose that SΓaΓS = S for all a ∈ S. Let I be a Γ-ideal of S. Let a ∈ I. Then a ∈ S. So SΓaΓS = S.

Let s ∈ S. Then s ∈ SΓaΓS ⇒ s = t_1ααβt_2 for some t_1, t_2 ∈ S, α, β ∈ Γ.

Therefore S ⊆ I. Clearly I ⊆ S and hence S = I.

Therefore S is the only Γ-ideal of S. Hence S is a simple Γ-semigroup.

THEOREM 3.5.8 : If S is a duo Γ-semigroup, then the following are equivalent for any element a ∈ S.

1) a is regular.
2) a is left regular.
3) a is right regular.
4) a is intra regular.
5) a is semisimple.

Proof : Since S is duo Γ-semigroup, aΓS^1 = S^1Γa.

We have aΓS^1Γa = aΓaΓS^1 = S^1ΓaΓa = < aΓa > = < a > Γ < a >.

(1) ⇒ (2) : Suppose that a is regular. Then a = aαβγa for some x ∈ S and α, β, γ ∈ Γ.

Therefore a ∈ aΓS^1Γa = aΓaΓS^1 ⇒ a = aγαδγ for some y ∈ S^1, γ, δ ∈ Γ.

Therefore a is left regular.

(2) ⇒ (3) : Suppose that a is left regular. Then a = aαβxy for some x ∈ S and α, β, γ ∈ Γ.

Therefore a ∈ aΓaΓS^1 = S^1ΓaΓa ⇒ a = yγαδa for some y ∈ S^1, γ, δ ∈ Γ.

Therefore a is right regular.

(3) ⇒ (4) : Suppose that a is right regular. Then for some x ∈ S, α, β, γ ∈ Γ; a = xαβγa.

Therefore a ∈ S^1ΓaΓa = < aΓa > ⇒ a = xαβγaγ for some x, y ∈ S^1 and α, β, γ ∈ Γ.

Therefore a is intra regular.
(4) $\Rightarrow$ (5): Suppose that $a$ is intra regular. Then $a = xax\beta a\gamma$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$. Therefore $a \in < a > \Gamma < a >$. Therefore $a$ is semisimple.

(5) $\Rightarrow$ (1): Suppose that $a$ is semisimple. Then $a \in < a > \Gamma < a > = a\Gamma S^1 \Gamma a$ $\Rightarrow a \in aax\beta a$ for some $x \in S^1$ and $\alpha, \beta \in \Gamma$. Therefore $a$ is a regular element. 

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