CHAPTER-VI

PAIRWISE GENERALIZED $b$-$R_0$ SPACES
IN BITOPOLOGICAL SPACES

6.1 DEFINITIONS AND PRELIMINARIES

In this section, we list some known definitions and results those will be used throughout this Chapter.

**DEFINITION 6.1.1.** A subset $A$ of a topological space $(X, \tau)$ is called $b$-open, if $A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$ and called $b$-closed if $X \setminus A$ is $b$-open.

**DEFINITION 6.1.2.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-$b$-open if $A \subseteq \text{int}(j \text{ cl}(A)) \cup j \text{ cl}(i \text{ int}(A))$, where $i, j = 1, 2$ and $i \neq j$.

The complement of $(i, j)$-$b$-open set is said to be $(i, j)$-$b$-closed.

Clearly, $A$ is $(i, j)$-$b$-closed if and only if $i \text{ cl}(j \text{ int}(A)) \cap j \text{ int}(i \text{ cl}(A)) \subseteq A$.

**DEFINITION 6.1.3.** Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. Then

(i) The $(i, j)$-$b$-closure of $A$ denoted by $(i, j)$-$bcl(A)$, is defined by the intersection of all $(i, j)$-$b$-closed sets containing $A$.

(ii) The $(i, j)$-$b$-interior of $A$, denoted by $(i, j)$-$bint(A)$, is defined by the union of all $(i, j)$-$b$-open sets contained in $A$.

In Chapter-IV, we have established the following definitions and results.

**DEFINITION 6.1.4.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-generalized $b$-closed (in short, $(i, j)$-gb-closed) set if $(j, i)$-$bcl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\tau_i$-open in $X$, for $i, j = 1, 2$ and $i \neq j$. 
Lemma 6.1.1. A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is \((i, j)\)-gb-open if and only if \( U \subset (j, i)\)-bint\( A \), whenever \( U \) is \( \tau_1 \)-closed and \( U \subset A \), for \( i, j = 1, 2 ; i \neq j \).

Definition 6.1.5. The \((i, j)\)-generalized \( b \)-closure of a subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is the intersection of all \((i, j)\)-gb-closed sets containing \( A \) and is denoted by \((i, j)\)-gbcl\( A \).

Lemma 6.1.2. For any subset \( A \) of a bitopological space \((X_2, \tau_1, \tau_2)\), \( A \subset (i, j)\)-gbcl\( A \).

Lemma 6.1.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space. If \( A \) is \((i, j)\)-gb-closed subset of \( X \), then \( A = (i, j)\)-gbcl\( A \).

Lemma 6.1.4. A point \( x \in (i, j)\)-gbcl\( A \) if and only if for every \((i, j)\)-gb-open set \( U \) containing \( x \), \( U \cap A \neq \emptyset \).

In this Chapter, we have introduced the following definitions.

Definition 6.1.6. A bitopological space \((X, \tau_1, \tau_2)\) is said to be \textit{pairwise generalized \( b \)-R}_\( o \) (in short, \textit{pairwise \( gb \)-R}_\( o \)) spaces if \((j, i)\)-gbcl\( \{x\} \subset U \), for every \((i, j)\)-gb-open set \( U \) containing \( x \) and \( i, j = 1, 2 ; i \neq j \).

Definition 6.1.7. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \subset X \). The intersection of all \((i, j)\)-gb-open sets containing \( A \) is called the \textit{(i, j)-gb-kernel} of \( A \) and is denoted by\((i, j)\)-gbker\( A \).

The \((i, j)\)-gb-kernel of a point \( x \in X \) is the set \((i, j)\)-gbker\( \{x\} = \cap \{ U : U \text{ is (i, j)-gb-open and } x \in U \} \).

\[ = \{ y : x \in (i, j)\text{-gbcl}\{y\} \}. \]
DEFINITION 6.1.8. A bitopological space \((X, \tau_1, \tau_2)\) is said to be \textit{pairwise generalized b-R} \(_1\) (in short, \textit{pairwise gb-R} \(_1\)) if for every pair of distinct points \(x\) and \(y\) of \(X\) such that \((i, j)\)-\(gbcl\)\(\{x\}\) \(\neq\) \((j, i)\)-\(gbcl\)\(\{y\}\), there exists a \((j, i)\)-gb-open set \(U\) and a \((i, j)\)-gb-open set \(V\) such that \(U \cap V = \emptyset\) and \((i, j)\)-\(gbcl\)\(\{x\}\) \(\subset U\), \((j, i)\)-\(gbcl\)\(\{y\}\) \(\subset V\), for \(i, j = 1, 2\) and \(i \neq j\).

6.2 MAIN RESULTS

We have established the following results in this Chapter.

THEOREM 6.2.1. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then the following statements are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_0\) space.

(b) For any \((i, j)\)-gb-closed set \(V\) and \(x \notin V\), there exist a \((j, i)\)-gb-open set \(U\) such that \(x \notin U\) and \(V \subset U\), for \(i, j = 1, 2\) and \(i \neq j\).

(c) For any \((i, j)\)-gb-closed set \(V\) and \(x \notin V\), \((j, i)\)-\(gbcl\)\(\{x\}\) \(\cap V = \emptyset\), for \(i, j = 1, 2\) and \(i \neq j\).

THEOREM 6.2.2. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then \(X\) is pairwise gb-\(R_0\) space if and only if for any two distinct points \(x\) and \(y\) of \(X\), either \((i, j)\)-\(gbcl\)\(\{x\}\) \(\cap (j, i)\)-\(gbcl\)\(\{y\}\) \(= \emptyset\) or \(\{x, y\} \subset (i, j)\)-\(gbcl\)\(\{x\}\) \(\cap (j, i)\)-\(gbcl\)\(\{y\}\), for \(i, j = 1, 2\) and \(i \neq j\).

THEOREM 6.2.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) be a subset of \(X\). Then \((i, j)\)-\(gbker\)(\(A\)) = \{ \(x \in X : (i, j)\)-\(gbcl\)\(\{x\}\) \(\cap A \neq \emptyset\) \}.

THEOREM 6.2.4. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then \(\cap \{(i, j)\)-\(gbcl\)\(\{x\}\) : \(x \in X\) = \emptyset\) if and only if \((i, j)\)-\(ker\)(\(A\)) \(\neq X\), for every \(x \in X\).
**THEOREM 6.2.5.** Let \((X,\tau_1,\tau_2)\) be a bitopological space. Then the following statements are equivalent:

(a) \((X,\tau_1,\tau_2)\) is pairwise gb-R_0 space.

(b) For any \(x \in X\), \((i,j)\)-gbcl\((\{x\})\) = \((j,i)\)-gb-ker\((\{x\})\), for \(i, j = 1, 2\) and \(i \neq j\).

(c) For any \(x \in X\), \((i,j)\)-gbcl\((\{x\})\) \(\subset\) \((j,i)\)-gb-ker\((\{x\})\), for \(i, j = 1, 2\) and \(i \neq j\).

(d) For any \(x, y \in X\), \(y \in (i,j)\)-gb-ker\((\{x\})\) if and only if \(x \in (j,i)\)-gb-ker\((\{y\})\), for \(i, j = 1, 2\) and \(i \neq j\).

(e) For any \(x, y \in X\), \(y \in (i,j)\)-gbcl\((\{x\})\) if and only if \(x \in (j,i)\)-gbcl\((\{y\})\), for \(i, j = 1, 2\) and \(i \neq j\).

(f) For any \((i,j)\)-gb-closed set \(V\) and \(x \notin V\), there exist a \((j,i)\)-gb-open set \(U\) such that \(x \notin U\) and \(V \subset U\), for \(i, j = 1, 2\) and \(i \neq j\).

(g) For each \((i,j)\)-gb-closed set \(V\), \(V = \cap\{U : U\) is \((j,i)\)-gb-open and \(V \subset U\}\), for \(i, j = 1, 2\) and \(i \neq j\).

(h) For each \((i,j)\)-gb-open set \(U\), \(U = \cup\{V : V\) is \((j,i)\)-gb-closed and \(V \subset U\}\), for \(i, j = 1, 2\) and \(i \neq j\).

(i) For every non-empty subset \(A\) of \(X\) and for any \((i,j)\)-gb-open set \(U\) such that \(A \cap U \neq \emptyset\), there exists a \((j,i)\)-gb-closed \(V\) such that \(A \cap V \neq \emptyset\) and \(V \subset U\), for \(i, j = 1, 2\) and \(i \neq j\).

(j) For any \((i,j)\)-gb-closed set \(V\) and \(x \notin V\), \((j,i)\)-gbcl\((\{x\})\) \(\cap\) \(V = \emptyset\), for \(i, j = 1, 2\) and \(i \neq j\).

**THEOREM 6.2.6.** In a pairwise gb-R_0 space \((X,\tau_1,\tau_2)\), for any \(x \in X\), \((i,j)\)-gbcl\((\{x\})\) \(\cap\) \((j,i)\)-gb-ker\((\{x\})\) = \(\{x\}\) holds for \(i, j = 1, 2\) and \(i \neq j\), then \((i,j)\)-gbcl\((\{x\})\) = \(\{x\}\).

**THEOREM 6.2.7.** If \((X,\tau_1,\tau_2)\) is a pairwise gb-R_0 space, then for any \(x, y \in X\), either \((i,j)\)-gbcl\((\{x\})\) \(\cap\) \((j,i)\)-gbcl\((\{x\})\) = \((i,j)\)-gbcl\((\{y\})\) \(\cap\) \((j,i)\)-gbcl\((\{y\})\) or \(\{(i,j)\)-gbcl\((\{x\})\) \(\cap\) \((j,i)\)-gbcl\((\{x\})\) \(\cap\) \((i,j)\)-gbcl\((\{y\})\) \(\cap\) \((j,i)\)-gbcl\((\{y\})\) = \emptyset\), for \(i, j = 1, 2\) and \(i \neq j\).
THEOREM 6.2.8. If \((X,\tau_1,\tau_2)\) is a pairwise \(gb-R_0\) space, then for any \(x, y \in X\), either \((i, j)\)-\(gb\)-ker\({\{x}\}\) \(\cap (j, i)\)-\(gb\)-ker\({\{y}\}\) or \((j, i)\)-\(gb\)-ker\({\{x}\}\) \(\cap (i, j)\)-\(gb\)-ker\({\{y}\}\}) = \emptyset\), for \(i, j = 1, 2\) and \(i \neq j\).

THEOREM 6.2.9. If \((X,\tau_1,\tau_2)\) is pairwise \(gb-R_1\) space, then it is pairwise \(gb-R_o\) space.

THEOREM 6.2.10. A bitopological space \((X,\tau_1,\tau_2)\) is pairwise \(gb-R_1\) if and only if for every \(x, y \in X\) such that \((i, j)\)-\(gbcl\)\({\{x}\}\) \(\neq (j, i)\)-\(gbcl\)\({\{y}\}\), there exists a \((i, j)\)-\(gb\)-open set \(U\) and a \((j, i)\)-\(gb\)-open set \(V\) such that \(x \in V\), \(y \in U\) and \(U \cap V = \emptyset\), for \(i, j = 1, 2\) and \(i \neq j\).

THEOREM 6.2.11. Let \((X,\tau_1,\tau_2)\) be a bitopological space. Then the following are equivalent.

(a) \((X,\tau_1,\tau_2)\) is pairwise \(gb-R_1\) space.

(b) For any \(x, y \in X\), \(x \neq y\) and \((i, j)\)-\(gbcl\)\({\{x}\}\) \(\neq (j, i)\)-\(gbcl\)\({\{y}\}\) implies that there exists a \((i, j)\)-\(gb\)-closed set \(G_1\) and a \((j, i)\)-\(gb\)-closed set \(G_2\) such that \(x \in G_1\), \(y \notin G_1\), \(y \in G_2\), \(x \notin G_2\), \(X = G_1 \cup G_2\), for \(i, j = 1, 2\) and \(i \neq j\).

6.3 PROOF OF THE RESULTS OF SECTION 6.2

PROOF OF THEOREM 6.2.1. (a) \(\Rightarrow\) (b) Let \(V\) be a \((i, j)\)-\(gb\)-closed set and \(x \notin V\).

By (a), we have

\((j, i)\)-\(gbcl\)\({\{x}\}\) \(\subset X \setminus V\).

Put

\[ U = X \setminus (j, i)\)-\(gbcl\)\({\{x}\}\). \]

Then \(U\) is \((j, i)\)-\(gb\)-open and \(V \subset X \setminus (j, i)\)-\(gbcl\)\({\{x}\}\) = \(U\).

Thus \(V \subset U\) and \(x \notin U\).
(b) ⇒ (c) Let \( V \) be a \((i,j)\)-gb-closed set and \( x \not\in V \).

By hypothesis, there exists a \((j,i)\)-gb-open set \( U \) such that \( x \not\in U \) and \( V \subset U \).

Which implies that,
\[
U \cap (j,i)\text{-gbcl}\{x\} = \emptyset,
\]
since \( U \) is \((j,i)\)-gb-open.

Hence \( V \cap (j,i)\text{-gbcl}\{x\} = \emptyset \).

(c) ⇒ (a) Let \( U \) be a \((i,j)\)-gb-open set such that \( x \in U \).

Now, \( X \setminus U \) is \((i,j)\)-gb-closed and \( x \not\in X \setminus U \).

By (c),
\[
(j,i)\text{-gbcl}\{x\} \cap (X \setminus U) = \emptyset.
\]

Which implies that,
\[
(j,i)\text{-gbcl}\{x\} \subset U.
\]

Hence \((X,\tau_1,\tau_2)\) is pairwise gb-R_0 space.

PROOF OF THEOREM 6.2.2. Let \((i,j)\)-gbcl\{x\} \cap (j,i)-gbcl\{y\} \neq \emptyset and \{x,y\} \not\subset (i,j)-\text{gbcl}\{x\} \cap (j,i)-\text{gbcl}\{y\}.

Let \( z \in (i,j)-\text{gbcl}\{x\} \cap (j,i)-\text{gbcl}\{y\} \) and \( x \notin (i,j)-\text{gbcl}\{x\} \cap (j,i)-\text{gbcl}\{y\} \).

Now, \( x \notin (j,i)-\text{gbcl}\{y\} \) implies that \( x \in X \setminus (j,i)-\text{gbcl}\{y\} \), which is a \((j,i)\)-gb-open set containing \( x \).

Since \( z \in (j,i)-\text{gbcl}\{y\} \), so \( (i,j)-\text{gbcl}\{x\} \not\subset X \setminus (j,i)-\text{gbcl}\{y\} \).

Hence \((X,\tau_1,\tau_2)\) is not pairwise gb-R_0 space.

Conversely, Let \( U \) be a \((i,j)\)-gb-open set such that \( x \in U \).

Suppose that
\[
(j,i)-\text{gbcl}\{x\} \not\subset U.
\]

Then there exists a \( y \in (j,i)-\text{gbcl}\{x\} \) such that \( y \not\in U \) and \((i,j)-\text{gbcl}\{y\} \cap U = \emptyset\), since \( X \setminus U \) is \((i,j)\)-gb-closed and \( y \in X \setminus U \).

Hence
\[
\{x,y\} \not\subset (i,j)-\text{gbcl}\{y\} \cap (j,i)-\text{gbcl}\{x\}
\]

and
\[
(i,j)-\text{gbcl}\{y\} \cap (j,i)-\text{gbcl}\{x\} \neq \emptyset.
\]

PROOF OF THEOREM 6.2.3. Let \( x \in (i,j)-\text{gb-ker}(A) \) and \((i,j)-\text{gbcl}\{x\} \cap A = \emptyset \).
Therefore,
\[(i,j)\text{-}gbcl(\{x\}) \subset X \setminus A \text{ and so } A \subset X \setminus (i,j)\text{-}gbcl(\{x\}).\]

But \(x \notin X \setminus (i,j)\text{-}gbcl(\{x\})\), which is a \((i,j)\text{-}gb\)-open sets containing \(A\).

Thus \(x \notin (i,j)\text{-}gb\text{-}\ker(A)\), a contradiction.

Consequently,
\[(i,j)\text{-}gbcl(\{x\}) \cap A \neq \emptyset.\]

Conversely, let \((i,j)\text{-}gbcl(\{x\}) \cap A \neq \emptyset\).

If possible, let \(x \notin (i,j)\text{-}gb\text{-}\ker(A)\).

Then there exists \(U \in GBO(i,j)\) such that \(x \notin U\) and \(A \subset U\).

Let \(y \in (i,j)\text{-}gbcl(\{x\}) \cap A\).

Then \(y \in (i,j)\text{-}gbcl(\{x\})\) and \(y \in A \subset U\).

Hence \(U \in GBO(i,j)\) such that \(y \in U\) and \(x \notin U\), which is a contradiction, since \(y \in (i,j)\text{-}gbcl(\{x\}) \in GBC(i,j)\).

Therefore,
\[x \in (i,j)\text{-}gb\text{-}\ker(A).\]

Hence
\[(i,j)\text{-}gb\text{-}\ker(A) = \{x \in X : (i,j)\text{-}gbcl(\{x\}) \cap A \neq \emptyset\}.\]

**PROOF OF THEOREM 6.2.4.** Assume that \(\cap\{ (i,j)\text{-}gbcl(\{x\}) : x \in X \} = \emptyset\).

Let \((i,j)\text{-}gb\text{-}\ker(\{x\}) = X\).

If there is some \(y \in X\), then \(X\) is the only \((i,j)\text{-}gb\)-open set containing \(y\).

Which shows that
\[y \in (i,j)\text{-}gbcl(\{x\}), \text{ for every } x \in X.\]

Therefore,
\[\cap\{ (i,j)\text{-}gbcl(\{x\}) : x \in X \} \neq \emptyset, \text{ a contradiction.}\]

Hence
\[(i,j)\text{-}gb\text{-}\ker(\{x\}) \neq X, \text{ for every } x \in X.\]

Conversely, assume that \((i,j)\text{-}gb\text{-}\ker(\{x\}) \neq X, \text{ for every } x \in X.\)

Let \(\cap\{ (i,j)\text{-}gbcl(\{x\}) : x \in X \} \neq \emptyset\).

If there is some \(y \in X\) such that \(y \in \cap\{ (i,j)\text{-}gbcl(\{x\}) : x \in X \}\), then every \((i,j)\text{-}gb\)-open set containing \(y\) must contain every point of \(X\).
This shows that, \( X \) is the only \((i,j)\)-gb-open set containing \( y \).

Therefore,

\[(i,j)\text{-}gb\text{-}ker\{x\} = X, \text{ a contradiction.}\]

Hence

\[\cap\{i,j\text{-}gbcl\{x\} : x \in X\} = \emptyset.\]

**PROOF OF THEOREM 6.2.5.** \((a) \Rightarrow (b)\) Let \( x, y \in X \).

Then by Definition 6.1.7,

\[y \in (j,i)\text{-}gb\text{-}ker\{x\} \iff x \in (j,i)\text{-}gbcl\{y\} .\]

Since \( X \) is pairwise gb-R\(_{0}\) space, therefore by Theorem 6.2.2, we have

\[x \in (j,i)\text{-}gbcl\{y\} \iff y \in (i,j)\text{-}gbcl\{x\} .\]

Thus we get

\[y \in (j,i)\text{-}gb\text{-}ker\{x\} \iff x \in (j,i)\text{-}gbcl\{y\} \iff y \in (i,j)\text{-}gbcl\{x\} .\]

Hence

\[(i,j)\text{-}gbcl\{x\} = (j,i)\text{-}gb\text{-}ker\{x\} .\]

\((b) \Rightarrow (c)\) It is obvious.

\((c) \Rightarrow (d)\) Let \( x, y \in X \) and \( y \in (i,j)\text{-}gb\text{-}ker\{x\} \).

Then by Definition 6.1.7,

\[x \in (i,j)\text{-}gbcl\{y\} .\]

Therefore by (c),

\[x \in (i,j)\text{-}gbcl\{y\} \subset (j,i)\text{-}gb\text{-}ker\{y\} .\]

Thus

\[x \in (j,i)\text{-}gb\text{-}ker\{y\} .\]

Similarly, we can prove the other part also.

\((d) \Rightarrow (e)\) Let \( x, y \in X \) and \( y \in (i,j)\text{-}gbcl\{x\} \).

Then by Definition 6.1.7,

\[x \in (i,j)\text{-}gb\text{-}ker\{y\} .\]

Therefore by (d),

\[y \in (j,i)\text{-}gb\text{-}ker\{x\} \text{ and so } x \in (j,i)\text{-}gbcl\{y\} .\]
Similarly, we can prove the other part also.

(e) ⇒ (f) Let \( V \) be a \((i, j)\)-gb-closed set and \( x \notin V \). Then for any \( y \in V \), we have

\[
(i, j)\text{-}\text{gbcl}(\{y\}) \subset V \quad \text{and} \quad x \notin (i, j)\text{-}\text{gbcl}(\{y\}).
\]

Therefore by (e),

\[
y \notin (j, i)\text{-}\text{gbcl}(\{x\}).
\]

That is there exists a \((j, i)\)-gb-open set \( U_y \) such that \( y \in U_y \) and \( x \notin U_y \).

Let \( U = \cup_{y \in V} \{ U_y: U_y \text{ is } (j, i)\text{-gb-} \text{open, } y \in U_y \text{ and } x \notin U_y \} \).

Hence, \( U \) is \((j, i)\)-gb-open set such that \( x \notin U \) and \( V \subset U \).

(f) ⇒ (g) Let \( V \) be a \((i, j)\)-gb-closed set in \( X \) and \( W = \cap \{ U: U \text{ is } (j, i)\text{-gb-} \text{open and } V \subset U \} \).

Clearly,

\[
V \subset W.
\]

Suppose that, \( x \notin V \).

Therefore by (f), there is a \((j, i)\)-gb-open set \( U \) such that \( x \notin U \) and \( V \subset U \).

So \( x \notin W \) and thus \( W \subset V \).

Hence

\[
V = W = \cap \{ U: U \text{ is } (j, i)\text{-gb-} \text{open and } V \subset U \}.
\]

(g) ⇒ (h) It is obvious.

(h) ⇒ (i) Let \( A \) be a non-empty subset of \( X \) and \( U \) be a \((i, j)\)-gb-open set in \( X \) such that \( A \cap U \neq \emptyset \).

Let \( x \in A \cap U \).

By (h),

\[
U = \cup \{ V: V \text{ is } (j, i)\text{-gb-} \text{closed and } V \subset U \}.
\]

Then there is a \((j, i)\)-gb-closed \( V \) such that \( x \in V \subset U \).

Therefore, \( x \in A \cap V \) and so \( A \cap V \neq \emptyset \).
(i) ⇒ (j) Let \( V \) be a \((i, j)\)-\(gb\)-closed set such that \( x \notin V \). Then \( X \setminus V \) is \((i, j)\)-\(gb\)-open set containing \( x \) and \( \{x\} \cap (X \setminus V) \neq \emptyset \). Therefore by (i), there is a \((j, i)\)-\(gb\)-closed set \( W \) such that \( W \subset X \setminus V \) and \( \{x\} \cap W \neq \emptyset \).

Hence

\[(j, i)\)-\(gbcl\}(\{x\}) \subset X \setminus V \] and so \((j, i)\)-\(gbcl\}(\{x\}) \cap V = \emptyset \).

(j) ⇒ (a) Follows from Theorem 6.2.1.

**PROOF OF THEOREM 6.2.6.** Since \((X, \tau_1, \tau_2)\) is pairwise \(gb\)-\(R_o\) space, therefore by Theorem 6.2.5 (b), we have

\[(i, j)\)-\(gbcl\}(\{x\}) = (j, i)\)-\(gb-ker\}(\{x\}) \].

Hence the result follows.

**PROOF OF THEOREM 6.2.7.** Let \((X, \tau_1, \tau_2)\) is a pairwise \(gb\)-\(R_o\) space.

Suppose that,

\[\{(i, j)\}-\text{\(gbcl\}}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{y\}) \neq \emptyset \].

Let \( z \in \{(i, j)\}-\text{\(gbcl\}}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{y\}) \).

Then

\[(i, j)\)-\(gbcl\}(\{z\}) \subset (i, j)\)-\(gbcl\}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{y\}) \]

and

\[(j, i)\)-\(gbcl\}(\{z\}) \subset (j, i)\)-\(gbcl\}(\{x\}) \cap (j, i)\)-\(gbcl\}(\{y\}) \).

Since \( z \in (i, j)\)-\(gbcl\}(\{x\}) \), we have by (e) of Theorem 6.2.5,

\[x \in (j, i)\)-\(gbcl\}(\{z\}) \].

Therefore,

\[(j, i)\)-\(gbcl\}(\{x\}) \subset (j, i)\)-\(gbcl\}(\{z\}) \subset (j, i)\)-\(gbcl\}(\{y\}) \].

Similarly,

\[z \in (j, i)\)-\(gbcl\}(\{x\}) \] implies \((i, j)\)-\(gbcl\}(\{x\}) \subset (i, j)\)-\(gbcl\}(\{y\}) \),

\[z \in (i, j)\)-\(gbcl\}(\{y\}) \] implies \((j, i)\)-\(gbcl\}(\{y\}) \subset (j, i)\)-\(gbcl\}(\{x\}) \)

and also

\[z \in (j, i)\)-\(gbcl\}(\{y\}) \] implies \((i, j)\)-\(gbcl\}(\{y\}) \subset (i, j)\)-\(gbcl\}(\{x\}) \)
Thus
\[(i, j)\text{-}gbcl(\{x\}) = (i, j)\text{-}gbcl(\{y\})\]
and
\[(j, i)\text{-}gbcl(\{x\}) = (j, i)\text{-}gbcl(\{y\}).\]
Hence the result follows.

**PROOF OF THEOREM 6.2.8.** The proof is similar to that of Theorem 6.2.7 which follows from Definition of \((i, j)\text{-}gb\text{-}ker(\{x\})\) and Theorem 6.2.5.

**PROOF OF THEOREM 6.2.9** Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(gb\text{-}R_1\) space.

Let \(U\) be a \((i, j)\text{-}gb\text{-open set containing } x.\)
Then, for each \(y \in X \setminus U,\)
\[(j, i)\text{-}gbcl(\{x\}) \neq (i, j)\text{-}gbcl(\{y\}).\]
Since \((X, \tau_1, \tau_2)\) is pairwise \(gb\text{-}R_1\), there exists a \((i, j)\text{-}gb\text{-open set } U_y\) and a \((j, i)\text{-}gb\text{-open set} V_y\) such that
\[U_y \cap V_y = \emptyset \text{ and } (i, j)\text{-}gbcl(\{y\}) \subset V_y, (j, i)\text{-}gbcl(\{x\}) \subset U_y.\]
Let \(A = \cup \{V_y : y \in X \setminus U\}.\)
Then \(X \setminus U \subset A, x \notin A\) and \(A\) is \((j, i)\text{-}gb\text{-open set.}\)
Therefore,
\[(j, i)\text{-}gbcl(\{x\}) \subset X \setminus A \subset U.\]
Hence \((X, \tau_1, \tau_2)\) is pairwise \(gb\text{-}R_0\) space.

**PROOF OF THEOREM 6.2.10.** Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(gb\text{-}R_1\) space.

Let \(x, y \in X\) such that \((i, j)\text{-}gbcl(\{x\}) \neq (j, i)\text{-}gbcl(\{y\}).\)
Then there exists a \((i, j)\text{-}gb\text{-open set } U\) and a \((j, i)\text{-}gb\text{-open set} V\) such that
\[x \in (i, j)\text{-}gbcl(\{x\}) \subset V \text{ and } y \in (j, i)\text{-}gbcl(\{y\}) \subset U.\]
Conversely, suppose that there exists a \((i, j)\text{-}gb\text{-open set } U\) and a \((j, i)\text{-}gb\text{-open set} V\) such that \(x \in V, y \in U\) and \(U \cap V = \emptyset.\)
Therefore,
\[(i, j)\text{-}gbcl(\{x\}) \cap (j, i)\text{-}gbcl(\{y\}) = \emptyset.\]
So by Theorem 6.2.2, \((X, \tau_1, \tau_2)\) is pairwise \(gb\text{-}R_0\) space.
Then

\((i,j)\)-\(gbcl\{x\}\) \(\subseteq V\) and \((j,i)\)-\(gbcl\{y\}\) \(\subseteq U\).

Hence \((X,\tau_1,\tau_2)\) is pairwise \(gb\)-\(R_1\) space.

**PROOF OF THEOREM 6.2.11.** \((a) \Rightarrow (b)\) Suppose that \((X,\tau_1,\tau_2)\) is pairwise \(gb\)-\(R_1\) space.

Let \(x, y \in X\) such that \((i,j)\)-\(gbcl\{x\}\) \(\neq (j,i)\)-\(gbcl\{y\}\).

Therefore by Theorem 6.2.10, there exists a \((i,j)\)-\(gb\)-open set \(V\) and a \((j,i)\)-\(gb\)-open set \(U\) such that

\[x \in U, \; y \in V \text{ and } U \cap V = \emptyset.\]

Then \(G_1 = X \setminus V\) is \((i,j)\)-\(gb\)-closed and \(G_2 = X \setminus U\) is \((j,i)\)-\(gb\)-closed set such that

\[x \in G_1, \; y \notin G_1, \; y \in G_2, \; x \notin G_2 \text{ and } X = G_1 \cup G_2.\]

\((b) \Rightarrow (a)\) Let \(x, y \in X\) such that \((i,j)\)-\(gbcl\{x\}\) \(\neq (j,i)\)-\(gbcl\{y\}\).

Therefore for any \(x, y \in X, x \neq y\), we have

\[(i,j)\)-\(gbcl\{x\}\) \(\cap (j,i)\)-\(gbcl\{y\}\) = \(\emptyset\).

Then by Theorem 6.2.2, \((X,\tau_1,\tau_2)\) is pairwise \(gb\)-\(R_o\) space.

By (b), there is a \((i,j)\)-\(gb\)-closed set \(G_1\) and a \((j,i)\)-\(gb\)-closed set \(G_2\) such that

\[x \in G_1, \; y \notin G_1, \; y \in G_2, \; x \notin G_2 \text{ and } X = G_1 \cup G_2.\]

Therefore,

\[x \in X \setminus G_2 = U, \text{ which is } (j,i)\)-\(gb\)-open\]

and

\[y \in X \setminus G_1 = V, \text{ which is } (i,j)\)-\(gb\)-open.\]

Which implies that

\[(i,j)\)-\(gbcl\{x\}\) \(\subseteq U, \; (j,i)\)-\(gbcl\{y\}\) \(\subseteq V\) and \(U \cap V = \emptyset\).

Hence the result.