7. CHARACTERISTIC CONDITION FOR CONNECTEDNESS OF ANTIMEDIAN AND MEDIAN SETS

7.1 Introduction

The graphs, in which the median sets always induce connected subgraphs for strictly positive weight functions were characterized by Bandelt and Chepoi in [16], it is also shown that the characteristic condition can be tested in polynomial time. Motivated by these results, we followed this direction to address the question, which have posed in the concluding section of the previous chapter: Is it possible to characterize graphs with connected antimeadians for arbitrary profiles?

Using the characterization of graphs with connected median discussed in [16] analogous results for the antimeadian can be obtained by considering the opposite extreme concepts such as weakly concave instead of weakly convex, antipseudopeakless instead of pseudopeakless. This can be achieved by effecting the following changes in the definition and in the proof: Replace "≤" with "≥", "maximum" with "minimum", and "median" with "antimeadian".

The main difference of these analogous results with that of Bandelt and Chepoi [16] is that the weight function cannot be restricted to take strictly nonnegative value. We proceed to prove that the analogous results in antimeadian case hold for arbitrary weight functions. These observations result in a characterization for graphs whose antimeadian and median sets always induce connected subgraphs for arbitrary weight functions, the result is in Section 7.4. We will also prove that the
characteristic condition can be tested in polynomial time.

The results of Bandelt and Chepoi [16] for median problem with respect to weight functions, is recalled in Section 7.2. It is proved by K. Balakrishnan et al. in [10] that there exists a correspondence between positive weight functions on the vertex set of a graph and profiles on a graph and hence the medians of positive weight functions are equivalent to medians of profiles. In [10, 8], it is further proved that graphs with connected medians are precisely those graphs where the so called plurality strategy on profiles will produce the same output, namely the median of the profile, irrespective of the starting vertex. In order to establish this, the authors used the results in [16]. The results in [10, 8] are recalled in Section 7.3. In the remainder of this section necessary definitions and basic observations are given.

A \textit{weight function} on a graph $G = (V, E)$ is a mapping $f$ from $V$ to the set of real numbers.

For a vertex $v$ of $G$ and a weight function defined on the vertex set of $G$,

$$D_f(v) = D(v, f) = \sum_{x \in V} d(v, x)f(x).$$

Note that $D_f$ is a weight function on $G$ as well. $AM(f)(M(f))$ denote the set of all vertices $v$ such that $D(v, f)$ is maximum(minimum).

\textbf{Definition 7.1.1:} Local Anti-median(Median)

A \textit{local anti-median(median)} of a weight function $f$ is a vertex $u$ such that $D_f$ has a local maximum(minimum) at $u$.

The set of all local anti-medians(medians) of a weight function $f$ is denoted by $AM_{loc}(f)$ ($M_{loc}(f)$).

\textbf{Definition 7.1.2:} Weakly Concave(Convex) Functions

A real-valued function $f$ defined on the vertex set $v$ of a graph $G$ is said to be \textit{weakly concave(convex)} if for any two vertices $u, v$ and a real number $\lambda$ between 0
and 1 such that $\lambda d(u, v)$ and $(1 - \lambda)d(u, v)$ are integers, there exists a vertex $x$ such that

$$d(u, x) = \lambda d(u, v), \ d(v, x) = (1 - \lambda) d(u, v),$$

$$f(x) \geq (1 - \lambda)f(u) + \lambda f(v)$$

$$(f(x) \leq (1 - \lambda)f(u) + \lambda f(v), \text{ respectively})$$

**Definition 7.1.3:** Anti-peakless (Peakless) Function

A real valued function $f$ defined on a path $P = (w_0, w_1, \ldots, w_p)$ is anti-peakless (Peakless) if $0 \leq i < j < k \leq p$ implies

$$f(w_j) \geq \min\{f(w_i), f(w_k)\}$$

$$(f(w_j) \leq \max\{f(w_i), f(w_k)\}, \text{ respectively})$$

and equality holds only if $f(w_i) = f(w_k)$.

**Definition 7.1.4:** Anti-pseudopeakless (Pseudopeakless)

A function $f$ defined on the vertex set of a graph $G$ is anti-pseudopeakless (pseudopeakless) if any two vertices of $G$ can be joined by a shortest path along which $f$ is anti-peakless (peakless).

Note that a real-valued function $f$ defined on the vertex set $v$ of a graph $G$ is anti-pseudopeakless (peakless) if for any two nonadjacent vertices $u, v$ there is a vertex $w \in I^\circ(u, v)$ such that $f(w) \geq \min\{f(u), f(v)\}$ ($f(w) \leq \max\{f(u), f(v)\}$) and equality holds only if $f(u) = f(v)$.

Recall the definition of interval from Section 1.2.1, here $I^\circ(u, v)$ denotes the interior of the interval between $u$ and $v$ i.e.,

$$I^\circ(u, v) = I(u, v) - \{u, v\}$$
7.2 Characterization of Graphs with Connected Median

In this Section we quote the results by Bandelt and Chepoi [16] which give a characterization of graphs with connected medians.

**Lemma 7.1 (Lemma 1 in [16]):** For a real-valued weight function $f$ defined on the vertex set of a graph $G$ the following conditions are equivalent:

(i) $f$ is weakly convex;

(ii) for any two non-adjacent vertices $u$ and $v$ there exists $w \in I^c(u, v)$ such that

$$d(u, v).f(w) \leq d(v, w).f(u) + d(u, w).f(v);$$

(iii) any two vertices $u$ and $v$ at distance 2 have a common neighbour $w$ with

$$2f(w) \leq f(u) + f(v).$$

**Observation 7.2:** There the Lemma is proved for strictly nonnegative weight functions but it can be observed that the conditions also holds for arbitrary weight functions.

Next proposition gives a characterization of graphs whose median sets are always connected:

**Proposition 7.3 (Proposition 1 in [16]):** For a graph $G$ and any weight function $f$ defined on the vertex set of $G$ the following conditions are equivalent:

(i) $M_{loc}(f) = M(f)$ for all weight functions $f$;

(ii) $D_f$ is weakly convex for all $f$;

(iii) $D_f$ is pseudo peakless for all $f$;

(iv) all level sets $\{x : D_f(x) \leq \lambda\}$ induce isometric subgraphs;
all median sets $M(f)$ induce isometric subgraphs;

all median sets $M(f)$ are connected.

Note also that the conditions hold for arbitrary weight functions.

### 7.3 Equivalence of Profiles and Rational Weight Functions

In [10, 8] the equivalence of profile and real valued weight function in perspective of median computation is established and another characterization of graphs with connected medians were given as the graphs for which the plurality strategy will always produce the median $M(\pi)$ of profiles $\pi$ independent of the position of the initial vertex.

The Plurality strategy on a graph $G$ can be described as follows:

**Algorithm 7.4 (Plurality Strategy):**

1. Start at an initial vertex $v$.

2. If we are in $v$ and $w$ is a neighbor of $v$ with $|\pi_{vw}| \leq |\pi_{vw}|$, then we move to $w$

3. We move only to a vertex already visited if there is no alternative

4. We stop when
   
   i. We are stuck at a vertex $v$

   or

   ii. We have visited vertices at least twice, and, for each vertex $v$ visited at least twice and each neighbor $w$ of $v$, either $w$ is also visited at least twice or $|\pi_{vw}| > |\pi_{vw}|$.

5. We park at the vertex where we get stuck or at each vertex visited twice and erect a traffic sign (that is, the set of vertices visited twice is selected as the result of the strategy).
Next we quote the results in [10] which establish the equivalence of profiles and real valued weight functions in perspective of median computation.

**Lemma 7.5 (Lemma 3 in [10]):** If \( \pi \) is any profile with associated weight function \( f_\pi \) then \( D(x, \pi) = D(x, f_\pi) \) for every \( x \) in \( V \). Furthermore \( M(f_\pi) = M(\pi) \).

**Observation 7.6 (Observation in [8]):** If \( f \) a weight function on \( G \) then for any positive real number \( k \), \( M(kf) = M(f) \). Also \( M_{loc}(kf) = M_{loc}(f) \). Also \( D(x, kf) \) has a strict local minimum at a vertex \( v \) if and only if \( D(x, f) \) has a strict local minimum at \( v \).

**Lemma 7.7 (Lemma 4 in [10]):** Given any rational weight function \( g \) there is a profile \( \pi \) such that \( f_\pi = kg \) for some positive integer \( k \). In other words, medians of profiles are exactly medians of rational weight functions.

Using the following Lemma it is observed in [8] that, in the results in Section 7.2, real valued weight functions can be replaced by rational weight functions and equivalently by profiles.

**Lemma 7.8 (Lemma 5 in [10]):** Let \( f \) be a weight function on \( G \) such that \( D(x, f) \) has a local minimum at vertex \( v_1 \), which is not a global minimum. Then there is a weight function \( g \) such that \( D(x, g) \) has a strict local minimum at \( v_1 \), which is not a global minimum. Furthermore if \( f \) is rational, then \( g \) is also rational.

**Lemma 7.9 (Lemma 6 in [10]):** If \( G \) has the property that for each rational weight function \( f \), every local minimum of \( D(x, f) \) is also a global minimum, the same property holds for any real valued weight function \( f \).

**Theorem 7.10 (Theorem 7 in [10]):** For a graph \( G \) the following are equivalent.

1. The function \( D(x, f) \) has no local minima, which is not a global minima for all weight functions.
2. All median sets $M(f)$ are connected for all weight functions $f$.

3. The function $D(x, f)$ has no local minima, which is not a global minima for all rational weight functions.

4. The function $D(x, \pi)$ has no local minima, which is not a global minima for all profiles $\pi$.

**Theorem 7.11 (Theorem 8 in [10]):** The following are equivalent for a connected graph $G$.

1. Plurality strategy produces $M(\pi)$ starting from an arbitrary vertex, for all profiles $\pi$.

2. $M(\pi)$ is connected for all profiles $\pi$.

3. The function $D(x, \pi)$, has no local minima which is not global for all profiles $\pi$.

4. Hill climbing produces $M(\pi)$ for all profiles $\pi$.

5. Steepest ascent hill climbing produces $M(\pi)$ for all profiles $\pi$.

### 7.4 Characteristic Condition for Connected Antimedians for Arbitrary Weight Functions

In this section we present a partial answer to the question: Is it possible to characterize graphs with connected antimedians for arbitrary profiles? First, we give the basic observations, which support the answer. For the completeness of this Chapter we give separate proof for the results in this Section, even though the proof goes in similar lines as that in [16] but by using the opposite extreme concepts like "weakly concave" instead of "weakly convex", "antipseudopeakless" in-
Lemma 7.12: For a real-valued function \( f \) defined on the vertex set of a graph \( G \), the following conditions are equivalent:

(i) \( f \) is weakly concave;

(ii) for any two non-adjacent vertices \( u \) and \( v \) there exists \( w \in I^o(u, v) \) such that

\[
d(u, v)f(w) \geq d(v, w)f(u) + d(u, w)f(v);
\]

(iii) any two vertices \( u \) and \( v \) at distance 2 have a common neighbour \( w \) with

\[
2f(w) \geq f(u) + f(v).
\]

Proof. (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i): Consider two vertices \( u \) and \( v \) at distance \( n \). Among all shortest paths connecting \( u \) and \( v \) select a path

\[
P = (u = w_0, w_1, \ldots, w_{n-1}, w_n = v)
\]

such that \( \sum_{i=0}^{n} f(w_i) \) is as large as possible. Condition (iii) implies that \( 2f(w_i) \geq f(w_{i-1}) + f(w_{i+1}) \) for each \( 1 < i < n \). Consider

\[
(0, f(w_0)), (1, f(w_1)), \ldots, (n, f(w_n))
\]
as points in the plane \( \mathbb{R}^2 \). Connecting the consecutive points by segments, we get a graph of a piecewise-linear function. Since it coincides on \( P \) with the function \( f \), it is concave. Hence we conclude that

\[
f(w_i) \geq (1 - \lambda)f(u) + (\lambda)f(v),
\]
where $\lambda = (n-i)/n$ and $(1-\lambda) = i/n$.

\[ \square \]

**Lemma 7.13:** For an anti-pseudopeakless function $f$ defined on the vertex set of a graph $G$, every local maximum of $f$ is global maximum.

**Proof.** Let $u$ be a global maximum and $v$ be a local maximum of $f$. Consider a shortest path $P$ between $u$ and $v$ along which $f$ is anti-peakless. Then for any neighbour $w$ of $v$ we have $f(w) \leq \min\{f(u), f(v)\}$, whence $f(u) = f(v) = f(w)$, as required.

**Lemma 7.14:** If the function $D_f$ is not weakly concave on the vertex set $V$ of a graph $G$ for some weight function $f$, then there exist a weight function $f'$ such that $AM(f')$ induces a disconnected subgraph in $G$.

**Proof.** If $D_f$ is not weakly concave, then by Lemma 7.12 there exists two vertices $u$ and $v$ at distance 2 such that $2D_f(w) < D_f(u) + D_f(v)$ for any $w \in I^\circ(u,v)$.

Let $\epsilon = (D_f(v) - D_f(u))/2 \geq 0$.

Let $\mu$ denote the maximum value of $D_f$. Define a new weight function $f'$ by

$$f'(u) = f(u) - (2\mu + \epsilon), f'(v) = f(v) - 2\mu$$

and $f'(x) = f(x)$ otherwise. Then

$$D_{f'}(v) = D_f(v) - 2(2\mu + \epsilon)$$

$$= D_f(u) - 4\mu$$

$$= D_{f'}(u)$$
and $AM(f') \subseteq I(u, v)$ because
\[
D_{f'}(x) \leq D_f(x) - 2\mu(d(u, x) + d(v, x)) \\
\leq D_f(x) - 6\mu \\
\leq D_f(u)
\]

for every vertex outside $I(u, v)$,

and for any $w \in I^o(u, v)$

\[
D_{f'}(w) = D_f(w) - 4\mu - \epsilon \\
\leq \frac{1}{2}(D_f(u) + D_f(v)) - 4\mu - \epsilon \\
= D_{f'}(u).
\]

Hence $AM(f') = \{u, v\}$, which completes the proof.

Now, let us present a partial answer to our question.

**Proposition 7.15:** For a graph $G$ and any arbitrary weight function defined on the vertex set of $G$ the following conditions are equivalent

(i) $AM_{loc}(f) = AM(f)$ for all weight functions $f$;

(ii) $D_f$ is weakly concave for all $f$;

(iii) $D_f$ is anti-pseudopeakless for all $f$;

(iv) all level sets $\{x : D_f(x) \geq \lambda\}$ induce isometric subgraphs;

(v) all antimedian sets $AM(f)$ induce isometric subgraphs;

(vi) all antimedian sets $AM(f)$ are connected.

**Proof.** The implications $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ and $(ii) \Rightarrow (i)$ are trivial. $(vi) \Rightarrow (ii)$ by Lemma 7.14. It remains to prove that $(i) \Rightarrow (ii)$. Let $AM_{loc}(f) = AM(f)$ for all weight functions $f$, assume the converse, $D_f$ is not
weakly concave. By Lemma 7.14 there exists a weight function $f'$ such that $AM(f')$ induces a disconnected subgraph in $G$. Thus we can select two vertices $u'$ and $v'$ at distance $l \geq 2$ in $AM(f')$ such that in any shortest $u', v'$-path $D_{f'}(x) < D_{f'}(v')$ for any vertex $x$ adjacent to $u'$ or $v'$ along the shortest $u', v'$-path. Select any $\epsilon$ satisfying

$$0 < (l - 1)\epsilon < \min_{x \in V - \{u, v\}}(D_{f'}(v) - D_{f'}(x))$$

and define a new weight function $g$ by

$$g(u) = f'(u) - \epsilon$$

and $g(x) = f'(x)$ otherwise. Then

$$D_g(u) = D_{f'}(u) = D_{f'}(v),$$

$$D_g(x) = D_{f'}(x) - d(x, u)\epsilon$$

$$\leq D_{f'}(x) - \epsilon$$

$$< D_{f'}(v) - l\epsilon$$

$$= D_g(v)$$

for all $x \in V - \{u, v\}$. Therefore both $u$ and $v$ are local maxima of $D_g$, but $v \not\in AM(g)$, a contradiction, hence $(i) \Rightarrow (ii)$. \qed

**Theorem 7.16:** Let $f$ be an arbitrary weight function defined on the vertex set of a graph $G$. Whether for any $f$, antmedian sets $AM(f)$ of a graph $G$ are connected can be determined in polynomial time.
Proof. From Lemma 7.12(iii) and Proposition 7.15 we infer that all antimedian sets are connected if and only if for each pair \( u, v \) of vertices at distance 2 the following system of linear inequalities is unsolvable in \( f: D^{uv} > 0 \) for any \( f \) with matrix

\[
D^{uv} = (d(u, x) + d(v, x) - 2d(w, x))_{w \in I^{o}(u, v), \ x \in V}
\]

Since LP problems are solvable in polynomial time, the proof is completed. \( \square \)

From the definitions of weakly concave and weakly convex functions Lemma 7.12 we have:

**Theorem 7.17:** For any graph \( G \) and arbitrary weight function \( f \) defined on the vertex set of \( G \), the following are equivalent

(i) For any arbitrary weight function \( f \), \( D_f \) is weakly concave

(ii) For any arbitrary weight function \( f \), \( D_f \) is weakly convex

Proof. \((i) \Rightarrow (ii)\): Assume the contrary, there exists a weight function \( f \) for which \( D_f \) is not weakly convex. By Lemma 7.1 there exists two vertices \( u \) and \( v \) at distance 2 such that for any common neighbour \( w \)

\[
2D_f(w) > D_f(u) + D_f(v).
\]

Define weight function \( f' \) as follows \( f'(x) = -f(x) \) for any \( x \) in the vertex set of \( G \). Then for any \( w \) in the neighbourhood of \( u \) and \( v \)

\[
2D_f(w) < D_f(u) + D_f(v),
\]

a contradiction, which completes the proof in this direction.

Similarly \((ii) \Rightarrow (i)\): Similarly assume \((i)\) does not hold, then we have a weight function \( f \), \( D_f \) is not weakly concave. By Lemma 7.12 there exists two vertices \( u \) and \( v \) at distance 2 such that for any common neighbour \( w \)

\[
2D_f(w) < D_f(u) + D_f(v).
\]
Define weight function $f'$ as follows $f'(x) = -f(x)$ for any $x$ in the vertex set of $G$. Then for any $w$ in the neighbourhood of $u$ and $v$ $2D_{f'}(w) > D_{f'}(u)D_{f'}(v)$ which is a contradiction since $D_{f'}$ is weakly concave, which completes the proof. □

Note that the theorem does not hold for strictly nonnegative weight functions.

From Propositions 7.3 and 7.15 we have the following remark:

**Remark 7.18:** For any graph $G$ and arbitrary weight function $f$ defined on the vertex set of $G$, the following are equivalent

(i) For any arbitrary weight function $f$, $D_f$ is weakly concave

(ii) For any arbitrary weight function $f$, $D_f$ is weakly convex

(iii) all antimedians $AM(f)$ are connected.

(iv) all median sets $M(f)$ are connected.

### 7.5 Concluding Remarks

In this chapter, a characterization for graphs with both connected antimedians and medians for arbitrary weight functions is obtained as those graphs for which any weight function is weakly concave. It is also shown that the characteristic condition is testable in polynomial time. It turned out that for such graphs, $D_f$ is weakly convex for any arbitrary weight function $f$. If Proposition 7.15 and Lemma 7.12 hold for strictly nonnegative weight functions the results can be extended to arbitrary profiles using the results in [10], which is quoted in Section 7.3. We conclude this chapter with the following question:

**Question 7.19:** For which graphs the following condition hold? If the function $D_f$ is not weakly concave on the vertex set $V$ of a graph $G$ for some strictly positive weight function $f$, then there exists a strictly positive weight function $f'$ such that $AM(f')$ induces a disconnected subgraph in $G$. 