Chapter 4

Pseudo-differential operator involving first Hankel- Clifford transformation on certain Beurling type function spaces

4.1 Introduction

The Beurling type function spaces $H_\mu(\omega)$ was introduced by Pathak and Shrestha [24]. From the definition of $H_\mu(\omega)$ it follows that the conventional Hankel transformation $h_\mu$ defined (1.5.1) is an automorphism on $H_\mu(\omega)$. For $\omega(x) = \log(1 + x)$ the space $H_\mu(\omega)$ reduces to $H_\mu(I)$ and for $\omega(x) = x^\alpha (0 < \alpha < 1)$, it is a Gevrey space of test functions. The generalized Hankel transformation of ultradistributions belonging to $H'_\mu(\omega)$ is defined by the adjoint operator method (1.6.3) and it is found that the generalized Hankel transformation is also an automorphism of $H'_\mu(\omega)$. Multiplication on $H_\mu(\omega)$ and convolution on $H'_\mu(\omega)$ are investigated.

Our objective in this chapter is to investigate a brief introduction to the first Hankel-Clifford transformation and its basic properties is given. Another Beurling type space $H_\mu(\omega)$ generalizing the Zemanian-type space $H_\mu(I)$ is defined. It is given that the first Hankel-Clifford transformation is an automorphism of $H_\mu(\omega)$ as well as the generalized first Hankel-Clifford transformation is an automorphism of $H'_\mu(\omega)$. Product and convolution on $H_\mu(\omega)$ are investigated. The spaces $U_\mu(\omega)$ and $G_\mu(\omega)$ related to $H_\mu(\omega)$ are introduced and studied the continuity of the pseudo-differential operator.
4.2 Some properties of first Hankel-Clifford transformation

Lemma 4.2.1 Let $\phi \in H_\mu(I)$, then we have

$$
\Delta_\mu^n \phi(x) = \sum_{j=0}^{n} b_j x^{\mu+j} D^{j+n} x^{-\mu} \phi(x), \quad \forall n \in \mathbb{N},
$$

where $b_j$ are constants depending on $\mu$ only and $\Delta_\mu$ is same as (1.10.2).

Proof: Using (1.10.2), we have

$$
\Delta_\mu \phi(x) = D R_\mu \phi(x)
$$

$$
= D(x^{\mu+1} D x^{-\mu} \phi(x))
$$

$$
= x^{\mu+1} D^2 x^{-\mu} \phi(x) + (\mu + 1)x^\mu D x^{-\mu} \phi(x)
$$

$$
= \sum_{j=0}^{1} b_j x^{\mu+j} D^{j+1} x^{-\mu} \phi(x).
$$

Similarly

$$
\Delta_\mu^2 \phi(x) = \Delta_\mu (\Delta_\mu \phi(x))
$$

$$
= x^{\mu+1} D^2 x^{-\mu} \{x^{\mu+1} D^2 x^{-\mu} \phi(x) + (\mu + 1)x^\mu D x^{-\mu} \phi(x)\}
$$

$$
+ (\mu + 1)x^\mu D x^{-\mu} \{x^{\mu+1} D^2 x^{-\mu} \phi(x) + (\mu + 1)x^\mu D x^{-\mu} \phi(x)\}
$$

$$
= x^{\mu+1} \{D^2 (x D^2 x^{-\mu} \phi(x)) + (\mu + 1)D^3 x^{-\mu} \phi(x)\}
$$

$$
+ (\mu + 1)x^\mu \{D (x D^2 x^{-\mu} \phi(x)) + (\mu + 1)D^2 x^{-\mu} \phi(x)\}
$$

$$
= x^{\mu+2} D^4 x^{-\mu} \phi(x) + 2(\mu + 2)x^{\mu+1} D^3 x^{-\mu} \phi(x) + (\mu + 1)(\mu + 2)x^\mu D^2 x^{-\mu} \phi(x)
$$

$$
= \sum_{j=0}^{2} b_j x^{\mu+j} D^{j+2} x^{-\mu} \phi(x).
$$

Continuing in this way, we have

$$
\Delta_\mu^n \phi(x) = \sum_{j=0}^{n} b_j x^{\mu+j} D^{j+n} x^{-\mu} \phi(x),
$$

where $b_j$ are constants depending on $\mu$ only.
Lemma 4.2.2  Let $\mu \geq 0$ and $q, k \in \mathbb{N}_0$. For $\varphi \in \mathcal{H}_\mu(I)$, we have

\[(i) R_{\mu+q-1}R_{\mu+q-2} \ldots R_{\mu+1}R_{\mu} \varphi(y) = y^{\mu+q} D_{\varphi}^q y^{-\mu} \varphi(y), \quad (4.2.2)\]

\[(ii) R_{\mu+k+q-1} \ldots R_{\mu+k+1} R_{\mu+k} \varphi(y) = y^k R_{\mu+q-1} \ldots R_{\mu+1} R_{\mu} \varphi(y), \quad (4.2.3)\]

\[(iii) R_{\mu+q-1} R_{\mu+q-2} \ldots R_{\mu+1} \varphi(y) = (-1)^q h_{1, \mu+q}(x^q \varphi)(y), \quad (4.2.4)\]

\[(iv) h_{1, \mu+q+k}(x^q R_{\mu+k+1} \ldots R_{\mu} \varphi)(y) = (-y)^k h_{1, \mu+q}(x^q \varphi)(y), \quad (4.2.5)\]

where $R_{\mu}$ is as (1.10.1) and $\hat{\varphi}(y) = (h_{1, \mu} \varphi)(y)$.

Proof: (i) Since

\[R_{\mu} \varphi(y) = y^{\mu+1} D_{\varphi} y^{-\mu} \varphi(y)\]

\[R_{\mu+1} R_{\mu} \varphi(y) = y^{\mu+1+1} D_{\varphi} y^{-\mu-1} R_{\mu} \varphi(y) = y^{\mu+2} D_{\varphi}^2 y^{-\mu} \varphi(y)\]

\[R_{\mu+2} R_{\mu+1} R_{\mu} \varphi(y) = y^{\mu+2+1} D_{\varphi} y^{-\mu-2} R_{\mu+1} R_{\mu} \varphi(y) = y^{\mu+3} D_{\varphi}^3 y^{-\mu} \varphi(y)\]

Continuing in this way, we have

\[R_{\mu+q-1} R_{\mu+q-2} \ldots R_{\mu+1} R_{\mu} \varphi(y) = y^{\mu+q} D_{\varphi}^q y^{-\mu} \varphi(y)\]

(ii) Since $R_{\mu} \varphi(y) = y^{\mu+1} D_{\varphi} y^{-\mu} \varphi(y)$, then we have

\[R_{\mu+k} y^{k} \varphi(y) = y^{\mu+k+1} D_{\varphi} y^{-\mu-k} y^{k} \varphi(y) = y^{k} (y^{\mu+1} D_{\varphi} y^{-\mu} \varphi(y)) = y^{k} R_{\mu} \varphi(y)\]

Similarly

\[R_{\mu+k+1} R_{\mu+k} y^{k} \varphi(y) = y^{\mu+k+1+1} D_{\varphi} y^{-\mu-k-1} y^{k} R_{\mu} \varphi(y) = y^{k} (y^{\mu+1+1} D_{\varphi} y^{-\mu-1} R_{\mu} \varphi(y)) = y^{k} R_{\mu+1} R_{\mu} \varphi(y)\]

Continuing in this way, we have

\[R_{\mu+k+q-1} \ldots R_{\mu+k+1} R_{\mu+k} y^{k} \varphi(y) = y^{k} R_{\mu+q-1} \ldots R_{\mu+1} R_{\mu} \varphi(y)\]
Using (1.8.1), (1.7.4) and Lemma 4.2.2(i), we have
\[ R_{\mu+q-1} R_{\mu+q-2} \ldots R_{\mu+1} \hat{\phi}(y) = y^{\mu+q} D^q_y y^{-\mu} y^{\mu} \int_0^\infty C_\mu(xy) \varphi(x) dx \]
\[ = y^{\mu+q} \int_0^\infty D^q_y C_\mu(xy) \varphi(x) dx \]
\[ = y^{\mu+q} \int_0^\infty x^q (-1)^q C_{\mu+q}(xy) \varphi(x) dx \]
\[ = (-1)^q h_{1,\mu+q}(x^q \varphi)(y). \]

(iv) Using (1.8.1), (1.10.1) and (1.7.5), then we have
\[ h_{1,\mu+q+1}(x^q R_\mu \varphi)(y) = y^{\mu+q+1} \int_0^\infty x^q C_{\mu+q+1}(xy)x^{\mu+1}D_x x^{-\mu} \varphi(x) dx \]
\[ = -y \int_0^\infty (xy)^{\mu+q} C_{\mu+q}(xy)x^{-\mu} \varphi(x) dx \]
\[ = -y h_{1,\mu+q}(x^q \varphi)(y). \]

Similarly
\[ h_{1,\mu+q+2}(x^q R_{\mu+1} R_\mu \varphi)(y) = y^{\mu+q+2} \int_0^\infty x^q C_{\mu+q+2}(xy)x^{\mu+2}D^2_x x^{-\mu} \varphi(x) dx \]
\[ = (-y)^2 \int_0^\infty (xy)^{\mu+q} C_{\mu+q}(xy)x^{-\mu} \varphi(x) dx \]
\[ = (-y)^2 y^{\mu+q} \int_0^\infty C_{\mu+q}(xy)x^q \varphi(x) dx \]
\[ = (-y)^2 h_{1,\mu+q}(x^q \varphi)(y). \]

Continuing in this way, we have
\[ h_{1,\mu+q+k}(x^q R_{\mu+k-1} \ldots R_\mu \varphi)(y) = (-y)^k h_{1,\mu+q}(x^q \varphi)(y). \]

4.3 The space $\mathcal{H}_\mu(\omega)$

As G. Björck [6], let $\omega$ be a continuous real-valued function defined on $I = (0, \infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ and it satisfying the following three properties.
as an even function then \( \omega \) satisfies the sub additivity property (a) for every \( x, y \in \mathbb{R} \).

A. Beurling [5] developed the functions of general theory of distributions that extends the Schwartz theory. Some aspects of that theory were presented and completed by Björck [6]. We now collect some definitions and properties for the purpose of the present chapter. Assume that \( \omega \) is a function in \( \mathcal{M} \). A function \( \varphi \) is said to be in the space \( \mathcal{H}_\mu(\omega) \), where \( \varphi \) is a smooth function and for every \( \mu \in \mathbb{R} \), \( n \in \mathbb{N}_0 \) and \( m \) a positive real number

\[
\alpha_{m,n}^\mu(\varphi) = \sup_{x \in I} e^{m \omega(x)} |D^q_x x^{-\mu} \varphi(x)| < \infty.
\]

On \( \mathcal{H}_\mu(\omega) \) we consider the topology generated by family \( \{ \alpha_{m,n}^\mu \}_m \in I, n \in \mathbb{N}_0 \) of seminorms. Various properties of the \( \mathcal{H}_\mu(\omega) \) type space can be found in [24].

**Theorem 4.3.1** For \( \mu \geq 0 \), the first Hankel-Clifford transformation \( h_{1,\mu} \) is an automorphism of \( \mathcal{H}_\mu(\omega) \).

**Proof:** Let \( \varphi(x) \) be any member of \( \mathcal{H}_\mu(\omega) \). Linearity of \( h_{1,\mu} \) is obvious. Let \( k \) and \( q \) be any pair of non-negative integers. By using Lemma 4.2.2(iii) and (iv) and noting that

\[
R_{\mu+q-1}\cdots R_{\mu+1} \hat{\varphi}(y) = (-1)^q h_{1,\mu+q}(y \varphi)(y)
\]

\[
= (-1)^q \left( -\frac{1}{y} \right)^k h_{1,\mu+q+k}(x^q R_{\mu+k-1} \cdots R_{\mu} \varphi)(y).
\]

Therefore

\[
(-y)^k R_{\mu+q-1}\cdots R_{\mu+1} \hat{\varphi}(y) = \int_0^\infty C_{\mu+q+k}(xy) (-x)^q \{ R_{\mu+k-1} \cdots R_{\mu} \varphi \} dx,
\]

using Lemma 4.2.2 (i), we have

\[
(-1)^q y^k \hat{\varphi}(y) = \int_0^\infty C_{\mu+q+k}(xy) x^q \{ x^{\mu+k} D^k_x x^{-\mu} \varphi(x) \} dx.
\]

44
Therefore

\[ (-1)^q y^k D_y^q \bar{\phi}(y) = \int_0^\infty x^{\mu+q} D_x^k x^{-\mu} \phi(x) \left( (xy)^k C_{\mu+q+k}(xy) \right) dx, \]  

(4.3.5)

so that

\[ (-1)^q \sum_{k=0}^\infty (-1)^k \frac{(my)^k}{k!} D_y^k \bar{\phi}(y) = \sum_{k=0}^\infty \frac{m^k}{k!} \int_0^\infty x^{\mu+q} D_x^k x^{-\mu} \phi(x) \left( (xy)^k C_{\mu+q+k}(xy) \right) dx, \]

where \( m > 0 \). Also, by property \((b)\) of the function \( \omega(x) \)

\[ \int_0^\infty \frac{\omega(\eta)}{1 + \eta^2} < \infty, \]  

(4.3.6)

for \( \epsilon > 0 \), there exists a constant \( c(\epsilon) \) such that

\[ \omega(\eta) < \epsilon \eta + c(\epsilon). \]  

(4.3.7)

Hence

\[ e^{m\omega(\eta)} \leq e^{m\epsilon \eta + mc(\epsilon)} \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} \eta^v. \]  

(4.3.8)

Now, for any choice of \( m \) and \( k \) and using (4.3.5), we have

\[ \alpha_{m,n}^\mu(\hat{\phi}) = \sup_{\eta \in I} e^{m\omega(\eta)} \left| D_\eta^n \eta^{-\mu} \bar{\phi}(\eta) \right| \]

\[ \leq e^{mc(\epsilon)} \sup_{\eta \in I} \left| \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} \eta^v \left( D_\eta D_\eta^n \eta^{-\mu} \bar{\phi}(\eta) \right) \right| \]

\[ \leq e^{mc(\epsilon)} \sup_{\eta \in I} \left| \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} \int_0^\infty x^{\mu+n} D_x^v x^{-\mu} \phi(x) \left( (x\eta)^v C_{\mu+n+v}(x\eta) \right) dx \right|. \]

Since \( |(x\eta)^v C_{\mu+n+v}(x\eta)| \) is bounded on \( 0 < x\eta < \infty \) by the positive constant \( A_{\mu,n,v} \).

Assume that \( N \) is an integer no less than \( \mu + n \). Then

\[ x^{\mu+n} < (1+x)^N, \text{ for } x > 0, \]  

(4.3.9)

so that

\[ \alpha_{m,n}^\mu(\hat{\phi}) \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} \int_0^\infty (1+x)^{N+1} |D_x^v x^{-\mu} \phi(x)| A_{\mu,n,v} \frac{1}{(1+x)^2} dx \]

\[ \leq e^{mc(\epsilon)} \sum_{v=0}^\infty \frac{(m\epsilon)^v}{v!} B_{\mu,n,v} \sum_{k=0}^{N+1} \binom{N+1}{k} \sup_x x^k |D_x^v x^{-\mu} \phi(x)|. \]
Moreover using (4.3.3), we have

\[ e^{\frac{k\omega(x)}{b} + \frac{ka}{b}}(1 + x)^k \geq e^{\frac{ka}{b}x^k}, \]

so that

\[ \sup_x |x^k D_x^\mu \phi(x)| \leq \sup_x \left| e^{-\frac{ka}{b}x^k} e^{\frac{k\omega(x)}{b} + \frac{ka}{b}} D_x^\mu \phi(x) \right| \]

\[ = e^{-\frac{ka}{b}x^k} \alpha_{k/b,\nu}^\mu(\phi) \]

\[ \leq e^{-\frac{ka}{b}x^k} \alpha_{(N+1)/b,\nu}^\mu(\phi). \]

Now choosing

\[ \epsilon < \left( m^\nu B_{\mu,n,\nu} \alpha_{(N+1)/b,\nu}^\mu \right)^{-1/\nu}, \quad \nu \geq 1, \]

we have for some \( C > 0 \)

\[ \alpha_{m,n}^\mu(\phi) \leq C e^{mc(\epsilon)} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \sum_{k=0}^{N+1} \left( \begin{array}{c} N+1 \\ k \end{array} \right) e^{-\frac{ka}{b}} \]

\[ = C e^{mc(\epsilon)+1} \sum_{k=0}^{N+1} \left( \begin{array}{c} N+1 \\ k \end{array} \right) e^{-\frac{ka}{b}} < \infty. \]

This proves that \( h_{1,\mu}(\phi) \) is a continuous operator on \( \mathcal{H}_\mu(\omega) \). Also from (1.8.1) and (1.8.2), we see that \( h_{1,\mu}^{-1} h_{1,\mu} \phi = \phi = h_{1,\mu} h_{1,\mu}^{-1} \phi \) for all \( \phi \in \mathcal{H}_\mu(\omega) \). It follows \( h_{1,\mu} \) is a one-one function of \( \mathcal{H}_\mu(\omega) \). Thus \( h_{1,\mu} \) is an automorphism on \( \mathcal{H}_\mu(\omega) \).

Now the generalized first Hankel-Clifford transformation \( h'_{1,\mu} \) on \( \mathcal{H}'_\mu(\omega) \) is defined to be the adjoint of \( h_{1,\mu} \) on \( \mathcal{H}_\mu(\omega) \), we have

\[ \langle h'_{1,\mu} f, \phi \rangle = \langle f, h_{1,\mu} \phi \rangle, \quad f \in \mathcal{H}'_\mu(\omega), \quad \phi \in \mathcal{H}_\mu(\omega). \]

**Theorem 4.3.2** For \( \mu \geq 0 \), the generalized first Hankel-Clifford transformation \( h'_{1,\mu} \) is an automorphism of \( \mathcal{H}'_\mu(\omega) \).

### 4.4 Product and convolution on \( \mathcal{H}_\mu(\omega) \)

Now assume that \( \omega \) is a function in \( \mathcal{M} \). We shall denote by \( \Omega_n \) the space of all \( C^\infty \)-functions \( \phi(x), \quad 0 < x < \infty, \) such that for each non-negative integer \( n \), there exists a non-negative integer \( r = r(n) \) for which

\[ e^{-r\omega(x)} |D_x^n \phi(x)| < \infty. \]
Here $\Omega_n$ is the space of multipliers for $\mathcal{H}_\mu(\omega)$. The Lemma 2.2.3 will be used in the sequel.

**Theorem 4.4.1** If $\phi \in \mathcal{H}_\mu(\omega)$, and $x^\mu \psi \in \mathcal{H}_\mu(\omega)$, then $\phi \psi \in \mathcal{H}_\mu(\omega)$.

**Proof:** For non-negative real number $m$ and non-negative integer $n$, we have, by definition (4.3.4),

$$\alpha^\mu_{m,n}(\phi \psi) = \sup_{x \in I} e^{m\omega(x)} |D^m_x x^{-\mu} \phi(x) \psi(x)|$$

Applying Leibniz’s Theorem, we have

$$\alpha^\mu_{m,n}(\phi \psi) \leq \sum_{r=0}^{n} \binom{n}{r} \sup_{x \in I} e^{m\omega(x)} |D^r_x x^{-\mu} \phi(x)| \cdot \sup_{x \in I} |D^{n-r}_x x^{-\mu} (x^\mu \psi(x))|$$

This completes the proof of theorem.

**Theorem 4.4.2** For $\mu \geq 0$ and for every $x \in I$, the mapping $z^\mu \phi \rightarrow y^\mu \tau_x \phi$ is continuous from $\mathcal{H}_\mu(\omega)$ into itself.

**Proof:** Let $z^\mu \phi \in \mathcal{H}_\mu(\omega)$, we show that

$$C_\mu(xt) \in \Omega_n.$$ 

From (1.7.4), we have

$$D^n_tC_\mu(xt) = x^n (-1)^n C_{\mu+n}(xt),$$

so that there exists $r > 0$ such that

$$e^{-r\omega(t)} |D^n_tC_\mu(xt)| < \infty, \text{ for every } x \in I.$$ 

Hence $C_\mu(xt) \in \Omega_n$ for fixed $x \in I$. But $(h_{1,\mu} z^\mu \phi)(t) \in \mathcal{H}_\mu(\omega)$ then $C_\mu(xt) (h_{1,\mu} z^\mu \phi)(t) \in \mathcal{H}_\mu(\omega)$. Since $h_{1,\mu}$ is an automorphism of $\mathcal{H}_\mu(\omega)$, therefore $y^\mu \tau_x \phi \in \mathcal{H}_\mu(\omega)$. 

This concludes the proof.
Theorem 4.4.3 If $\varphi, \psi \in \mathcal{H}_\mu(\omega)$, then $x^\mu(\varphi \# \psi) \in \mathcal{H}_\mu(\omega)$.

Proof: By using (4.3.4), we have

$$
\alpha_{m,n}^\mu(h_{1,\mu}(x^\mu(\varphi \# \psi))(t)) = \sup_{i \in I} e^{m_0(t)} |D_i^n t^{-\mu} h_{1,\mu}(x^\mu(\varphi \# \psi))(t)|,
$$

using Lemma 2.2.3(ii), we have

$$
\alpha_{m,n}^\mu(h_{1,\mu}(x^\mu(\varphi \# \psi))(t)) = \sup_{i \in I} e^{m_0(t)} |D_i^n t^{-\mu} h_{1,\mu}(y^\mu \varphi)(t) h_{1,\mu}(z^\mu \psi)(t)|.
$$

By using Leibnitz’s Theorem, we have

$$
\alpha_{m,n}^\mu(h_{1,\mu}(x^\mu(\varphi \# \psi))(t)) \leq \sum_{r=0}^n \binom{n}{r} \alpha_{m,r}^\mu(h_{1,\mu}(y^\mu \varphi)) \alpha_{0,n-r}^\mu(h_{1,\mu}(z^\mu \psi)) < \infty.
$$

Hence $h_{1,\mu}(x^\mu(\varphi \# \psi))(t) \in \mathcal{H}_\mu(\omega)$. Since $h_{1,\mu}$ is an automorphism of $\mathcal{H}_\mu(\omega)$ therefore $x^\mu(\varphi \# \psi) \in \mathcal{H}_\mu(\omega)$.

4.5 The space $\mathcal{U}_\mu(\omega)$

In the sequel $\omega$ is a function in $\mathcal{M}$. We now introduce the function space $\mathcal{U}_\mu(\omega)$. A function $\varphi \in C^\infty(I)$ is said to be in the space $\mathcal{U}_\mu(\omega)$, when $\varphi$ and $h_{1,\mu}(\varphi)$ are smooth functions and for every $\mu \in \mathbb{R}, n \in \mathbb{N}_0$ and $m$ is a positive real number

$$
\alpha_{m,n}^\mu(\varphi) = \sup_{x \in I} e^{m_0(x)} |D^n x^{-\mu} \varphi(x)| < \infty, \quad (4.5.1)
$$

and

$$
\beta_{m,n}^\mu(\varphi) = \sup_{x \in I} e^{m_0(x)} |D^n x^{-\mu} (h_{1,\mu} \varphi)(x)| < \infty. \quad (4.5.2)
$$

On $\mathcal{U}_\mu(\omega)$ we consider the topology generated by family $\{\alpha_{m,n}^\mu, \beta_{m,n}^\mu\}_{m \in I, n \in \mathbb{N}_0}$ of seminorms. The pseudo-differential operator $h_{1,\mu,a}$ is defined by (2.3.3) exists for every $\varphi \in \mathcal{H}_\mu(\omega) \subset \mathcal{H}_\mu(I)$. For this purpose we define the symbol class $\mathcal{S}_{p,\delta}^{\infty,\omega}$.

Definition 4.5.1 The function $a(x,y) : C^\infty(I \times I) \to \mathbb{C}$ belongs to class $\mathcal{S}_p^{\infty,\omega}$ if and only if $\forall \ p, q \in \mathbb{N}_0, \ m \in I$ and $\sigma > 0$.
Proof: (i) Let \( \varphi \in \mathcal{U}_\mu(\omega) \) and using Leibnitz’s Theorem, we have

\[
\alpha_{m,n}^\mu(h_{1,\mu,a}\varphi) = \sup_{x \in I} e^{m\omega(x)} \left| D_x^n x^{-\mu}(h_{1,\mu,a}\varphi)(x) \right|
\]

\[
= \sup_{x \in I} e^{m\omega(x)} \left| D_x^n \int_0^\infty C_\mu(xy) a(x,y) \dot{\varphi}(y) dy \right|
\]

\[
= \sup_{x \in I} e^{m\omega(x)} \left| \sum_{r=0}^n \binom{n}{r} \int_0^\infty D_x^n-r C_\mu(xy) D_x^r a(x,y) \dot{\varphi}(y) dy \right|
\]

\[
\leq \int_0^\infty \sum_{r=0}^n \binom{n}{r} \left| D_x^n-r C_\mu(xy) \right| e^{m\omega(x)} \left| D_x^r a(x,y) \right| \left| \dot{\varphi}(y) \right| dy.
\]

Using inequality (1.7.4) and definition (4.5.3), the above expression can be estimated by

\[
\alpha_{m,n}^\mu(h_{1,\mu,a}\varphi) \leq \int_0^\infty \sum_{r=0}^n \binom{n}{r} \left| y^{n-r}(-1)^n C_{\mu+n-r}(xy) \right| C_{\sigma,m,r} C^r (1+y)^{\delta r} \times e^{\sigma\omega(y)} e^{-m\omega(y)} \left| \dot{\varphi}(y) \right| dy.
\]

By the property \((c)\) of the function \(\omega(x)\), we have

\[
(1+y) \leq e^{-a/b} e^{\omega(y)/b},
\]

and from [3, 28] we know that there exists a constant \(Q_\mu\), being a positive constant and independent of \(r\) such that

\[
\left| C_{\mu+n-r}(xy) \right| \leq Q_\mu, \quad 0 < x, y < \infty.
\]

Therefore (4.5.4) can be bounded by

\[
Q_\mu \sum_{r=0}^n \binom{n}{r} C_{\sigma,m,r} C^r \int_0^\infty e^{(\sigma-m)\omega(y)} e^{-a\delta r/b} e^{\omega(y)\delta r/b} y^{n-r} e^{m\omega(y)} \left| \dot{\varphi}(y) \right| dy
\]

\[
\leq Q_\mu \sum_{r=0}^n \binom{n}{r} C_{\sigma,m,r} C^r \int_0^\infty e^{(\sigma-m)\omega(y)} y^{\mu+n-r} e^{-a\delta r/b} \sup_{y \in I} e^{(m+\delta r/b)\omega(y)} \left| y^{-\mu} \dot{\varphi}(y) \right| dy.
\]
Using (4.5.2) we have

\[
\alpha_{m,n}^{\mu}(h_{1,\mu,a}\varphi) \leq Q_{\mu} \sum_{r=0}^{n} \binom{n}{r} C_{\sigma,m,r} r! e^{-a\delta r/b} \beta_{\nu}^{\mu} e^{(\sigma-m)\omega(y)} y^\mu y^{-r} dy < \infty,
\]

by choosing \( m > \sigma \). Therefore \( \alpha_{m,n}^{\mu}(h_{1,\mu,a}\varphi) < \infty \).

(ii) Let \( \Phi(x) = (h_{1,\mu,a}\varphi)(x) \). Then to complete the proof of the theorem we need to show that \( \Phi(x) \) satisfies (4.5.2). Now by using (4.3.8) and (4.3.5), we have

\[
\beta_{m,n}^{\mu}(\Phi) = e^{m\omega(x)} \left| D^\nu_x x^{-\mu} \Phi(x) \right|
\leq e^{mc(\epsilon)} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} y^v \left| D^\nu_x x^{-\mu} \Phi(x) \right|
\leq e^{mc(\epsilon)} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} (-1)^{q+k} \int_0^\infty y^{\mu+n} D^v_y y^{-\mu} \Phi(y)
\times \left( (xy)^v C_{\mu+n+v}(xy) \right) dy.
\]  

(4.5.6)

Since \( (z)^v C_{\mu+n+v}(z) \) is bounded on \( 0 < z < \infty \) by \( Q_{\mu} \), applying definition (2.3.3) and Leibnitz’s Theorem, right-hand side of the (4.5.6) can be estimated by

\[
e^{mc(\epsilon)} Q_{\mu} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^\infty y^{\mu+n} D^v_y \int_0^\infty C_{\mu}(yt) a(y,t) \tilde{\Phi}(t) dt \bigg| dy
\leq e^{mc(\epsilon)} Q_{\mu} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^\infty \int_0^\infty y^{\mu+n}
\times \sum_{s=0}^{v} \binom{v}{s} \left| D^v_y y^{-s} C_{\mu}(yt) \right| \left| D^s_y a(y,t) \right| \left| \tilde{\Phi}(t) \right| dt dy
\leq e^{mc(\epsilon)} Q_{\mu} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^\infty \int_0^\infty y^{\mu+n}
\times \sum_{s=0}^{v} \binom{v}{s} \left| t^{v-s} (-1)^{v-s} C_{\mu+v-s}(yt) \right| \left| D^s_y a(y,t) \right| \left| \tilde{\Phi}(t) \right| dt dy
\leq e^{mc(\epsilon)} Q_{\mu}^2 \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^\infty \int_0^\infty y^{\mu+n} \sum_{s=0}^{v} \binom{v}{s} t^{v-s}
\times \left| D^s_y a(y,t) \right| \left| \tilde{\Phi}(t) \right| dt dy.
\]  

(4.5.7)
Using inequality, (4.3.9) right-hand side of (4.5.7) can be bounded by
\[ e^{mc(t)} Q_\mu^2 \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^{\infty} \int_0^{\infty} (1+y)^{N+2} \times \sum_{s=0}^{v} \left( \frac{v}{s} \right) t^{v-s} |D_y^s a(y,t)| |\bar{\phi}(t)| dt dy. \]

Using inequality, definition (4.5.3) the right-hand side of (4.5.8) can be bounded by
\[ e^{mc(t)} Q_\mu^2 \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^{\infty} e^{(-a(N+2)/b)} e^{((N+2)\omega(y))/b} \times \sum_{s=0}^{v} \left( \frac{v}{s} \right) t^{v-s} \leq \sum_{s=0}^{v} \left( \frac{v}{s} \right) t^{v-s}, \] (4.5.8)

Using inequality, definition (4.5.3) the right-hand side of (4.5.8) can be bounded by
\[ e^{mc(t)} Q_\mu^2 \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \int_0^{\infty} e^{(-a(N+2)/b)} \sum_{s=0}^{v} \left( \frac{v}{s} \right) t^{v-s} C_{\sigma(N+2)/b,s} \times c^s s! (1+t)^{\delta s} e^{\sigma \omega(t)} |t^{-\mu} \bar{\phi}(t)| dt. \]

Since \( \delta s \leq v \) the above expression can be estimated by
\[ e^{mc(t)} Q_\mu^2 e^{-a(N+2)/b} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} e^{-a/b} C_{\sigma(N+2)/b,s} \sum_{s=0}^{v} \left( \frac{v}{s} \right) c^s s! \times \sup_{t \in I} \left| (1+t)^{\mu+2v+2} e^{\sigma \omega(t)} \left| t^{-\mu} \bar{\phi}(t) \right| \right| \int_0^{\infty} \frac{dx}{(1+t)^2} \]
\[ \leq e^{mc(t)} Q_\mu^2 e^{-a(N+2)/b} \sum_{v=0}^{\infty} \frac{(me)^v}{v!} e^{-a/b} C_{\sigma(N+2)/b,s} \sum_{s=0}^{v} \left( \frac{v}{s} \right) c^s s! \times e^{-a((\mu+2v+2)/b)} \sup_{t \in I} \left( \frac{e^{(\mu+2v+2)/b} + \sigma}{t} \right) \omega(t) \left| t^{-\mu} \bar{\phi}(t) \right|. \] (4.5.9)
Using property (4.5.2) we can estimate the right-hand side of (4.5.9)

\[ Q_\mu^2 e^{\left( mc \varepsilon - a(N+\mu+5)/b \right)} \sum_{\nu=0}^{\infty} \frac{(me)^\nu}{\nu!} e^{-2a\nu/b} C_{\sigma,(N+2)/b,s} \sum_{s=0}^{\nu} \left( \frac{v}{s} \right) e^{\gamma s!/s!} \]

\[ \times \left( \beta_\mu \left( \frac{(\mu+2\nu+2)}{b} \right) \right)_0 (\varphi) \]

\[ \leq Q_\mu^2 e^{\left( mc \varepsilon - a(N+\mu+5)/b \right)} \sum_{\nu=0}^{\infty} \left( (me)e^{-2a\nu/b}(1+c) \right)^\nu \]

\[ \times \left[ \left( \beta_\mu \left( \frac{(\mu+2\nu+2)}{b} \right) \right)_0 (\varphi) C_{\sigma,(N+2)/b,s} \right]^{1/v} \]

Now choosing \( \varepsilon < \left( \beta_\mu \left( \frac{(\mu+2\nu+2)}{b} \right) \right)_0 (\varphi) C_{\sigma,(N+2)/b,s} \left( me^{-2a/b}(1+c) \right)^{-1} \), we find that the last series is convergent. This completes the proof theorem.

### 4.6 The space \( \mathcal{G}_\mu (\omega) \)

We now introduce a new family of seminorms on \( \mathcal{G}_\mu (\omega) \) that is equivalent to \( \mathcal{U}_\mu (\omega) \) and that will be very useful in the sequel. Now assume that \( \omega \) is a function in \( \mathcal{M} \). A function \( \varphi \in C^\infty (I) \) is said to be in the space \( \mathcal{G}_\mu (\omega) \), where \( \varphi \) and \( h_{1,\mu}(\varphi) \) are smooth functions and for every \( \mu \in \mathbb{R}, n \in \mathbb{N}_0 \) and \( m \) is a positive real number

\[ \Gamma_{m,n}^\mu (\varphi) = \sup_{x \in I} e^{m\omega(x)} \left| \Delta_\mu^n \varphi(x) \right| < \infty, \quad (4.6.1) \]

and

\[ \Upsilon_{m,n}^\mu (\varphi) = \sup_{x \in I} e^{m\omega(x)} \left| \Delta_\mu^n (h_{1,\mu}\varphi)(x) \right| < \infty, \quad (4.6.2) \]

where \( \Delta_\mu \) represents the Bessel operator \( D_x^{\mu+1} D_x^{-\mu} \) defined in (1.10.2). The family \( \{ \Gamma_{m,n}^\mu, \Upsilon_{m,n}^\mu \}_{m \in \mathbb{N}, n \in \mathbb{N}_0} \) of seminorms generates the topology of \( \mathcal{G}_\mu (\omega) \).

**Theorem 4.6.1** The pseudo-differential operator \( h_{1,\mu,a} \) is a continuous linear map of \( \mathcal{G}_\mu (\omega) \) into itself for \( \mu \geq 0 \).
Proof: (i) Let $\phi \in \mathcal{G}_\mu(\omega)$. Then in view of Lemma 4.2.1, we have

$$
\Gamma_{m,n}^\mu(h_{1,\mu,\alpha}\phi) = \sup_{x \in I} e^{m\omega(x)} \left| \Delta_n^\mu(h_{1,\mu,\alpha}\phi)(x) \right|
$$

$$
= \sup_{x \in I} e^{m\omega(x)} \left| \sum_{j=0}^{n} b_j x^{\mu+j} I^{n+j} x^{-\mu}(h_{1,\mu,\alpha}\phi)(x) \right|. \quad (4.6.3)
$$

Using (2.3.3) we can write the right-hand side in the form

$$
\sup_{x \in I} e^{m\omega(x)} \left| \sum_{j=0}^{n} b_j x^{\mu+j} \int_0^{\infty} C_\mu(xy) a(x,y) \hat{\phi}(y) \, dy \right|
$$

$$
= \sup_{x \in I} e^{m\omega(x)} \left| \sum_{j=0}^{n} b_j x^{\mu+j} \int_0^{\infty} D^{n+j} \{ C_\mu(xy) a(x,y) \} \hat{\phi}(y) \, dy \right|
$$

$$
= \sup_{x \in I} e^{m\omega(x)} \left| \sum_{j=0}^{n} b_j x^{\mu+j} \int_0^{\infty} \sum_{r=0}^{n+j} \binom{n+j}{r} D^{n+j-r} a(x,y) D^{r} \{ C_\mu(xy) \} \hat{\phi}(y) \, dy \right|
$$

$$
= \sup_{x \in I} e^{m\omega(x)} \left| \sum_{j=0}^{n} b_j x^{\mu+j} \int_0^{\infty} \sum_{r=0}^{n+j} \binom{n+j}{r} D^{n+j-r} a(x,y) (-y)^r C_{\mu+r}(xy) \hat{\phi}(y) \, dy \right|
$$

$$
\leq \sum_{j=0}^{n} \sum_{r=0}^{n+j} \binom{n+j}{r} \int_0^{\infty} x^{j-r} e^{m\omega(x)} |D^{n+j-r} a(x,y)| \left| (xy)^{\mu+r} C_{\mu+r}(xy) \right| |y^{-\mu} \hat{\phi}(y)| \, dy.
$$

Since by property (c), $x \leq (1+r) \leq e^{-a/b} e^{\omega(x)/b}$, using definition (4.5.3), the above expression can be estimated by

$$
Q_\mu \sum_{j=0}^{n} \sum_{r=0}^{n+j} \binom{n+j}{r} e^{-a(j-r)/b} C_{\sigma,m+\frac{a}{b} r} e^{n+j-r} (n+j-r)!
$$

$$
\times \int_0^{\infty} (1+y)^{\delta(n+j-r)} e^{\sigma\omega(y)} |y^{-\mu} \hat{\phi}(y)| \, dy.
$$

$$
\leq Q_\mu \sum_{j=0}^{n} \sum_{r=0}^{n+j} \binom{n+j}{r} e^{-an/b} C_{\sigma,m+\frac{a}{b} r} e^{2n} 2n!
$$

$$
\times \int_0^{\infty} e^{-2na\delta/b} e^{2n\delta\omega(y)/b} e^{\sigma\omega(y)} e^{-m\omega(y)} e^{m\omega(y)} |y^{-\mu} \hat{\phi}(y)| \, dy
$$

$$
\leq Q_\mu \sum_{j=0}^{n} \sum_{r=0}^{n+j} \binom{n+j}{r} e^{-an/b} C_{\sigma,m+\frac{a}{b} r} e^{2n} 2n! e^{-2na\delta/b}
$$

$$
\times \int_0^{\infty} e^{(\sigma-m)\omega(y)} \times \sup_{y \in I} e^{(m+2n\delta/b)\omega(y)} |y^{-\mu} \hat{\phi}(y)| \, dy
$$

53
by choosing $m > \sigma$. Therefore $\Gamma_{m,n}^{\mu}(h_{1,\mu,\alpha} \varphi) < \infty$.

(ii) Let $\Phi(x) = (h_{1,\mu,\alpha} \varphi)(x)$, then as in Theorem 4.5.1(ii) and using inequality (4.3.8), we have

$$Y_{m,n}^{\mu}(\Phi) = \sup_{x \in I} e^{\omega(x)} |\Delta_{\mu,x}^{n}(h_{1,\mu} \Phi)(x)| \leq e^{mc(\varepsilon)} \sum_{j=0}^{n} |b_j| \sum_{v=0}^{\infty} \frac{(me)^{\mu+v+j}}{v!} \left(\begin{array}{c} \mu + v + j \\ k \end{array}\right) |(-1)^{n+j+k} \int_{0}^{\infty} y^{\mu+n+j} \right|$$

$$\times \left|D_{x}^{n+j} \sum_{i=0}^{\mu+n+j} \Phi \int_{0}^{\infty} \int_{0}^{\infty} y^{\mu+2n} \right|$$

Using (4.3.5) and (2.3.3), the above expression can be estimated as

$$\leq e^{mc(\varepsilon)} \sum_{j=0}^{n} |b_j| \sum_{v=0}^{\infty} \frac{(me)^{\mu+v+j}}{v!} \left(\begin{array}{c} \mu + v + j \\ k \end{array}\right) Q_{\mu} \int_{0}^{\infty} \int_{0}^{\infty} y^{\mu+2n} \right|$$

$$\times \left|D_{y}^{k} \left[C_{\mu}(yt)\alpha(y,t)\hat{\varphi}(t)\right] dt \right| dy$$

$$\leq e^{mc(\varepsilon)} \sum_{j=0}^{n} |b_j| \sum_{v=0}^{\infty} \frac{(me)^{\mu+v+j}}{v!} \left(\begin{array}{c} \mu + v + j \\ k \end{array}\right) Q_{\mu} \int_{0}^{\infty} \int_{0}^{\infty} y^{\mu+2n} \right|$$

$$\times \left|\sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array}\right) \left[D_{y}^{k-r} C_{\mu}(yt)\right] D_{x}^{r} \alpha(y,t) \right| |\hat{\varphi}(t)| dt dy.$$
Again, using the property (c) of the function $\omega(x)$, the inequality $(1 + y) \leq e^{-a/b} e^{\omega(y)/b}$ and definition (4.5.3), we have

$$Q_m^2 e^{mc(\varepsilon)} \sum_{k=0}^{n} \left| b_j \right| \sum_{v=0}^{\infty} \frac{(me)^v}{v!} \sum_{k=0}^{\infty} \left( \frac{\mu + v + j}{k} \right) \sum_{r=0}^{k} \left( \frac{k}{r} \right) e^{-a(N+2)/b} e^{(N+2)\omega(y)/b} \times \int_0^\infty t^{k-r} \left| D_x a(y, t) \right| |\hat{\phi}(t)| dt$$

$$\leq Q_m^2 e^{mc(\varepsilon)} \sum_{j=0}^{n} \sum_{v=0}^{\infty} e^{\mu + v + j} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \left( \frac{\mu + v + j}{k} \right) \left( \frac{k}{r} \right) |b_j| \frac{(me)^v}{v!} e^{-a(N+2)/b} \times \int_0^\infty t^{k-r} \sigma_{\omega(N+2)/b} c^r r!(1 + t)^{\delta_r} e^{\sigma\omega(t)} |\hat{\phi}(t)| dt.$$ 

Since $\delta r \leq v$ and using again property (c) of the function $\omega(x)$ the inequality $(1 + t) \leq e^{-a/b} e^{\omega(t)/b}$, we have

$$Q_m^2 e^{mc(\varepsilon)} \sum_{j=0}^{n} \sum_{v=0}^{\infty} \frac{\mu + v + j}{k} \left( \frac{\mu + v + j}{k} \right) \left( \frac{k}{r} \right) |b_j| \frac{(me)^v}{v!} \times e^{-a(N+n+4+2\mu)/b} C_{\sigma, (N+2)/b} \mu \beta^\mu \left( \frac{\sigma + n + 2\nu + 2\mu}{b} \right) \phi$$

$$\leq Q_m^2 e^{mc(\varepsilon)} e^{-a(N+n+4+2\mu)/b} \sum_{j=0}^{n} \sum_{v=0}^{\infty} \left| b_j \right| \frac{(me)^v}{v!} (2 + c)^{\mu + v + j} e^{(-2\nu)\beta^\mu} C_{\sigma, (N+2)/b, \mu + v + j}$$

$$\times (\mu + v + j)! \beta^\mu \left( \frac{\sigma + n + 2\nu + 2\mu}{b} \right) \phi$$

$$\leq Q_m^2 e^{mc(\varepsilon)} - a(N+n+4+2\mu)/b \sum_{j=0}^{n} \left( me e^{-2\nu} (2 + c) \right)^v \times \frac{(\mu + v + n)!}{v!}$$

$$\times \sum_{j=0}^{n} \left| b_j \right| (2 + c)^{\mu + j} \left[ C_{\sigma, (N+2)/b, \mu + v + j} \beta^\mu \left( \frac{\sigma + n + 2\nu + 2\mu}{b} \right) \phi \right]^{1/v}.$$  

(4.6.6)

Setting $K = 2 + c$ and using the result $\frac{\Gamma(v+\alpha)}{\Gamma(v+\beta)} = O(\nu^{\alpha-\beta})$, $\nu \rightarrow \infty$, the expression (4.6.6) can also estimated by

$$AQ_m^2 \exp [mc(\varepsilon) - a(N+n+4+2\mu)/b] \left[ \sum_{v=0}^{\infty} \left( me e^{-2\nu} K \right)^{v} v^{\mu+n} \right]$$

$$\times \sum_{j=0}^{n} \left| b_j \right| K^{\mu+j} \left[ (Q_{\nu, N})^\mu \right]^{v},$$
for some positive constant \( A \), where \( Q_{n,v,N}^\mu = C_{\sigma, (N+2)/b, \mu + v + j} \beta_{(\sigma + n + 2v + 2\mu)/b}^\mu (\varphi) \), we have

\[
AQ_{\mu}^2 \exp[mc(\varepsilon) - a(N + n + 4 + 2\mu)/b] \left[ \sum_{\nu=0}^{\infty} \exp \left[ \nu \log \left( m\varepsilon e^{\frac{-2a}{b}} K(Q_{n,v,N}^\mu)^{\frac{1}{\nu}} \right) \right] e^{(\mu+n)\log v} \right] \times \sum_{j=0}^{n} |b_j| K^{\mu+j}.
\]

Since \( \log v < v \), we can further estimate it as follows:

\[
A Q_{\mu}^2 \exp[mc(\varepsilon) - a(N + n + 4 + 2\mu)/b] \left[ \sum_{\nu=0}^{\infty} \exp \left[ \nu \left( \log \left( m\varepsilon K(Q_{n,v,N}^\mu)^{\frac{1}{\nu}} \right) - 2a/b + \mu + n \right) \right] \right] \times \sum_{j=0}^{n} |b_j| K^{\mu+j} < \infty,
\]

as the infinite series can be made convergent by choosing

\[
\varepsilon < \left( (mK)^{-1}(Q_{n,v,N}^\mu)^{\frac{1}{\nu}} \exp (2a/b - \mu - n) \right).
\]

This completes the proof theorem.

*****