Chapter 2

Pseudo-differential operators involving Hankel-Clifford transformations

2.1 Introduction

The Schwartz’s theory of Fourier transformation of tempered distributions has been exploited by many authors to study the properties of the pseudo-differential operator (p.d.o.), see for instance Zaidman [37], Wong [36]. The Zemanian’s theory of the Hankel transformation was exploited by Pathak and Pandey [22] to study a class of pseudo-differential operators associated with Bessel operator $S_{\mu}$.

In this chapter, Pseudo-differential operator associated with the symbol $a(x, y)$ whose derivatives satisfy certain growth condition is defined and the Zemanian-type spaces $H_{\mu}(I)$ and $S(I)$ are introduced. It is shown that the pseudo-differential operator is continuous linear map of the Zemanian-type spaces $H_{\mu}(I)$ and $S(I)$ into itself. An integral representation of pseudo-differential operator is obtained. Using the Hankel-Clifford convolution it is shown that pseudo-differential operator satisfies a certain $L_{\mu}^1$ norm inequality. Properties of Sobolev-type space $G_{\mu,2}^S(I)$ are studied.

2.2 Properties of the Hankel-Clifford transformations

In this section properties of the Hankel-Clifford transformations are investigated.
Proposition 2.2.1 For any $\varphi \in \mathcal{H}_{\mu}(I)$ and for all non-negative integer $r$, we have

$$\Delta_{\mu}^{r}\varphi(x) = \sum_{i=0}^{r} b_{i} x^{r-i} D^{2r-i} \varphi(x),$$

(2.2.1)

where $\Delta_{\mu}$ is as (1.10.2) and $b_{i}$ are constants depending only on $\mu$.

Proof: At first we prove for $r = 1$,

$$\Delta_{\mu} \varphi(x) = DR_{\mu} \varphi(x) = xD^2 \varphi(x) + (1 - \mu) D \varphi(x)$$

$$\Delta_{\mu} \varphi(x) = \sum_{i=0}^{1} b_{i} x^{1-i} D^{2-i} \varphi(x).$$

Similarly, for $r = 2$.

$$\Delta_{\mu}^{2} \varphi(x) = \Delta_{\mu} \Delta_{\mu} \varphi(x) = \Delta_{\mu} \left( xD^2 \varphi(x) + (1 - \mu) D \varphi(x) \right)$$

$$= \left( xD^2 + (1 - \mu) D \right) \left( x\varphi''(x) + (1 - \mu) \varphi'(x) \right)$$

$$= x \left( x\varphi''(x) + (1 - \mu) \varphi'(x) \right) + (1 - \mu) D \left( x\varphi''(x) + (1 - \mu) \varphi'(x) \right)$$

$$= x \left( x\varphi'''(x) + (1 - \mu) \varphi''(x) \right) + (1 - \mu) \left( x\varphi''(x) + \varphi''(x) + (1 - \mu) \varphi''(x) \right)$$

$$= x^2 \varphi'''(x) + 2x(1 - \mu) \varphi''(x) + ((1 - \mu) + (1 - \mu)^2) \varphi''(x)$$

$$\Delta_{\mu}^{2} \varphi(x) = \sum_{i=0}^{2} b_{i} x^{2-i} D^{4-i} \varphi(x).$$

Continuing in this way, we have

$$\Delta_{\mu}^{r} \varphi(x) = \sum_{i=0}^{r} b_{i} x^{r-i} D^{2r-i} \varphi(x).$$

Similarly, we can obtain this result by using (1.10.3).

$$(\Delta_{\mu}^*)^{r} \varphi(x) = \sum_{i=0}^{r} a_{i} x^{r-i} D^{2r-i} \varphi(x),$$

(2.2.2)

where $a_{i}$ are constants depending only on $\mu$.

Let $\Delta(x, y, z)$ be the area of triangle with sides $x, y, z$ [10, 35]. For $\mu \geq 0$, set

$$D_{\mu}(x, y, z) = \frac{\Delta_{\mu}^{2\mu-1}}{2^{2\mu} (xyz) \Gamma(\mu + 1/2) \sqrt{\pi}},$$

(2.2.3)

if $\Delta$ exists and zero otherwise. We note that $D_{\mu}(x, y, z) \geq 0$ and it is symmetric in $x, y, z$. 
Lemma 2.2.1  
For $\mu \geq 0$, we have
\[
\int_0^\infty C_\mu(zt)z^\mu D_\mu(x,y,z)dz = C_\mu(xt)C_\mu(yt),
\]  
(2.2.4)

and
\[
\Gamma(\mu + 1) \int_0^\infty D_\mu(x,y,z)x^\mu dx = 1.
\]  
(2.2.5)

Proof: From Watson [35, p.411], we have
\[
\int_0^\infty J_\mu(xt)J_\mu(yt)J_\mu(zt)t^{-\mu+1} dt = \frac{2^{\mu-1}\Delta^{2\mu-1}}{(xyz)^\mu \Gamma(\mu + 1/2) \sqrt{\pi}},
\]  
(2.2.6)

where $\Delta$ is as (2.2.3).

Let $x = 2\sqrt{u}$, $y = 2\sqrt{v}$, $z = 2\sqrt{w}$, we have
\[
\int_0^\infty J_\mu(2\sqrt{ut})J_\mu(2\sqrt{vt})J_\mu(2\sqrt{wt})t^{-\mu+1} dt = \frac{2^{\mu-1}\Delta^{2\mu-1}}{2^\mu (uvw)^{\mu/2} \Gamma(\mu + 1/2) \sqrt{\pi}},
\]

now we putting $t = \sqrt{s}$, we have
\[
\int_0^\infty J_\mu(2\sqrt{us})J_\mu(2\sqrt{vs})J_\mu(2\sqrt{ws})(\sqrt{s})^{-\mu} ds = \frac{\Delta^{2\mu-1}}{2^\mu (uvw)^{\mu/2} \Gamma(\mu + 1/2) \sqrt{\pi}}
\]
\[
= P_\mu(u,v,w).
\]

Now
\[
\int_0^\infty (us)^{-\mu/2}J_\mu(2\sqrt{us})(vs)^{-\mu/2}J_\mu(2\sqrt{vs})(ws)^{-\mu/2}J_\mu(2\sqrt{ws})s^\mu ds
\]
\[
= \frac{\Delta^{2\mu-1}}{2^\mu (uvw)^{\mu} \Gamma(\mu + 1/2) \sqrt{\pi}}
\]
\[
= D_\mu(u,v,w),
\]

by using (1.7.3), we have
\[
\int_0^\infty C_\mu(us)C_\mu(vs)C_\mu(ws)s^\mu ds = D_\mu(u,v,w)
\]
\[
w^{-\mu} \left[w^\mu \int_0^\infty C_\mu(ws)\{C_\mu(us)C_\mu(vs)s^\mu\} ds\right] = D_\mu(u,v,w).
\]
Therefore by using inversion formula of first Hankel-Clifford transformation (1.8.2) and (1.8.1) we have

\[ h_{1,\mu}^{-1} \left[ C_\mu(us)C_\mu(vs)s^\mu \right](w) = D_\mu(u,v,w)w^\mu \]

\[ C_\mu(us)C_\mu(vs) = h_{1,\mu} \left[ D_\mu(u,v,w)w^\mu \right](s) = s^\mu \int_0^\infty C_\mu(ws)D_\mu(u,v,w)w^\mu dw \]

\[ C_\mu(us)C_\mu(vs) = \int_0^\infty C_\mu(ws)D_\mu(u,v,w)w^\mu dw, \]

by replacing \( u \) by \( x \), \( v \) by \( y \), \( w \) by \( z \) and \( s \) by \( t \), then we have

\[ \int_0^\infty C_\mu(zt)D_\mu(x,y,z)z^\mu dz = C_\mu(xt)C_\mu(yt), \]

now using (1.7.3) and setting \( t = 0 \), we have

\[ \Gamma(\mu + 1) \int_0^\infty D_\mu(x,y,z)x^\mu dx = 1. \]

Let \( \varphi \) and \( \psi \in L^1_\mu(I) \), then \((\tau_\varphi)(y)\) and \((\varphi#\psi)(x)\) are defined by

\[ (\tau_\varphi)(y) = \int_0^\infty \varphi(z)z^\mu D_\mu(x,y,z)dz, \quad 0 < x, y < \infty, \]  

(2.2.7)

\[ (\varphi#\psi)(x) = \int_0^\infty \psi(y)(\tau_\varphi)(y)y^\mu dy, \quad 0 < x < \infty. \]  

(2.2.8)

**Lemma 2.2.2** Let \( \varphi \) and \( \psi \in L^1_\mu(I) \), then we have

\[ ||\varphi#\psi||_{L^1_\mu} \leq \frac{1}{\Gamma(\mu + 1)}||\varphi||_{L^1_\mu(I)}||\psi||_{L^1_\mu(I)}. \]  

(2.2.9)

**Proof:** Using (2.2.8), (1.13.2) and (2.2.5), we have

\[ ||\varphi#\psi||_{L^1_\mu} = \int_0^\infty |(\varphi#\psi)(x)|x^\mu dx \]

\[ = \int_0^\infty x^\mu dx \int_0^\infty \psi(y)(\tau_\varphi)(y)y^\mu dy \]

\[ = \int_0^\infty x^\mu dx \int_0^\infty \psi(y)y^\mu dy \int_0^\infty \varphi(z)z^\mu D_\mu(x,y,z)dz \]

\[ \leq \int_0^\infty |\varphi(z)|z^\mu dz \int_0^\infty D_\mu(x,y,z)x^\mu dx \int_0^\infty |\psi(y)|y^\mu dy \]

\[ \leq \frac{1}{\Gamma(\mu + 1)} \int_0^\infty |\varphi(z)|z^\mu dz \Gamma(\mu + 1) \int_0^\infty D_\mu(x,y,z)x^\mu dx \int_0^\infty |\psi(y)|y^\mu dy \]

\[ = \frac{1}{\Gamma(\mu + 1)}||\varphi||_{L^1_\mu(I)}||\psi||_{L^1_\mu(I)}. \]
Lemma 2.2.3 \ Let \( \phi \) and \( \psi \) be functions of \( L^1_{\mu}(I) \), then

(i) \( h_{1,\mu}(y^\mu \tau_x \phi)(t) = C_{\mu}(xt)(h_{1,\mu}z^\mu \phi)(t) \),

or

\( h_{2,\mu}(\tau_x \psi)(t) = C_{\mu}(xt)(h_{2,\mu} \psi)(t) \).

(ii) \( h_{1,\mu}(x^\mu(\phi \# \psi))(t) = t^{-\mu}h_{1,\mu}(y^\mu \psi)(t)h_{1,\mu}(z^\mu \phi)(t) \),

or

\( h_{2,\mu}(\phi \# \psi)(t) = h_{2,\mu}(\phi)(t)h_{2,\mu}(\psi)(t) \).

Proof: (i) For \( \phi \in L^1_{\mu}(I) \) and using (2.2.7), we have

\[ h_{1,\mu}(y^\mu \tau_x \phi)(t) = t^\mu \int_0^\infty C_{\mu}(yt)z^\mu(\tau_x \phi)(y)dy \]

\[ = t^\mu \int_0^\infty C_{\mu}(yt) \left( \int_0^\infty \phi(z)z^\mu D_{\mu}(x,y,z)dz \right) y^\mu dy \]

\[ = \int_0^\infty \phi(z)z^\mu t^\mu \left[ \int_0^\infty C_{\mu}(yt)D_{\mu}(x,y,z)y^\mu dy \right] dz, \]

using (2.2.4), we have

\[ h_{1,\mu}(y^\mu \tau_x \phi)(t) = \int_0^\infty \phi(z)z^\mu t^\mu C_{\mu}(xt)C_{\mu}(zt)dz \]

\[ = C_{\mu}(xt)(h_{1,\mu}z^\mu \phi)(t), \]

or

\[ h_{2,\mu}(\tau_x \psi)(t) = C_{\mu}(xt)(h_{2,\mu} \psi)(t). \]

(ii) For \( \phi, \psi \in L^1_{\mu}(I) \) and using (2.2.8) and (2.2.7), we have

\[ h_{1,\mu}(x^\mu(\phi \# \psi))(t) = t^\mu \int_0^\infty C_{\mu}(xt)(\phi \# \psi)(x)x^\mu dx \]

\[ = t^\mu \int_0^\infty C_{\mu}(xt) \left( \int_0^\infty \psi(y)(\tau_x \phi)(y)y^\mu dy \right) x^\mu dx \]

\[ = t^\mu \int_0^\infty C_{\mu}(xt) \left[ \int_0^\infty \psi(y)y^\mu dy \left( \int_0^\infty \phi(z)z^\mu D_{\mu}(x,y,z)dz \right) \right] x^\mu dx \]

\[ = t^\mu \int_0^\infty \psi(y)y^\mu dy \int_0^\infty \phi(z)z^\mu dz \int_0^\infty C_{\mu}(xt)D_{\mu}(x,y,z)x^\mu dx, \]

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using (2.2.4), we have

\[
\begin{align*}
    h_{1, \mu}(x^\mu (\varphi \# \psi))(t) &= t\mu \int_0^\infty \psi(y)y^\mu dy \int_0^\infty \varphi(z)z^\mu dz \left[ C_\mu(yt)C_\mu(zt) \right] \\
    &= t^{-\mu} \left[ \mu \int_0^\infty C_\mu(yt) \psi(y) y^\mu dy \right] \left[ \mu \int_0^\infty C_\mu(zt) \varphi(z) z^\mu dz \right] \\
    &= t^{-\mu} h_{1, \mu}(y^\mu \psi)(t) h_{1, \mu}(z^\mu \varphi)(t),
\end{align*}
\]

or

\[
    h_{2, \mu}(\varphi \# \psi)(t) = h_{2, \mu}(\varphi)(t) h_{2, \mu}(\psi)(t).
\]

### 2.3 Pseudo-differential operator involving Hankel-Clifford transformations

A linear differential operator \( P(x, \Delta_\mu) \) on \( I \) is given by

\[
P(x, \Delta_\mu) = \sum_{r=0}^{n} a_r(x) \Delta_\mu^r,
\]

where the coefficients \( a_r(x) \) are functions defined on \( I = (0, \infty) \) and \( \Delta_\mu = DR_\mu = xD^2 + (1 - \mu)D, D = d/dx \). If we replace \( \Delta_\mu^r \) in (2.3.1) by monomial \((-y)^r\) in \( I \), then we obtain the so called symbol

\[
P(x, y) = \sum_{r=0}^{n} a_r(x) (-y)^r,
\]

of operator (2.3.1). In order to get the another representation of the operator \( P(x, \Delta_\mu) \), let us take any function \( \varphi \in \mathcal{H}_\mu(I) \). Then by (2.3.1), (2.3.2), (1.10.4) and the inversion formula of first Hankel-Clifford transformation (1.8.2) for Zemanian-type functions, we
have

\[
(P(x, \Delta_{\mu}) \varphi)(x) = \sum_{r=0}^{n} a_r(x) \left( \Delta_{\mu}^r \varphi \right)(x)
\]

\[
= \sum_{r=0}^{n} a_r(x) h_{1, \mu}^{-1} (h_{1, \mu} \varphi)(x)
\]

\[
= \sum_{r=0}^{n} a_r(x) h_{1, \mu}^{-1} ((-y)^r (h_{1, \mu} \varphi))(x)
\]

\[
= x^{\mu} \int_{0}^{\infty} C_{\mu}(xy) \left( \sum_{r=0}^{n} a_r(x) (-y)^r \right) \hat{\varphi}(y) dy
\]

\[
= x^{\mu} \int_{0}^{\infty} C_{\mu}(xy) P(x,y) \hat{\varphi}(y) dy,
\]

where \(C_{\mu}(xy)\) is as (1.7.3). If we replace the symbol \(P(x,y)\) by more general symbol \(a(x,y)\) which is no longer polynomial in \(y\). The operator so obtained is called pseudo-differential operator and denoted by \(h_{1, \mu, a}\) and defined as below:

**Definition 2.3.1** Let \(a(x,y)\) be a complex valued function belonging to the space \(C^\infty(I \times I), \ I = (0, \infty)\). Then the pseudo-differential operator \(h_{1, \mu, a}\) associated to \(a(x,y)\) is defined by

\[
(h_{1, \mu, a} \varphi)(x) = x^{\mu} \int_{0}^{\infty} C_{\mu}(xy) a(x,y) \hat{\varphi}(y) dy, \ \varphi \in \mathcal{H}_{\mu}(I), \quad (2.3.3)
\]

where \(\hat{\varphi}(y)\) is as (1.8.1).

Similarly, the pseudo-differential operator \(h_{2, \mu, a}\) is defined by

\[
(h_{2, \mu, a} \psi)(x) = \int_{0}^{\infty} C_{\mu}(xy) a(x,y) \psi(y) y^{\mu} dy, \ \psi \in S(I), \quad (2.3.4)
\]

where \(\hat{\psi}(y)\) is as (1.8.6).

**Definition 2.3.2** Let \(m \in (-\infty, \infty)\). Then we define \(H^m\) to be the set of all functions \(a(x,y) \in C^\infty(I \times I)\) such that for any three non-negative integers \(k, l, q\), there exists a positive constant \(D_{m,l,k,q}\) such that

\[
(1 + x)^k \left| D^k_x D^l_y a(x,y) \right| \leq D_{m,l,k,q} (1 + y)^{m-q}, \quad x, y \in I. \quad (2.3.5)
\]

For pseudo-differential operator involving Fourier transformation and Hankel transformation we refer to [21, 36] and [22, 30] respectively.
Theorem 2.3.1 Let the symbol $a(x,y) \in H^m$ with $m < -1$, then for $\mu \in \mathbb{N}_0$, the pseudo-differential operator $h_{1,\mu,a}$ is a continuous linear mapping from $\mathcal{H}_\mu(I)$ into itself.

Proof: For $\varphi \in \mathcal{H}_\mu(I)$ and $k, q \in \mathbb{N}_0$, and using definition (2.3.3), (1.7.4) and (2.3.5), then we have

$$\left| x^\mu D^k x^{-\mu} (h_{1,\mu,a} \varphi)(x) \right| = \left| x^\mu \int_0^\infty \left( \frac{d}{dx} \right)^k C_\mu(xy)a(x,y) (h_{1,\mu} \varphi)(y) dy \right|$$

$$\leq \int_0^\infty \sum_{r=0}^k \binom{k}{r} \left| \left( \frac{d}{dx} \right)^r C_\mu(xy) (1 + x)^q \right| \left( \frac{d}{dx} \right)^{k-r} a(x,y) (h_{1,\mu} \varphi)(y) dy$$

$$\leq \int_0^\infty \sum_{r=0}^k \binom{k}{r} \left| (-1)^r C_{\mu+r}(xy) \right| D_{m,q,k-r}(1+y)^m$$

$$\times \sup_{y \in I} \left| y^r(h_{1,\mu} \varphi)(y) \right| dy$$

$$\leq \sum_{r=0}^k \binom{k}{r} \left| C_{\mu+r}(xy) A_{m,q,k,r} \right| y^{\mu+r}_{r+\mu,0}(h_{1,\mu} \varphi) \int_0^\infty (1+y)^m dy$$

$$< \infty.$$ 

The integral in $y$ is convergent for $m < -1$. This implies that $h_{1,\mu,a} \varphi \in \mathcal{H}_\mu(I)$. Moreover, if we assume that $\{ \varphi_n \}$ is sequence in $\mathcal{H}_\mu(I)$ which converges to zero in $\mathcal{H}_\mu(I)$ then from [28], $h_{1,\mu} \varphi_n \longrightarrow 0$ in $\mathcal{H}_\mu(I)$. Therefore $h_{1,\mu,a} \varphi_n$ also converges to zero in $\mathcal{H}_\mu(I)$. This proves the continuity.

Theorem 2.3.2 Let the symbol $a(x,y) \in H^m$ with $m < -1$, then for $\mu \in \mathbb{N}_0$, the pseudo-differential operator $h_{2,\mu,a}$ is a continuous linear mapping from $S(I)$ into itself.

Proof: The proof of this theorem is similar as the proof of Theorem 2.3.1.

2.4 An integral representation

The function $a_y(\eta)$ associated with the symbol $a(x,y)$ and defined by

$$a_y(\eta) = (h_{1,\mu} \left[ C_\mu(xy)a(x,y) \right]) (\eta), \ x \in I, \quad (2.4.1)$$

will play a fundamental role in our investigation. An estimate for $a_y(\eta)$ is given in the following Lemma:
Lemma 2.4.1 Let the symbol $a(x,y) \in H^m$, then the function $a_y(\eta)$ defined by (2.4.1), satisfies the inequality

$$|a_y(\eta)| \leq A_{\mu,m,k,r}(1+y)^{m+2r}\eta^\mu(1+\eta^r)^{-1},$$

(2.4.2)

where $k$, $r \in \mathbb{N}_0$, $\mu \geq 0$ and $A_{\mu,m,k,r}$ is positive constant.

Proof: For $r \in \mathbb{N}_0$, and using (2.4.1), (1.10.4) and (2.2.1), then we have

$$(-\eta)^r a_y(\eta) = (-\eta)^r (h_{1,\mu}[C_\mu(xy)a(x,y)])(\eta)$$

$$= h_{1,\mu} \left( \Delta^\mu_\eta [C_\mu(xy)a(x,y)] \right)(\eta)$$

$$= \eta^\mu \int_0^\infty C_\mu(x\eta) \sum_{i=0}^r b_ix^{r-i}D^{2r-i} [C_\mu(xy)a(x,y)]dx$$

$$= \eta^\mu \int_0^\infty C_\mu(x\eta) \sum_{i=0}^r b_ix^{r-i} \sum_{j=0}^{2r-i} \binom{2r-i}{j}$$

$$\times D^{2r-i-j}C_\mu(xy)D^ja(x,y)dx.$$  \hspace{1cm} (2.4.3)

Now using the equation (1.7.4), then from above equation (2.4.3), we have

$$(-\eta)^r a_y(\eta) = \eta^\mu \int_0^\infty C_\mu(x\eta) \sum_{i=0}^r b_ix^{r-i} \sum_{j=0}^{2r-i} \binom{2r-i}{j}$$

$$\times D^ja(x,y)y^{2r-i-j}(-1)^{2r-i-j}C_{\mu+2r-i-j}(xy)dx.$$

Using Definition 2.3.2, we have

$$|(-\eta)^r a_y(\eta)| \leq |\eta^\mu| \int_0^\infty \left| C_\mu(x\eta) \right| \left| \sum_{i=0}^r b_ix^{r-i} \sum_{j=0}^{2r-i} \binom{2r-i}{j} D^ja(x,y) \right|$$

$$\times y^{2r-i-j}(-1)^{2r-i-j}C_{\mu+2r-i-j}(xy)dx$$

$$\leq |\eta^\mu| \int_0^\infty \left| C_\mu(x\eta) \right| \left| \sum_{i=0}^r b_i \sum_{j=0}^{2r-i} \binom{2r-i}{j} D^ja(x,y)y^{2r-i-j} \right|$$

$$\times |(-1)^{2r-i-j}|C_{\mu+2r-i-j}(xy)dx$$

$$\leq |\eta^\mu| \int_0^\infty \left| C_\mu(x\eta) \right| \sum_{i=0}^r b_i \sum_{j=0}^{2r-i} \binom{2r-i}{j} D_{m,j,k} y^{2r-i-j}(1+y)^m$$

$$\times \left| C_{\mu+2r-i-j}(xy) \right| x^{r-i}(1+x)^{-k}dx$$

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\[
\leq |\eta^\mu| \sum_{i=0}^{r} b_i \sum_{j=0}^{2r-i} \binom{2r-i}{j} B_{\mu,m,j,k} y^{2r-i-j} (1+y)^m \\
\times \int_0^\infty x^{r-i}(1+x)^{-k} dx
\]
\[
\leq \eta^\mu \sum_{i=0}^{r} b_i \sum_{j=0}^{2r-i} \binom{2r-i}{j} B_{\mu,m,j,k} y^{2r-i-j} (1+y)^m \\
\times \beta ((r-i+1,k-(r-i+1)) ),
\]
where \( B_{\mu,m,j,k} \) is a positive constant and \( \beta \) denotes the Beta function.

For \( k > r + 1 \), therefore there exists a constant \( A_{\mu,m,r,k} \) such that

\[
|a_y(\eta)| \leq A_{\mu,m,k,r}(1+y)^{m+2r} \eta^\mu (1+\eta^r)^{-1}, \quad \forall \ r > 0.
\]

An integral representation for the pseudo-differential operator \( h_{1,\mu,a} \) is now obtained.

**Theorem 2.4.1** For any symbol \( a(x,y) \in H^m \) the associated operator \( (h_{1,\mu,a}\varphi)(x) \), \( \varphi \in \mathcal{K}_\mu(I) \) can be represented by

\[
(h_{1,\mu,a}\varphi)(x) = x^\mu \int_0^\infty C_\mu(x\eta) \left[ \int_0^\infty a_y(\eta)(h_{1,\mu}\varphi)(y) dy \right] d\eta; \quad (2.4.4)
\]

when all involved integrals are convergent.

**Proof:** Since

\[
a_y(\eta) = \eta^\mu \int_0^\infty C_\mu(x\eta) \left[ C_\mu(xy)a(x,y) \right] dx,
\]

by inversion formula of first Hankel-Clifford transformation (1.8.2), we have

\[
x^\mu \int_0^\infty a_y(\eta)C_\mu(x\eta) d\eta = C_\mu(xy)a(x,y).
\]

Using (2.3.3) and above equation, we have

\[
(h_{1,\mu,a}\varphi)(x) = x^\mu \int_0^\infty C_\mu(xy)a(x,y)(h_{1,\mu}\varphi)(y) dy \\
= x^\mu \int_0^\infty (h_{1,\mu}\varphi)(y) dy \left[ x^\mu \int_0^\infty a_y(\eta)C_\mu(x\eta) d\eta \right] \\
= x^{2\mu} \int_0^\infty C_\mu(x\eta) d\eta \left[ \int_0^\infty (h_{1,\mu}\varphi)(y)a_y(\eta) dy \right]. \quad (2.4.6)
\]
Now, using estimate (2.4.1) for $a_y(\eta)$, the above change in the order of integration can be justified and existence of the last integral can be proved. Since $(h_{1,\mu}(\phi)) \in \mathcal{H}_\mu(I)$, we have

$$|(h_{1,\mu}(\phi))(y)| \leq C y^\mu (1+y)^{-l}, \quad \forall \ l > 0. \quad (2.4.7)$$

Hence,

$$|(h_{1,\mu,a}(\phi))(x)| \leq |x^{2\mu}| \int_0^\infty \int_0^\infty |C_\mu(x\eta)|$$
$$\times \ A_{\mu,m,k,r}(1+y)^{m+2r} \eta^\mu (1+\eta^r)^{-1} C y^\mu (1+y)^{-l} dy d\eta$$
$$\leq D'_{\mu,m,k,r} x^{2\mu} \int_0^\infty (1+\eta)^\mu (1+\eta^r)^{-1} d\eta$$
$$\times \ \int_0^\infty (1+y)^{m+2r-l} dy. \quad (2.4.8)$$

The above integrals are convergent, since $\mu \geq 0$ and $l$ and $r$ can be chosen sufficiently large.

## 2.5 An $L^1_\mu$ - Norm inequality

In the proof of Theorem 2.5.1, we shall need the following estimate for the first Hankel-Clifford transformation of $a(x,y)$. We write

$$A_y(\eta) = h_{1,\mu}(a(x,y))(\eta), \ x \in I. \quad (2.5.1)$$

**Lemma 2.5.1** For $\mu \geq 0$ and $r,k \in \mathbb{N}_0$, there exists a constant $C_{\mu,r,m,k} > 0$ such that

$$|A_y(\eta)| \leq C_{\mu,r,m,k}(1+y)^m (1+\eta^r)^{-1} \eta^\mu. \quad (2.5.2)$$

**Proof:** As in the proof of Lemma 2.4.1, we have

$$(-\eta)^r A_y(\eta) = (-\eta)^r h_{1,\mu}(a(x,y))(\eta)$$
$$= h_{1,\mu}(\Delta_\mu a(x,y))(\eta)$$
$$= \eta^\mu \int_0^\infty C_\mu(x\eta) \Delta_\mu a(x,y) dx$$
$$= \eta^\mu \int_0^\infty C_\mu(x\eta) \sum_{i=0}^{r} b_i x^{r-i} D^{2r-i} a(x,y) dx$$
$$= \eta^\mu \int_0^\infty C_\mu(x\eta) \sum_{i=0}^{r} b_i D_{m,2r-i,k}(1+y)^m x^{r-i}(1+x)^{-k} dx.$$
Hence of all those elements \( \phi \)

**Definition 2.5.1 (Sobolev-type space)** The space \( H^{k,m} \) is defined to be the set of all those elements \( \phi \in H^m(I) \) which satisfy

\[
\| \phi \|_{H^k} = \| \sum_{i=0}^{k} b_i D_{\mu,2r-i,k}(1 + y)^m (1 + x)^{r-i-k} \|_{L^1} < \infty.
\]

**Theorem 2.5.1** Let \( \mu > 0 \). Then for all \( m \in \mathbb{N}_0 \) there exists \( C > 0 \) such that

\[
\| h_{1,\mu,a}(\phi) \|_{H^k} \leq C \sum_{i=0}^{m} \binom{m}{i} \| \phi \|_{H^k}, \quad \phi \in H^m(I).
\]

**Proof:** From Theorem 2.4.1 and using (2.4.1) and Definition 2.5.1 and relation (2.2.4), we have

\[
\eta^\mu \int_0^\infty C\mu(x\eta)(h_{1,\mu,a})(x)dx = \int_0^\infty a_y(\eta)(h_{1,\mu,a})(y)dy.
\]

Hence

\[
\eta^\mu \int_0^\infty C\mu(x\eta)(h_{1,\mu,a})(x)dx = \int_0^\infty \left[ \eta^\mu \int_0^\infty C\mu(x\eta)(C\mu(x)y)a(x,y)dx \right] (h_{1,\mu,a})(y)dy
\]

\[
\int_0^\infty C\mu(x\eta)(h_{1,\mu,a})(x)dx = \int_0^\infty (h_{1,\mu,a})(y) \left( \int_0^\infty C\mu(xz)g^\mu D_{\mu}(\eta,y,z)a(x,y)dz \right) dx dy
\]

\[
\eta^{-\mu} h_{1,\mu}(h_{1,\mu,a})(\eta) = \int_0^\infty (h_{1,\mu,a})(y)dy \int_0^\infty \left( z^\mu \int_0^\infty C\mu(xz)a(x,y)dx \right) D_{\mu}(\eta,y,z)dz
\]
Now, an application of the estimate (2.5.2) to (2.5.5) and using (2.2.7) and (2.2.8) yields

\[
|\eta^{-\mu} h_{1,\mu} (h_{1,\mu,a} \varphi)(\eta)| \leq C_{\mu,r,m,k} \int_0^\infty (1 + y)^m (h_{1,\mu} \varphi)(y) dy \\
\times \int_0^\infty z^\mu (1 + z')^{-1} D_\mu (\eta, y, z) dz \\
\leq C_{\mu,r,m,k} \sum_{l=0}^m \binom{m}{l} \int_0^\infty y^l (h_{1,\mu} \varphi)(y) dy \\
\times \int_0^\infty z^\mu (1 + z')^{-1} D_\mu (\eta, y, z) dz \\
\leq C_{\mu,r,m,k} \sum_{l=0}^m \binom{m}{l} \left( y^{l-\mu} (h_{1,\mu} \varphi)^{(1 + z')^{-1}} \right)(\eta).
\]

Therefore

\[
\int_0^\infty \eta^{-\mu} |\eta^{-\mu} h_{1,\mu} (h_{1,\mu,a} \varphi)(\eta)| d\eta \\
\leq C_{\mu,r,m,k} \sum_{l=0}^m \binom{m}{l} \int_0^\infty \eta^\mu \left( y^{l-\mu} (h_{1,\mu} \varphi)^{(1 + z')^{-1}} \right)(\eta) d\eta.
\]

Now applying (2.2.9) to (2.5.6), we get

\[
||| \eta^{-\mu} h_{1,\mu} (h_{1,\mu,a} \varphi)(\eta)|||_{L^1_{\mu}(I)} \leq C_{\mu,r,m,k} \sum_{l=0}^m \binom{m}{l} \frac{1}{\Gamma(\mu + 1)} \\
\times ||| y^{l-\mu} (h_{1,\mu} \varphi)|||_{L^1_{\mu}(I)} ||| (1 + z')^{-1} |||_{L^1_{\mu}(I)} \\
\leq C \sum_{l=0}^m \binom{m}{l} ||| y^{l-\mu} (h_{1,\mu} \varphi)|||_{L^1_{\mu}(I)},
\]

for \( r > \mu + 1 \) and a constant \( C > 0 \).

Hence

\[
||h_{1,\mu,a} \varphi||_{\xi_{\mu}} \leq C \sum_{l=0}^m \binom{m}{l} ||\varphi||_{\xi_{\mu}}.
\]

(2.5.7)
2.6 The Sobolev-type space $G_{\mu,2}^s(I)$

**Definition 2.6.1** Let $s \in \mathbb{R}$, we denote by $G_{\mu,2}^s(I)$ the space of all $f \in \mathcal{H}_\mu'(I)$, the dual of the test function space $\mathcal{H}_\mu(I)$, such that
\[
(1 + y^2)^{s/2}(h_{1,\mu}f)(y) \in L^2(I),
\]
which induces the norm
\[
\|f\|_{G_{\mu,2}^s} = \left( \int_0^\infty (1 + y^2)^s|\langle h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}, \quad \mu \geq 0.
\]

**Theorem 2.6.1** If $P(\Delta_\mu)$ is a linear differential operator with constant coefficients of order $r$, and $f \in G_{\mu,2}^s(I)$, then $(P(\Delta_\mu))f \in G_{\mu,2}^{s-r}(I)$ and the map $P(\Delta_\mu) : G_{\mu,2}^s(I) \to G_{\mu,2}^{s-r}(I)$ is continuous.

**Proof:** Let
\[
P(\Delta_\mu) = \sum_{r=0}^n a_r \Delta_\mu^r, \quad a_r \in \mathbb{C}.
\]
Then for $f \in \mathcal{H}_\mu'(I)$.

Now, we have
\[
\|(P(\Delta_\mu))f\|_{G_{\mu,2}^{s-r}} = \left( \int_0^\infty (1 + y^2)^{s-r}|\langle h_{1,\mu}(P(\Delta_\mu)f)(y)\rangle|^2dy \right)^{1/2}
\]
\[
= \left( \int_0^\infty (1 + y^2)^{s-r}|\langle (P(-y))h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}
\]
\[
= \left( \int_0^\infty (1 + y^2)^{s-r} \sum_{r=0}^n a_r (-y)^r\langle h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}
\]
\[
= \left( \int_0^\infty (1 + y^2)^{s-r} \sum_{r=0}^n a_r y^r\langle h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}
\]
\[
\leq \sum_{r=0}^n |a_r| \left( \int_0^\infty (1 + y^2)^{s-r}(1 + y^2)^r|\langle h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}
\]
\[
\leq \sum_{r=0}^n |a_r| \left( \int_0^\infty (1 + y^2)^{s}|\langle h_{1,\mu}f(y)\rangle|^2dy \right)^{1/2}
\]
\[
\leq \sum_{r=0}^n |a_r| \|f\|_{G_{\mu,2}^s}, \quad (2.6.2)
\]
which proves that \((P(\Delta \mu)f) \in G^{r}_{\mu,2}\).

To show that continuity assume that \(\{f_j\}_{j \in \mathbb{N}}\) is a sequence in \(G^{r}_{\mu,2}\), which converges to zero in \(G^{r}_{\mu,2}\). Then from above inequality it follows that

\[
\|(P(\Delta \mu)f_j)\|_{G^{r}_{\mu,2}} \leq \sum_{r=0}^{n} |a_r| \|f\|_{G^{r}_{\mu,2}} \to 0 \text{ as } j \to \infty,
\]

which implies the continuity of the operator \(P(\Delta \mu)\).

Theorem 2.6.2 For a non-negative integer \(r\) and \(I = (0, \infty)\), we have

\[
G^{r}_{\mu,2}(I) = \{ f \in L^2(I) : h_{1,\mu}(\Delta^r_{\mu}f) \in L^2(I) \},
\]

(2.6.3)

where \(\Delta_{\mu}\) is defined as (1.10.2).

Proof: Let \(f \in G^{r}_{\mu,2}(I)\), then \(f \in \mathcal{H}^{r}_{1,\mu}(I)\). Hence its first Hankel-Clifford transformation exists and we have

\[
h_{1,\mu}(\Delta^r_{\mu}f) = (-y)^r h_{1,\mu}f.
\]

(2.6.4)

Since \(f \in G^{r}_{\mu,2}(I)\), we have

\[
\int_{0}^{\infty} (1+y^2)^r |(h_{1,\mu}f)(y)|^2 dy < \infty.
\]

Now for \(h_{1,\mu}(\Delta^r_{\mu}f) \in L^2(I)\),

\[
\int_{0}^{\infty} |h_{1,\mu}(\Delta^r_{\mu}f)|^2 dy = \int_{0}^{\infty} |(-y)^r(h_{1,\mu}f)(y)|^2 dy
\]

\[
= \int_{0}^{\infty} |y^r(h_{1,\mu}f)(y)|^2 dy
\]

\[
\leq \int_{0}^{\infty} (1+y^2)^r |(h_{1,\mu}f)(y)|^2 dy < \infty.
\]

(2.6.5)

Next, we assume that \(h_{1,\mu}(\Delta^r_{\mu}f) \in L^2(I)\), \(r \in \mathbb{N}_0\), then we have

\[
\int_{0}^{\infty} (1+y^2)^r |(h_{1,\mu}f)(y)|^2 dy \leq \sum_{n=0}^{r} \binom{r}{n} \int_{0}^{\infty} |(y^n(h_{1,\mu}f)(y)|^2 dy
\]

\[
\leq \sum_{n=0}^{r} \binom{r}{n} \int_{0}^{\infty} |(-y)^n(h_{1,\mu}f)(y)|^2 dy
\]

\[
= \sum_{n=0}^{r} \binom{r}{n} \int_{0}^{\infty} |h_{1,\mu}(\Delta^n_{\mu}f)|^2 dy < \infty.
\]

(2.6.6)
This is complete proof of theorem.

**Remark:** Similar results of all Lemmas and Theorems of Section 2.4, 2.5, and 2.6 may be proved using second Hankel-Clifford transformation for $\psi \in S(I)$.

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