CHAPTER 4

APPLICATION OF AR (1) WITH CHANGE POINT ON TOTAL POPULATION OF INDIA FROM OFFICIAL STATISTICS,
CENSUS OF INDIA

4.1 INTRODUCTION

In applications, we are frequently faced with time series data which, for a variety of different reasons, have characteristics not compatible with the usual assumption of linearity or / and Gaussian errors – exponential autoregressive model studied by Bell and Smith (1986) are for instance, very powerful in the analysis of such time series. Statistical analysis of several closely related models arising in water quality analysis. Bell and Smith (1986) concerned with the autoregressive scheme $X_i = \beta X_{i-1} + \varepsilon_i$ where $0 < \beta < 1$ and $\varepsilon$’s are i.i.d. and non-negative. The estimation and testing problem was considered by them for three parametric models Gaussian, uniform and exponential. For large series, non normality may not be of importance due to the additivity of filtering processes. However, for small series, the effects may be important. Hence, models other than Gaussian, in particular, the exponential, are studied.

In this chapter, we propose an AR(1) model with one change point, where the error distribution is supposed to be the changing exponential distribution, since, the errors are assumed to be distributed as exponential, we refer it to “AR(1) with exponential
errors”. This AR(1) model can be used as a model for water quality analysis. Let the initial level of pollutant be $X_0$. Let random quantities $\varepsilon_1, \varepsilon_2, \ldots$. Of that pollutant are “dumped” at regular fixed intervals into the relevant body of water and between successive “dumping” a proportion $(1 - \beta)$ of the pollutant $(0 \leq \beta \leq 1)$ is “washed away”. If $X_i$ is level of pollutant at time $t$ then $X_i = \beta_i X_{i-1} + \varepsilon_i$, where $\varepsilon_i$ are assumed to have an exponential distribution. Now, it may happen at some unknown time $m$ that the random quantity $\varepsilon_i$ are “dumped” in to the relevant body of water changes the mean value $\theta$ from one value to another and the proportion $\beta$ of the pollutant that is not “washed away” is also changed. Then we are often interested in estimation of the change point parameter $m$, which indexes when or where the unknown change occurred.

In section 4.2, we have developed a change point model related to AR(1) with exponential. In section 4.3, we obtained posterior densities of $\beta_1, \beta_2, \theta_1, \theta_2$ and $m$ for this model. We derive Bayes estimators of $\beta_1, \beta_2, \theta_1, \theta_2$ and $m$ under symmetric loss functions and asymmetric loss functions in section 4.4 and 4.5 respectively. In section 4.6, we have presented a numerical study to illustrate the above technique on generated observations and census data. In section 4.7, we studied the sensitivity of Bayes estimates with respect to change in prior of the parameters. We also studied of the effects of different loss functions on Bayes estimates. In section 4.8, we have done simulation study by generating 10,000 different random samples. With section 4.9 we concluded the chapter.
4.2 PROPOSED AR(1) MODEL

Let \{ \varepsilon_n, n \geq 1 \} be a random sequence having exponential distribution viz.,

The p. d. f. of the distribution is given by,

\[
f ( \varepsilon | \theta_1 ) = \frac{1}{\theta_1} e^{-\varepsilon/\theta_1}, \quad i = 1, 2, 3, \ldots, m
\]

\[
f ( \varepsilon | \theta_2 ) = \frac{1}{\theta_2} e^{-\varepsilon/\theta_2}, \quad i = m + 1, \ldots, n.
\] (4.1)

Further, let \{ X_n, n \geq 1 \} be a sequence of random variables defined as,

\[
X_i = \begin{cases} 
\beta_1 X_{i-1} + \varepsilon_i, & i = 1, 2, 3, \ldots, m \\
\beta_2 X_{i-1} + \varepsilon_i, & i = m + 1, \ldots, n
\end{cases}
\] (4.2)

with \( X_0 \) fixed constant and \( 0 \leq \beta_1, \beta_2 < 1 \). (4.2) is the first order auto regressive process, AR(1), \( m \) is unknown change point in the sequence which is to be estimated.

For \( \beta_1 = \beta_2 = \beta \), \( m = n \), the model (4.2) reduces to the model studied by Bell and Smith (1986).

Since, the AR(1) process defined in (4.2) is Markovian, the joint p. d. f. of \( X_0, X_1, \ldots, X_n \) is given by,

\[
f ( X_0, X_1, \ldots, X_n ) =
\]

\[
\left( \frac{1}{\theta_1} \right)^m e^{-s_m + \beta_1 s_m}/\theta_1 \left( \frac{1}{\theta_2} \right)^{n-m} e^{-\beta_2 (s_n - s_m)/\theta_2} e^{-(s_n-s_m)/\theta_2}.
\] (4.3)

\[
S_k = \sum_{i=1}^{k} X_i; \quad S_k^* = \sum_{i=1}^{k} X_{i-1} \quad 0 \leq \beta_1, \beta_2 < 1
\]

\[
X_i > \beta_1, X_{i-1}, i = 1, 2, \ldots, m, \quad X_i > \beta_2, X_{i-1}, i = m+1, \ldots, n.
\] (4.4)
Then, the likelihood function given the sample information $x=(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n)$ is

$$L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) = \frac{1}{\theta_1^m} e^{-A/\theta_1} (\theta_2)^{-n-m} e^{-B/\theta_2}.$$

$$\theta_1, \theta_2 > 0, x_0 > 0. \quad (4.5)$$

Where,

$$A = S_m - \beta_1 S^*_m$$

$$B = S_n - S_m - \beta_2 (S^*_n - S^*_m) \quad (4.6)$$

4.3 POSTERIOR DENSITIES

The ML methods, as well as other classical approaches are based only on the empirical information provided by the data. However, when there is some technical knowledge on the parameters of the distribution available, a Bayes procedure seems to an attractive inferential method. The Bayes procedure is based on a posterior density, say, $g(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X)$, which is proportional to the product of the likelihood function $L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X)$, with a prior joint density, say, $g(\beta_1, \beta_2, \theta_1, \theta_2, m)$ representing uncertainty on the parameters values.

$$g(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) =$$

$$\frac{L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) \cdot g(\beta_1, \beta_2, \theta_1, \theta_2, m)}{\sum_{m=1}^{n-1} f_{\beta_1} f_{\beta_2} f_{\theta_1} f_{\theta_2} L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) \cdot g(\beta_1, \beta_2, \theta_1, \theta_2, m) \cdot d\beta_1 d\beta_2 d\theta_1 d\theta_2}.$$
4.3.1 Using Informative priors for $\beta_1$, $\beta_2$, $\theta_1$ and $\theta_2$

In this section, we derive posterior density of change point $m$, $\beta_1$, $\beta_2$, $\theta_1$ and $\theta_2$ of the model explained in (4.2) under informative beta prior.

As in Broemeling et al.(1987), we suppose the marginal prior distribution of $m$ to be discrete uniform over the set $\{1, 2, \ldots, n-1\}$.

We assume prior distributions for $\beta_1$ and $\beta_2$ with respective means $\mu_1$ and $\mu_2$ and standard deviation $\sigma$ to be beta distributions.

\[
g_1(\beta_1) = \int \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} | \beta (a_1, b_1), \quad 0 < \beta_1 < 1, a_1 > 0, b_1 > 0
\]

\[
g_1(\beta_2) = \int \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} | \beta (a_2, b_2), \quad 0 < \beta_2 < 1, a_2 > 0, b_2 > 0
\]

\[
g(m) = \frac{1}{n-1}
\]

If the prior information is given in terms of $\mu_1$, $\mu_2$ and $\sigma$, then the beta parameters can be obtained by solving –

\[
a_i = \sigma^{-1} \int (1 - \mu_i) \mu_i^2 - \mu_i \sigma, \quad i = 1, 2.
\]

\[
b_i = \mu_i^{-1}(1 - \mu_i) a_i, \quad i = 1, 2. \tag{4.7}
\]

We also assume the joint prior density of $\theta_1$ and $\theta_2$ be

\[
G(\theta_1, \theta_2) \propto \frac{1}{\theta_1 \theta_2} \tag{4.8}
\]
The joint prior density of $\beta_1$, $\beta_2$, $\theta_1$, $\theta_2$ and $m$ say, $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m)$ is

$$g_1(\beta_1, \beta_2, \theta_1, \theta_2, m) = k_1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1}$$

(4.9)

where, $k_1 = \frac{1}{\beta(a_1, b_1) \beta(a_2, b_2)(n-1)}$

(4.10)

Using the likelihood function (4.5) with the joint prior density (4.8), the joint posterior density of $\beta_1$, $\beta_2$, $\theta_1$, $\theta_2$ and $m$ say, $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m \mid x)$, is

$$g_1(\beta_1, \beta_2, \theta_1, \theta_2, m \mid x) = \frac{L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid x) \cdot g_1(\beta_1, \beta_2, \theta_1, \theta_2, m \mid x)}{h_1(x)}$$

$$h_1(x)$$ is the marginal density of $X$ given by,

$$h_1(x) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid x) \cdot g_1(\beta_1, \beta_2, \theta_1, \theta_2, m) \, d\theta_1 \, d\theta_2 \, d\beta_1 \, d\beta_2$$

$$= k_1 \sum_{m=1}^{n-1} \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \int_0^\infty \theta_1^{-(m+1)} e^{-A/\theta_1} \, d\theta_1 \, d\beta_1$$

$$= k_1 \sum_{m=1}^{n-1} T_1(m)$$

(4.11)
where, \( k_1 \) is as given in (4.10).

\[
T_1(m) = \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \int_0^\infty \theta_1^{-(m+1)} e^{-A/\theta_1} \, d\theta_1 \, d\beta_1
\]

\[
\int_0^1 \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} \int_0^\infty \theta_2^{-(n-m+1)} e^{-B/\theta_2} \, d\theta_2 \, d\beta_2. \tag{4.13}
\]

Writing,

\[
l_1(m) = \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \int_0^\infty \theta_1^{-(m+1)} e^{-A/\theta_1} \, d\theta_1 \, d\beta_1
\]

\[
= \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \frac{\Gamma_m}{A^m} \, d\beta_1
\]

\[
= \Gamma m \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \frac{1}{(s_m^{*} - s_m^{*})^m} \, d\beta_1
\]

\[
= \left\{ \left( \frac{1}{a_1} \right) (S_m - S_m^{*})^{-m} \left( 1 - \frac{s_m^{*}}{s_m} \right)^m \text{Appel}_1 F_1 \left[ a_1, 1 - b_1, m, 1 + a_1, 1, \frac{s_m^{*}}{s_m} \right] \right\}. \tag{4.14}
\]

And,

\[
l_2(m) = \int_0^1 \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} \, d\beta_2 \int_0^\infty \theta_2^{-(n-m+1)} e^{-B/\theta_2} \, d\theta_2
\]

\[
= \int_0^1 \Gamma(n-m) B^{n-m} \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} \, d\beta_1,
\]

\[
= \int_0^1 \frac{\Gamma(n-m)}{(s_n - s_m^*) - \beta_2(s_n^* - s_m^*)}^{n-m} \beta_1^{a_1-1} (1 - \beta_2)^{b_2-1} \, d\beta_1,
\]

\[
= \Gamma(n-m) \int_0^1 \frac{\beta_2^{a_2-1} (1 - \beta_2)^{b_2-1}}{(s_n - s_m^*) - \beta_2(s_n^* - s_m^*)}^{n-m} \, d\beta_1,
\]

\[
= \left( \frac{1}{a_2} \right) \left[ (S_n - S_m) - (S_n^* - S_m^*) \right]^{-(n-m)} \left( 1 - \frac{s_n^* - s_m^*}{s_n - s_m} \right)^{(n-m)}
\]

\[
\text{Appel}_1 F_1 \left[ a_2, 1 - b_2, n - m, 1 + a_2, 1, \frac{s_n^* - s_m^*}{s_n - s_m} \right]. \tag{4.15}
\]
So using this results in (4.13), it reduce to

\[ T_1(m) = I_1(m) I_2(m) \]

\[ = \Gamma(m) \Gamma(n - m) \left\{ \left( \frac{1}{a_1} \right) \left( S_m - S_m^* \right)^{-m} \left( 1 - \frac{s_m^*}{s_m} \right)^m \right\} \]

\[ = Appel_{1F1}[a_1, 1 - b_1, m, 1 + a_1, 1, \frac{s_m^*}{s_m}] \}

\[ \left\{ \left( \frac{1}{a_2} \right) \left[ (S_n - S_m) - (S_n^* - S_m^*) \right]^{(n-m)} \left( 1 - \frac{s_n^* - s_m^*}{s_n - s_m} \right)^{(n-m)} \right\} \]

\[ Appel_{1F1}[a_2, 1 - b_2, n - m, 1 + a_2, 1, \frac{s_n^* - s_m^*}{s_n - s_m}] \} \}

Where,

\[ _1F_1 \text{is the Appel hypergeometric function and it has the following integral representation,} \]

\[ \int_0^t \frac{x^a \left( \frac{1-x}{c} \right)^{b-1}}{(d-x)^a} \, dx = \frac{1}{c^a} \left[ (d-t)^a (1 - \frac{t}{d})^a \right] \]

\[ Appel_{1F1}(a, 1 - b, \nu, 1 + a, \frac{t}{c}, \frac{t}{a}) \]

(4.17)

\[ Appel_{1F1}(a, b_1, b_2, c, x, y) \text{ is Appel Hypergeometric function (Appell (1925)) defined as} \]

\[ Appel_{1F1}(a, b_1, b_2, c, x, y) = \frac{f_c}{f_a f_{(c-a)}} \int_0^1 u^{a-1} (1 - u)^{c-a-1} (1 - ux)^{-b_1} (1 - uy)^{-b_2} \, du, \]

Real[a] > 0, Real (c-a) > 0,
Integrating $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X)$ on $(\beta_1, \beta_2)$ and $(\theta_1, \theta_2)$, leads to the posterior distribution of change point $m$.

Marginal posterior density of change point $m$ is given by

$$g_1(m \mid X) = \frac{T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}$$  \hspace{1cm} (4.18)

$T_1(m)$ is as given in (4.13).

Marginal posterior density of $\beta_1, \beta_2$ is obtained from the joint posterior density of $\theta_1, \theta_2, \beta_1, \beta_2$ and $m$ given in (4.11) by integrating $\theta_1$ and $\theta_2$ and summing over $m$.

Hence, we get,

$$g_1(\beta_1, \beta_2 \mid X) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty g_1(\theta_1, \theta_2, \beta_1, \beta_2, m \mid X) \, d\theta_1 \, d\theta_2$$

$$= k_1 \sum_{m=1}^{n-1} \beta_1^{a_1-1}(1 - \beta_1)^{b_1-1} \beta_2^{a_2-1}(1 - \beta_2)^{b_2-1}$$

$$\int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} \, d\theta_1 \int_0^\infty \theta_2^{(n-m+1)} e^{-B/\theta_2} \, d\theta_2 / h_1(X)$$  \hspace{1cm} (4.19)

$$\int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} \, d\theta_1,$$

$$= \frac{\Gamma m}{A^m}$$  \hspace{1cm} (4.20)

$$\int_0^\infty \theta_2^{(n-m+1)} e^{-B/\theta_2} \, d\theta_2,$$

$$= \frac{\Gamma(n-m)}{B^{n-m}}$$  \hspace{1cm} (4.21)
So using $I_3(m)$ and $I_4(m)$ results in (4.19), it reduce to

$$g_1(\beta_1, \beta_2 \mid x) =$$

$$k_1 \sum_{m=1}^{n-1} \beta_1^{a_1-1} \beta_2^{a_2-1} (1 - \beta_1)^{b_1-1} (1 - \beta_2)^{b_2-1} \frac{\Gamma m}{A^m} \frac{\Gamma (n-m)}{B^{n-m}} / h_1(X). \quad (4.22)$$

Marginal posterior density of $\beta_1$ is obtained from the (4.19), by integrating out $\beta_2$. So we get,

$$g_1(\beta_1 \mid X) = \int_0^1 g(\beta_1, \beta_2 \mid x) \, d\beta_2$$

$$g_1(\beta_1 \mid X) = k_1 \sum_{m=1}^{n-1} \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \frac{\Gamma m}{A^m} \int_0^1 \frac{\beta_2^{a_2-1} (1 - \beta_2)^{b_2-1}}{B^{n-m}} \, d\beta_2 / h_1(X),$$

$$= k_1 \sum_{m=1}^{n-1} \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \frac{\Gamma m}{A^m} \left[ \frac{1}{a_2} \right] ((S_n - S_m) - (S_n^* - S_m^*))^{-(n-m)} (1 - \frac{s_n^* - s_m^*}{s_n - s_m})^{(n-m)} \text{Appel } _1F_1 \left[ a_2, 1 - b_2, n - m, 1 + a_2, 1, \frac{s_n^* - s_m^*}{s_n - s_m} \right] / h_1(X). \quad (4.23)$$

Where,

$$\text{Appel } _1F_1 \left[ a_2, 1 - b_2, n - m, 1 + a_2, 1, \frac{s_n^* - s_m^*}{s_n - s_m} \right]$$

is an apple function defined in (4.17) and $h_1(X)$ is as given in (4.12).

Marginal posterior density of $\beta_2$ is obtained from the (4.19), by integrating out $\beta_1$. So we get,

$$g_1(\beta_2 \mid x) = \int_0^1 g(\beta_1, \beta_2 \mid x) \, d\beta_1$$

$$= k_1 \sum_{m=1}^{n-1} \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} \frac{\Gamma m}{B^{n-m}} \int_0^1 \frac{\beta_1^{a_1-1} (1 - \beta_1)^{b_1-1}}{A^m} \, d\beta_1 / h_1(X),$$
\[= k_1 \sum_{m=1}^{n-1} \beta_2^{a_2-1}(1 - \beta_2)^{b_2-1} \frac{\Gamma(m) \Gamma(n-m)}{B^{n-m}} \]

\[\left\{ \left( \frac{1}{a_1} \right) \left( S_m - S_m^* \right)^{-m} \left( 1 - \frac{S_m^*}{S_m} \right)^m \text{Appel} \quad _1F_1 \left[ a_1, 1 - b_1, m, 1 + a_1, 1, \frac{S_m^*}{S_m} \right] \right\} \]

/ \ h_1(X). \quad (4.24)

Where,

\[\text{Appel} \quad _1F_1 \left[ a_1, 1 - b_1, m, 1 + a_1, 1, \frac{S_m^*}{S_m} \right] \] is an appel function defined in (4.17) and \( h_1(X) \) is as given in (4.12).

Marginal posterior density of \( \theta_1 \) and \( \theta_2 \), say \( g_1(\theta_1 | \mathbf{X}) \) and \( g_1(\theta_2 | \mathbf{X}) \) are obtained from joint posterior density \( g_1(\beta_1, \beta_2, \theta_1, \theta_2, m | \mathbf{x}) \) given in (4.11) by integrating,

we get

\[g_1(\theta_1 | \mathbf{X}) = \int_0^1 \int_0^1 \int_0^\infty g_1(\beta_1, \beta_2, \theta_1, \theta_2, m | \mathbf{x}) \, d\beta_1 \, d\beta_2 \, d\theta_2\]

\[= k_1 \sum_{m=1}^{n-1} \theta_1^{(m+1)} \Gamma(n - m) \int_0^1 \beta_1^{a_1-1}(1 - \beta_1)^{b_1-1} \, d\beta_1 \int_0^1 \beta_2^{a_2-1}(1 - \beta_2)^{b_2-1} \, d\beta_2 \, h_1(X).\]

\[= k_1 \sum_{m=1}^{n-1} \theta_1^{(m+1)} \int_0^1 \frac{\beta_1^{a_1-1}(1 - \beta_1)^{b_1-1}}{e^{-A/A_1}} \, I_2(m) \, h_1(X). \quad (4.29)

Where,

\( I_2(m) \) is as given in (4.15).
And

Marginal posterior density of $\theta$ and $\theta_2$, say $g_1(\theta_2 | X)$ are obtained from joint posterior density $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m | X)$ by integrating, we get

$$g_1(\theta_2 | X) = \int_0^1 \int_0^1 \int_0^\infty g_1(\beta_1, \beta_2, \theta_1, \theta_2, m | x) \, d\beta_1 \, d\beta_2 \, d\theta_1$$

$$= k_1 \sum_{m=1}^{n-1} \theta_2^{n-m+1} (n-m+1) \Gamma m \int_0^1 \frac{\beta_1^{\alpha_1-1} (1-\beta_1)^{\beta_1-1}}{A^m} \, d\beta_1 \int_0^1 \frac{\beta_2^{\alpha_2-1} (1-\beta_2)^{\beta_2-1}}{e^{-B/\theta_2}} \, d\beta_2 / h_1(X),$$

$$= k_1 \sum_{m=1}^{n-1} \theta_2^{n-m+1} \int_0^1 \frac{\beta_2^{\alpha_2-1} (1-\beta_2)^{\beta_2-1}}{e^{-B/\theta_2}} \, l_1(m) / h_1(X). \quad (4.30)$$

$l_1(m)$ is as given in (4.14) and $h_1(X)$ is as given in (4.12).

### 4.3.2 Using Non Informative priors for $\beta_1, \beta_2, \theta_1$ and $\theta_2$

A non-informative prior is a prior that reflects indifference to all values of the parameter, and adds no information to that contained in the empirical data. Thus, a Bayes inference based upon non-informative prior has generally a theoretical interest only, since, from an engineering view point, the Bayes approach is very attractive for it allows incorporating expert opinion or technical knowledge in the estimation procedure. However, such a Bayes inference acquires large interest in solving prediction problems when it is extremely difficult, if at all possible, to find a classical solution for the prediction problem, because classical prediction intervals are numerically equal to the Bayes ones based on the non-informative prior density. Hence, the Bayes approach based
on prior ignorance can be viewed as mathematical method for obtaining classical prediction intervals.

If no further information on $\beta_1$, $\beta_2$, $\theta_1$ and $\theta_2$ are available, and if they are assumed to be a prior independent random variables, then a discrete uniform for $m$ as explained in section 4.3.1 and the non-informative density for $\theta_1$ and $\theta_2$ can be used. The non-informative prior is a density, which adds no information to that contained in the empirical data.

In this section, we derive posterior density of $m$, $\beta_1$, $\beta_2$, $\theta_1$ and $\theta_2$ of model explained in 4.2 under non-informative beta prior.

We consider non informative priors of $\beta_1$, and $\beta_2$ to be

$$g ( \beta_1 ) = 1, \ 0 < \beta_1 < 1$$

$$g ( \beta_2 ) = 1, \ 0 < \beta_2 < 1$$

$$g ( m ) = \frac{1}{(n-1)}, \ m=1,2,\ldots,n-1.$$ 

Let the joint prior density of $\theta_1$ and $\theta_2$ be

$$g ( \theta_1, \theta_2 ) \propto \frac{1}{\theta_1 \theta_2} \quad \text{(4.31)}$$

Then, joint prior density of $\beta_1$, $\beta_2$, $\theta_1$, $\theta_2$ and $m$ is,

$$g_2(\beta_1, \beta_2, \theta_1, \theta_2, m) \propto \frac{1}{\theta_1 \theta_2 (n-1)} \quad \text{(4.32)}$$
Joint posterior density of $\theta_1, \theta_2, \beta_1, \beta_2$ and $m$ say, $g_2(\beta_1, \beta_2, \theta_1, \theta_2, m | \mathbf{X})$ is obtained using joint prior density $g_2(\beta_1, \beta_2, \theta_1, \theta_2, m)$ and likelihood function given in

$$g_2(\beta_1, \beta_2, \theta_1, \theta_2, m | \mathbf{X}) =$$

$$= \frac{1}{n-1} \theta_1^{(m+1)} e^{-s_m + \beta_1 s_m / \theta_1} \theta_2^{(n-m+1)} e^{-s_n + \beta_2 (s_n - s_m) / \theta_2} / h_2(\mathbf{X}) \tag{4.33}$$

Where,

$h_2(\mathbf{X})$ is the marginal density of $\mathbf{X}$ given by

$$h_2(\mathbf{X}) =$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 L(\beta_1, \beta_2, \theta_1, \theta_2, m | \mathbf{X}) \cdot g_2(\beta_1, \beta_2, \theta_1, \theta_2, m) d\theta_1 d\theta_2 d\beta_1 d\beta_2$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^1 \left[ \int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} d\theta_1 \right] d\beta_1 \left[ \int_0^\infty \theta_2^{(n-m+1)} e^{-B/\theta_2} d\theta_2 \right] d\beta_2$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} T_2(m) \tag{4.34}$$

Where,

$$T_2(m) = \int_0^1 \frac{r^m}{(s_m - s_m^*)^m} d\beta_1 \int_0^1 \frac{r(n-m)}{(s_n - s_m - \beta_2 s_n^* - s_m^*)^n-m} d\beta_2 \tag{4.35}$$

Writing,

$$l_1^*(m) = \sum_{m=1}^{n-1} \int_0^1 \left[ \int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} d\theta_1 \right] d\beta_1$$

$$= \Gamma m \int_0^1 \frac{1}{A^m} d\beta_1$$

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\[ (4.36) \]

\[ I_2^*(m) = \left[ \int_0^{\infty} \theta_2^{-(n-m+1)} e^{-\beta_2/\theta_2} d\theta_2 \right] d\beta_2 \]

\[ = \Gamma(n - m) \int_0^1 \frac{1}{(s_n - s_m)^{n-m}} \quad (4.37) \]

So using this results in (4.35), it reduce to

\[ T_2(m) = I_1^*(m) I_2^*(m) \quad (4.38) \]

Marginal posterior density of change point \( m \) is given by

\[ g_2(m \mid X) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 g_2(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) d\theta_1 d\theta_2 d\beta_1 d\beta_2 \]

\[ = \frac{1}{(n-1)} T_2(m) / \sum_{m=1}^{n-1} T_2(m) \quad (4.39) \]

Where,

\( T_2(m) \) is as given in (4.35).
Joint posterior density of $\beta_1, \beta_2$ is obtained from the joint posterior density of $\theta_1, \theta_2, \beta_1, \beta_2$ and $m$ given in (4.33) by integrating $\theta_1$ and $\theta_2$ and summing over $m$. Hence we get,

$$g_2(\beta_1, \beta_2 / x) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty g_1(\theta_1, \theta_2, \beta_1, \beta_2, m | x) \, d\theta_1 \, d\theta_2$$

$$= \frac{1}{(n-1)} \sum_{m=1}^{n-1} \left[ \int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} \, d\theta_1 \right] \left[ \int_0^\infty \theta_2^{(n-m+1)} e^{-B/\theta_2} \, d\theta_2 \right]$$

$$= \frac{1}{(n-1)} \sum_{m=1}^{n-1} \int_0^\infty \theta_1^{(m+1)} e^{-A/\theta_1} \, d\theta_1 \int_0^\infty \theta_2^{(n-m+1)} e^{-B/\theta_2} \, d\theta_2$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{\Gamma(m+1)}{\Gamma(n-m)} \left[ \frac{1}{(B)^{n-m}} \right]$$

The marginal posterior density of $\beta_1$ is obtained using (4.33) and integrating out $\theta_1, \theta_2$ and $\beta_2$, we get

$$g_2(\beta_1 | X) = \sum_{m=1}^{n-1} \int_0^1 \int_0^\infty g_2(\beta_1, \beta_2, \theta_1, \theta_2, m | X) \, d\theta_1 \, d\theta_2 \, d\beta_2$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{\Gamma(m+1)}{\Gamma(n-m)} \left[ \frac{1}{(B)^{n-m}} \right]$$

The marginal posterior density of $\beta_2$ is obtained using (4.33) and integrating out $\theta_1, \theta_2$ and $\beta_1$, we get

$$g_2(\beta_2 | X) = \sum_{m=1}^{n-1} \int_0^1 \int_0^\infty g_2(\beta_1, \beta_2, \theta_1, \theta_2, m | X) \, d\theta_1 \, d\theta_2 \, d\beta_1$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{\Gamma(m+1)}{\Gamma(n-m)} \left[ \frac{1}{(B)^{n-m}} \right]$$
The marginal posterior density of $\theta_1$ is obtained using (4.33) and integrating out $\beta_1, \beta_2, \theta_2$, we get

\[ g_2(\theta_1 | X) = \sum_{m=1}^{n-1} \int_0^1 \int_0^\infty g_2(\beta_1, \beta_2, \theta_1, \theta_2, m | X) \cdot d\theta_2 d\beta_1 d\beta_2 \]

\[ = \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^1 \bar{\theta}_1^{(m+1)} e^{-s/m} d\beta_1 \int_0^1 \frac{r(n-m)}{((s_n-s_m) - \beta_2(s_n^* - s_m^*))^{n-m}} d\beta_2 / h_2(X), \]

\[ = \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^1 \bar{\theta}_1^{(m+1)} e^{-s_m/\beta_1 s_m^*/\theta_1} d\beta_1 \int_0^1 \frac{r(n-m)}{((s_n-s_m) - \beta_2(s_n^* - s_m^*))^{n-m}} d\beta_2 / h_2(X), \]

Using (4.37) result in above, we get

\[ g_2(\theta_1 | X) = \frac{1}{n-1} \sum_{m=1}^{n-1} (\theta_1)^{-(m+1)} \frac{\beta_1 e^{-s_m/\theta_1}}{s_m^*} \left( e^{s_m/\theta_1} - 1 \right) I_2^*(m) / h_2(X), \]

\[ = \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{\theta_1^{-m} e^{-s_m/\theta_1}}{s_m^*} \left( e^{s_m/\theta_1} - 1 \right) I_2^*(m) / h_2(X). \]

$I_2^*(m)$ is as given in (4.37).
Marginal posterior density of $\theta_2$ is obtained using (4.33) and integrating out $\beta_1, \beta_2$ and $\theta_1$, we get

$$g_2(\theta_2 \mid X) = \sum_{m=1}^{n-1} \int_0^1 \int_0^{\infty} g_2(\beta_1, \beta_2, \theta_1, \theta_2, m \mid X) \, d\theta_1 d\beta_1 d\beta_2$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^1 \theta_2^{(n-m+1)} e^{-\theta_2} \, d\beta_2 \int_0^1 \frac{r(m)}{(s_m - \beta_1 s_m^*)^{m}} d\beta_1 / h_2(X),$$

$$= \sum_{m=1}^{n-1} \int_0^1 \theta_2^{(n-m+1)} e^{-s_n + s_m + \beta_2(s_n^* - s_m^*)/\theta_2} \, d\beta_2 \int_0^1 \frac{r(m)}{(s_m - \beta_1 s_m^*)^{m}} d\beta_1 / h_2(X),$$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \theta_2^{(n-m+1)} e^{-\frac{(s_n - s_m)}{\theta_2}} \left[ e^{-\frac{(s_n^* - s_m^*)}{\theta_2}} - 1 \right] l_1^*(m) / h_2(X). \quad (4.45)$$

$l_1^*(m)$ is as given in (4.36).

### 4.4 BAYES ESTIMATES UNDER SYMMETRIC LOSS FUNCTIONS

In this section, we derive Bayes estimators of change point $m$, $\beta_1, \beta_2, \theta_1$ and $\theta_2$ under symmetric loss function using both prior considerations explained in section 4.3.1 and 4.3.2.

#### 4.4.1 Using Informative priors for $\beta_1, \beta_2, \theta_1$ and $\theta_2$

The Bayes estimate of a generic parameter(or function thereof) $\alpha$ based on a Squared Error Loss (SEL) function $L_1(\alpha, d) = (\alpha - d)^2$, where $d$ is decision rule to estimate $\alpha$, is the posterior mean. As a consequence, the SEL function relative to an integer parameter,
\[ L'(m, v) \propto (m - v)^2, \quad m, v = 0, 1, 2, \ldots \]

Hence, the Bayesian estimate of an integer-valued parameter under the SEL function \( L'(m, v) \) is no longer the posterior mean and can be obtained by numerically minimizing the corresponding posterior loss. Generally, such a Bayesian estimate is equal to the nearest integer value to the posterior mean. So, we tell the nearest value to the posterior mean as Bayes Estimate.

The Bayes estimate of unknown change point \( m \), using informative priors, discussed in the section 4.3.1, under SEL is given by,

\[
m^* = \sum_{m=1}^{n-1} m T_1(m) \mid \sum_{m=1}^{n-1} T_1(m)
\]

(4.46)

Where,

\( T_1(m) \) is as given in (4.13).

**4.4.2 Using Non-Informative priors for \( \beta_1, \beta_2, \theta_1 \) and \( \theta_2 \)**

The Bayes estimate, posterior mean, of unknown change point \( m \), using informative prior considerations, discussed in the section (4.3.2), under SEL is given by

\[
m^{**} = \sum_{m=1}^{n-1} m T_2(m) \mid \sum_{m=1}^{n-1} T_2(m)
\]

(4.47)

Where,

\( T_2(m) \) has same meaning as in (4.35) respectively.

Other Bayes estimators of \( \alpha \) based on the loss functions
\[ L_2 (\alpha, d) = |\alpha - d| \]

\[ L_3 (\alpha, d) = \begin{cases} 0, & \text{if } |\alpha - d| < \epsilon, \epsilon > 0 \\ 1, & \text{otherwise} \end{cases} \]

is the posterior median and posterior mode, respectively.

### 4.5 BAYES ESTIMATES UNDER ASYMMETRIC LOSS FUNCTIONS

The Loss function \( L(\alpha, d) \) provides a measure of the financial consequences arising from a wrong decision rule \( d \) to estimate an unknown quantity \( \alpha \). The choice of the appropriate loss function depends on financial considerations only, and is independent from the estimation procedure used. The use of symmetric loss function was found to be generally inappropriate, since for example, an over estimation of the reliability function is usually much more serious than an under estimation.

A useful asymmetric loss, known as the Linex loss function was introduced by Varian (1975). Under the assumption that the minimal loss at \( d \), the Linex loss function can be expressed as,

\[ L_4 (\alpha, d) = \exp [q_1 (d - \alpha)] - q_1 (d - \alpha) - 1, \ q_1 \neq 0. \]

The sign of the shape parameter \( q_1 \) reflects the deviation of the asymmetry, \( q_1 > 0 \) if over estimation is more serious than under estimation, and vice-versa, and the magnitude of \( q_1 \) reflects the degree of asymmetry.
Minimizing expected loss function $E_m [L_4 (m, d)]$ and using posterior distribution (4.18) and (4.39), we get the estimate $m$ by means of the nearest integer value to (4.48) and (4.49), say $m^*_L$, as under. We get the Bayes estimators of $m$ using Linex loss function respectively as,

$$m^*_L = - \frac{1}{q_1} \ln \left[ \sum_{m=1}^{n-1} e^{-a_1 m T_1 (m) / \sum_{m=1}^{n-1} T_1 (m)} \right],$$  \hspace{1cm} (4.48)

$$m^{**}_L = - \frac{1}{q_1} \ln \left[ \sum_{m=1}^{n-1} e^{-a_1 m T_2 (m) / \sum_{m=1}^{n-1} T_2 (m)} \right],$$  \hspace{1cm} (4.49)

Where, $T_1 (m)$ and $T_2 (m)$ are as given in (4.13) and (4.35).

Minimizing expected loss function $E_m [L_4 (\beta_1, d)]$ and using posterior distributions (4.23) and (4.24) respectively, we get the Bayes estimators of $\beta_1$ and $\beta_2$ using Linex loss function as,

$$\beta^*_1 = - \frac{1}{q_1} \ln \left[ E e^{-q_1 \beta_1} \right] = - \frac{1}{q_1} \ln \left[ \sum_{m=1}^{n-1} \int_0^1 \beta_1^{a_2-1} (1-\beta_1)^{b_2-1} e^{-q_1 \beta_1} \frac{d\beta_1}{\Gamma(m)} \int_0^1 \beta_2 \Theta_2^{-(n-m+1)} e^{-B/\Theta_2} d\Theta_2 \right] / h_1 (X),$$

$$= - \frac{1}{q_1} \ln \left[ \sum_{m=1}^{n-1} \int_0^1 \beta_1^{a_2-1} (1-\beta_1)^{b_2-1} e^{-q_1 \beta_1} \frac{d\beta_1}{\Gamma(m)} \int_0^1 \beta_2 \Theta_2^{-(n-m+1)} e^{-B/\Theta_2} d\Theta_2 \right] / h_1 (X),$$

$$= - \frac{1}{q_1} \ln \left[ \sum_{m=1}^{n-1} \int_0^1 \beta_1^{a_2-1} (1-\beta_1)^{b_2-1} e^{-q_1 \beta_1} \frac{d\beta_1}{\Gamma(m)} \int_0^1 \beta_2 \Theta_2^{-(n-m+1)} e^{-B/\Theta_2} d\Theta_2 \right] / h_1 (X),$$  \hspace{1cm} (4.50)

$$\beta^*_2 = - \frac{1}{q_1} \ln \left[ E e^{-q_1 \beta_2} \right]$$
\[= - \frac{1}{q_1} \ln \left( \sum_{m=1}^{n-1} \int_0^1 \frac{\beta_2^{a_2-1}(1-\beta_2)^{b_2-1} e^{-q_1\beta_2}}{\beta^{n-m}} d\beta_2 \right) \Gamma(n - m)\]

\[\int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \int_0^\infty \theta_1^{-(m+1)} e^{-A/\theta_1} d\theta_1 d\beta_1 / h_1(X),\]

\[= - \frac{1}{q_1} \ln \left( \sum_{m=1}^{n-1} \int_0^1 \frac{\beta_2^{a_2-1}(1-\beta_2)^{b_2-1} e^{-q_1\beta_2}}{\beta^{n-m}} d\beta_2 \right) \Gamma(n - m) I_1(m) / h_1(X). \quad (4.51)\]

\(I_1(m)\) and \(I_2(m)\) are same as (4.14) and (4.15) respectively.

Another loss function, called General Entropy loss function (GEL), proposed by Calabria and Pulcini (1996) is given by,

\[L_5(\alpha, d) = \frac{d}{\alpha}^{q_2} - q_2 I_1(d / \alpha) - 1,\]

Whose minimum occurs at \(d = \alpha\), minimizing expectation \(E[L_5(m, d)]\) and using posterior density \(g_i(m \mid x), i = 1,2.\)

### 4.5.1 Using Informative priors for \(\beta_1, \beta_2, \theta_1\) and \(\theta_2\)

We get the estimate \(m\) by means of the nearest integer value to (4.52), say \(m_{iE}^*\), as under. We get the Bayes estimates \(m_{iE}^*, i=1,2\) of \(m\) using General Entropy loss function as

\[m_{iE}^* = \left[ E_1[m^{-q_3}] \right]^{-1/q_3} = \left[ \sum_{m=1}^{n-1} m^{-q_3} T_1(m) / \sum_{m=1}^{n-1} T_1(m) \right]^{-1/q_3}, \quad (4.52)\]

Where,

\(T_1(m)\) is as given in (4.18).
Putting \( q_3 = -1 \) in (4.52), we get Bayes estimate, posterior mean of \( m \).

Minimizing expected loss function \( E[L_5(\beta_i, d)] \) and using posterior distributions (4.23) and (4.24), we get the Bayes estimates of \( \beta_i \) using informative priors, under General Entropy loss function respectively as,

\[
\beta_{1E}^* = E \left[ \beta_1^{-q_3} \right] = \int_0^1 g_1(\beta_1 | X) \, d\beta_1
\]

\[
E \left[ \beta_1^{-q_3} \right] = \int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \, d\beta_1
\]

\[
\int_0^1 \beta_2^{a_2-1} (1 - \beta_2)^{b_2-1} \int_0^\infty \theta_2^{-(n-m+1)} e^{-B/\theta_2} \, d\theta_2 \, d\beta_2 / h_1(X),
\]

\[
= k_1 \sum_{m=1}^{n-1} \Gamma m \frac{\beta_1^{a_1-q_3-1} (1-\beta_1)^{b_1-1}}{A_m} \, d\beta_1
\]

\[
= \frac{1}{a_1-q_3} \left( (S_m - S_m^*)^{-m} \left( 1 - \frac{s_m^*}{s_m} \right)^m \right) Appel \, _1F_1 \left[ a_1 - q_3, 1 - b_1, m, 1 + a_1 - q_3, 1, \frac{s_m^*}{s_m} \right]
\]

Hence, we get Bayes estimates of \( \beta_1 \) under General Entropy Loss function as,

\[
\beta_{1E}^* = k_1 \{ \sum_{m=1}^{n-1} \Gamma m \Gamma(n - m) \frac{1}{a_1-q_3} \left( (S_m - S_m^*)^{-m} \left( 1 - \frac{s_m^*}{s_m} \right)^m \right) Appel \, _1F_1 \left[ a_1 - q_3, 1 - b_1, m, 1 + a_1 - q_3, 1, \frac{s_m^*}{s_m} \right] \}_2(m) / h_1(X) \}^{-1/q_3} \quad (4.53)
\]
\[
\beta_{2E}^* = E \left[ \beta_2^{-q_3} \right]^{-1/q_3} = \int_0^1 g_1(\beta_2 | X) \, d\beta_2
\]

\[
E \left[ \beta_2^{-q_3} \right] = \]

\[
= k_1 \sum_{m=1}^{n-1} \Gamma(n - m) \int_0^1 \beta_2^{a_2 - q_3 - 1} (1 - \beta_2)^{b_2 - 1} \frac{1}{\beta^{n-m}} \, d\beta_2
\]

\[
\int_0^1 \beta_1^{a_1-1} (1 - \beta_1)^{b_1-1} \int_0^{\infty} \theta_1^{-(m+1)} e^{-\theta_1 / \theta_1} \, d\theta_1 \, d\beta_1 / h_1(X)
\]

\[
\int_0^1 \beta_2^{a_2 - q_3 - 1} (1 - \beta_2)^{b_2 - 1} \frac{1}{\beta^{n-m}} \, d\beta_2
\]

\[
= \frac{1}{a_2 - q_3} \left\{ [(S_n - S_m) - (S_n^* - S_m^*)]^{-(n-m)} \cdot \text{Appel} \, _1F_1 \left[ a_2 - q_3, 1 - b_2, n - m, 1 + a_2 - q_3, 1 - \frac{s_n^* - s_m^*}{s_n - s_m} \right] \right\}.
\]

Hence, we get Bayes estimates of \( \beta_2 \) under General Entropy Loss function as,

\[
\beta_{2E}^* = k_1 \sum_{m=1}^{n-1} \Gamma m \Gamma(n - m) \frac{1}{a_2 - q_3} \left\{ [(S_n - S_m) - (S_n^* - S_m^*)]^{-(n-m)} \cdot \text{Appel} \, _1F_1 \left[ a_2 - q_3, 1 - b_2, n - m, 1 + a_2 - q_3, 1 - \frac{s_n^* - s_m^*}{s_n - s_m} \right] \right\} l_1(m) / h_1(X)^{-1/q_3}
\]

(4.54)
Minimizing expected loss function \( E[L_5(\theta_1, d)] \) and using posterior distributions (4.29) and (4.30), We get the Bayes estimate \( \theta_{1E} \) and \( \theta_{2E} \) of \( \theta_1 \) using informative priors, under General Entropy Loss function as,

\[
\theta_{1E}^* = E \left[ \frac{1}{q_3} \theta_1^{-q_3} \right] = \left[ \int_0^{\infty} \theta_1^{-q_3} g_1(\theta_1 | X) \ d\theta_1 \right]^{-\frac{1}{q_3}}
\]

\[
E \left[ \theta_1^{-q_3} \right] = \frac{\Gamma(m + q_3)}{A^{m+q_3}}
\]

Hence, we get Bayes estimates of \( \theta_1 \) under General Entropy Loss function as,

\[
\theta_{1E} = k_1 \sum_{m=1}^{n-1} \Gamma(m + q_3) \int_0^1 \frac{\beta_1^{a_1-1} (1-\beta_1)^{b_1-1}}{A^{m+q_3}} d\beta_1 l_2(m) / h_1(X) \]

\[
= k_1 \sum_{m=1}^{n-1} \Gamma(m + q_3) \left( \frac{1}{a_1} (S_m - S_m^*)^{-(m+q_3)} \left( 1 - \frac{s_m^*}{S_m} \right)^{(m+q_3)} \right) \text{Appel} _1 F_1 \left[ a_1, 1 - b_1, m + q_3, 1 + a_1, 1, \frac{s_m^*}{S_m} \right] l_2(m) / h_1(X) \right]^{-\frac{1}{q_3}}.
\]

(4.55)
The Bayes estimates of $\theta_2$, using non informative priors, under General Entropy Loss function, is obtained using the marginal posterior density of $\theta_2$, we get

$$\theta_{2E} = E \left[ \theta_2^{q_3} \right]^{-1}$$

$$\theta_{2E}^* = E \left[ \theta_2^{q_3} \right]^{-1} \left[ \int_0^{\infty} \theta_2^{q_3} g_1(\theta_2 | X) \ d\theta_2 \right]^{-1}$$

$$E \left[ \theta_2^{q_3} \right] =$$

$$= \left\{ k_1 \sum_{m=1}^{n-1} \int_0^{\infty} \theta_2^{(n-m+1+q_3)} \int_0^1 \beta_2^{a_2-1}(1-\beta_2)^{b_2-1} e^{-B/\beta_2} d\beta_2 \ d\theta_2 \right\}$$

$$= \int_0^1 \beta_1^{a_1-1}(1-\beta_1)^{b_1-1} \int_0^{\infty} \theta_1^{-(m+1)} e^{-A/\theta_1} d\theta_1 \ d\beta_1 \ / \ h_1(X)$$

$$= \int_0^{\infty} \theta_2^{(n-m+1+q_3)} e^{-B/\beta_2} d\theta_2,$$

$$= \frac{\Gamma(n-m+q_3)}{B^{n-m+q_3}}.$$

Hence, we get Bayes estimates of $\theta_2$ under General Entropy Loss function as,

$$\theta_{2E}^* = k_1 \sum_{m=1}^{n-1} \int_0^1 \beta_2^{a_2-1}(1-\beta_2)^{b_2-1} \frac{\Gamma(n-m+q_3)}{B^{n-m+q_3}} d\beta_2 l_1(m) / h_1(X) \right]^{-1}$$

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= \{k_1 \sum_{m=1}^{n-1} \Gamma(n - m + q_3) \{ \frac{1}{a_2} \left[ (S_n - S_m) - (S^*_n - S^*_m) \right] \}^{-(n-m+q_3)}

\left(1 - \frac{S_n - S_m}{S^*_n - S^*_m}\right)^{(n-m+q_3)}\ Appel \ _1 F_1 \left[ a_2, 1 - b_2, n - m + q_3, 1 + a_2, 1 \frac{S_n - S_m}{S^*_n - S^*_m} \right] I_1(m) /
\ h_1(X) \}^{-\frac{1}{q_3}}. \tag{4.56}

\subsection*{4.5.2 Using Non-Informative priors for \( \beta_1, \beta_2, \theta_1 \) and \( \theta_2 \)}

We get the estimate \( m \) by means of the nearest integer value to (4.51), say \( m^{**}_E \), as under. We get the Bayes estimates \( m^{**}_E, i=1,2 \) of \( m \) using General Entropy loss function as

\[ m^{**}_E = \left[ E_1[m^{-q_3}] \right]^{-1/q_3} = \left[ \sum_{m=1}^{n-1} m^{-q_3} T_2(m) / \sum_{m=1}^{n-1} T_2(m) \right]^{-1/q_3}, \tag{4.57} \]

Where,

\( T_2(m) \) is as given in (4.35).

Putting \( q_3 = -1 \) in (4.51), we get Bayes estimate, posterior mean of \( m \).

Minimizing expected loss function \( E[L_5(\beta_i, d)] \) and using posterior distributions (4.41) and (4.42), We get the Bayes estimate \( \beta^{**}_{iE} \) of \( \beta_i, i = 1,2 \) using non-informative priors, under General Entropy Loss function is given as,

\[ \beta^{**}_{iE} = \left( E (\beta_i^{-q_3}) \right)^{-1/q_3}, i = 1.2. \]

\[ \beta^{**}_{1E} = \left( E (\beta_1^{-q_3}) \right)^{-1/q_3} = \int_0^1 g_2(\beta_1|X) \ d\beta_1 \]
\[ [E(\beta_1^{-q_3})] = \]

\[ = \left( \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma m \int_0^1 \beta_1^{-q_3} \frac{1}{(s_m - \beta_1 s_m^*)^m} d\beta_1 \int_0^\infty \theta_2^{(n-m+1)} e^{-\beta_2 \theta_2} d\theta_2 \right) \frac{1}{h_2(X)} \],

\[ \int_0^1 \beta_1^{-q_3} \frac{1}{(s_m - \beta_1 s_m^*)^m} d\beta_1 \]

\[ = (s_m)^{-m} \frac{1}{2} \int_1 [1-q_3, m, 2 - q_3, s_m^m](1-q_3)^{-1} \]

Hence, we get Bayes estimates of \( \beta_1 \) under General Entropy Loss function as,

\[ \beta_{1E}^{**} = \left( \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma m (s_m)^{-m} \right) \frac{1}{2} \int_1 [1-q_3, m, 2 - q_3, s_m^m](1-q_3)^{-1} \frac{l_2^*(m)}{h_2(X)} \]^{-1/q_3}

(4.58)

\( l_2^*(m) \) is as given in (4.37).

\[ \beta_{2E}^{**} = [E(\beta_2^{-q_3})]^{-1/q_3} = \int_0^1 \beta_2 g_2(\beta_2 | X) d\beta_2 \]

\[ [E(\beta_2^{-q_3})] = \]

\[ \left( \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma (n - m) \right) \frac{1}{2} \int_0^1 \beta_2^{-q_3} \frac{1}{[s_n - \beta_2 (s_n^* - s_m^*)]^{(n-m)}} d\beta_2 \int_0^\infty \theta_2^{(m+1)} e^{-A/\theta_2} d\theta_2 \] \frac{1}{h_2(X)} \]
\[ \int_0^1 \beta_2^{-q_3} \frac{1}{[s_n-s_m-\beta_2(s_n^* - s_m^*)]^{-(n-m)}} \, d\beta_2 \]

\[ = (s_n - s_m)^{-(n-m+1)} \quad {}_2F_1[\ 1 - q_3, n - m + 1, 2 - q_3, s_n^* - s_m^* ](1 - q_3)^{-1} \]

Hence, we get Bayes estimates of \( \beta_2 \) under General Entropy Loss function as,

\[ \beta_{2E}^{**} = \]

\[ \left\{ \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma(n - m) \left( s_n - s_m \right)^{-(n-m)} \quad {}_2F_1[\ 1 - q_3, n - m, 2 - q_3, s_n^* - s_m^* ](1 - q_3)^{-1} / h_2(X) \right\}^{-1/q_3} \]

Minimizing expected loss function \( E[L_5(\theta, d)] \) and using posterior distributions (4.44) and (4.45), We get the Bayes estimate \( \theta_{1E}^* \) and \( \theta_{2E}^* \) of \( \theta_{i}^{**} \) using non-informative priors, under General Entropy Loss function as,

\[ \theta_{1E}^{**} = E \left[ \theta_1^{-q_3} \right]^{-1/q_3} = \left[ \int_0^\infty \theta_1^{-q_3} g_2(\theta_1 \mid X) \, d\theta_1 \right]^{-1/q_3} \]

\[ E \left[ \theta_1^{-q_3} \right] = \left\{ \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^1 \theta_1^{(m+q_3)} \frac{e^{-s_m/\theta_1}}{s_m} \left( e^{s_m/\theta_1} - 1 \right) d\theta_1 \right\} \]

\[ \int_0^1 \theta_1^{(m+q_3)} e^{-s_m/\theta_1} \left( e^{s_m/\theta_1} - 1 \right) d\theta_1 \]

\[ = \left[ \left( -\frac{1}{s_m^*} \right)^{-1+m+q_3} + \left( -\frac{1}{s_m^*-s_m^*} \right)^{-1+m+q_3} \right] \frac{\Gamma(-1+m+q_3)}{s_m^*}. \]
Hence, we get Bayes estimates of $\theta_1$ under General Entropy Loss function as,

$$
\theta_{1E}^* = \left( \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{r(1+m+q_3)}{s_m^*} \right)^{-1/q_3} \left[ \left( \frac{1}{s_m} \right)^{-1+1+m+q_3} + \left( \frac{1}{s_m^*-s_m^*} \right)^{-1+1+m+q_3} \right] l_2^*(m) / h_2(X)^{-1/q_3}.
$$

(4.60)

$s_m - s_m^* > 0, s_m > 0, m + q_3 > 0,$

$$
\theta_{2E}^* = E \left[ \theta_2^{-q_3} \right]^{-1/q_3} = \left[ \int_0^\infty \theta_2^{-q_3} g_2(\theta_2 \mid X) \ d\theta_2 \right]^{-1/q_3}
$$

$$
E \left[ \theta_2^{-q_3} \right] = \frac{1}{n-1} \sum_{m=1}^{n-1} \int_0^\infty \theta_2^{-q_3} \left( \frac{1}{s_m} \right)^{n-m+q_3+1} \left( \frac{s_n-s_m}{\theta_2} \right) \left( \frac{s_n-s_m^*}{s_m^*-s_m^*} \right) \left( \frac{e^{(s_n-s_m)/\theta_2}}{\theta_2^2 e^{(s_n-s_m^*)/\theta_2^*} - 1} \right) \ d\theta_2 l_1^*(m)/h_2(X) \right],
$$

$$
\int_0^\infty \theta_2^{-q_3} \left( \frac{1}{s_m} \right)^{n-m+q_3} \left( \frac{1}{s_m^*} \right)^{n-m+q_3} \left( \frac{e^{(s_n-s_m)/\theta_2}}{\theta_2^2 e^{(s_n-s_m^*)/\theta_2^*} - 1} \right) \ d\theta_2
$$

$$
= \left[ \left( \frac{1}{s_n-s_m} \right)^{-1+1+n-m+q_3} + \left( \frac{1}{s_n-s_m^*-s_m^*} \right)^{-1+1+n-m+q_3} \right] \frac{r(-1+n-m+q_3)}{(s_n^*-s_m^*)}.
$$

Hence, we get Bayes estimates of $\theta_2$ under General Entropy Loss function as,
\[ \theta_{2,k}^* = \]
\[ \left( \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma(m) \frac{r(-1+n-m+q_3)}{(s^*_n - s^*_m)} \left[ - \left( \frac{1}{s_n - s_m} \right)^{-1+n-m+q_3} + \left( \frac{1}{s_n - s_m - (s^*_n - s^*_m)} \right)^{-1+n-m+q_3} \right] I_1'(m)/h_2(X) \right)^{-1/q_3}. \]

\[ s_m - s^*_m > 0, s_m > 0, m + q_3 > 0, \quad (4.61) \]

Where,

\[ h_1(X) \text{ and } h_2(X) \] are same as in (4.12) and (4.34) respectively.

### 4.6 NUMERICAL STUDY

#### 4.6.1 Application of AR (1) model to generated data:

Let us consider AR(1) model as

\[ X_1 = \begin{cases} 
0.1X_{i-1} + \varepsilon_i, & i = 1, 2, \ldots, 12 \\
0.8X_{i-1} + \varepsilon_i, & i = 13, 14, \ldots, 20 
\end{cases} \]

Where, \( \varepsilon_i \)'s are independently distributed exponential distributions given in (4.2) with \( \theta_1 = 0.6, \theta_2 = 1.0 \). We have generated 20 random observations from proposed AR(1) model given in (4.2). The first twelve observations are from exponential distribution with \( \theta_1 = 0.6 \) and next eight are from exponential distribution with \( \theta_2 = 1.0 \). \( \beta_1 \) and \( \beta_2 \) themselves were random observations from beta distributions with prior means \( \mu_1 = 0.1, \mu_2 = 0.8 \) and common standard deviation \( \sigma = 0.1 \) respectively, resulting in \( a_1 = 0.001, b_1 = 0.09, a_2 = 0.48 \) and \( b_2 = 0.12 \). These observations are given in table-4.1.
Table -4.1: Generated observations from proposed AR(1) model.

<table>
<thead>
<tr>
<th>I</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_i$</td>
<td>0.91</td>
<td>1.08</td>
<td>1.18</td>
<td>0.26</td>
<td>2.08</td>
<td>0.25</td>
<td>0.17</td>
<td>0.88</td>
<td>1.21</td>
<td>0.44</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_i$</td>
<td>0.90</td>
<td>0.99</td>
<td>1.07</td>
<td>0.15</td>
<td>2.05</td>
<td>0.04</td>
<td>0.14</td>
<td>0.86</td>
<td>1.12</td>
<td>0.32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_i$</td>
<td>0.38</td>
<td>0.41</td>
<td>0.86</td>
<td>1.68</td>
<td>3.01</td>
<td>2.53</td>
<td>4.36</td>
<td>5.53</td>
<td>4.57</td>
<td>4.61</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_i$</td>
<td>0.33</td>
<td>0.43</td>
<td>0.48</td>
<td>0.99</td>
<td>1.67</td>
<td>0.12</td>
<td>2.33</td>
<td>2.04</td>
<td>0.15</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Table –4.2: The values of Bayes estimates of change point

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>Bayes estimates of change point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Posterior Median</td>
</tr>
<tr>
<td>Informative</td>
<td>12.02</td>
</tr>
<tr>
<td>Non informative</td>
<td>12.22</td>
</tr>
</tbody>
</table>

We have calculated posterior mean of $m$, $\theta_1$, $\theta_2$, $\beta_1$, and $\beta_2$ and the posterior median of $m$. posterior mode appears to be a bad estimator of $m$. For a comparative purpose point of view, estimators under the non-informative prior are also calculated. The results are shown in table-4.2.
Table -4.3: The Bayes estimates using Asymmetric loss functions.

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>Shape Parameter</th>
<th>Bayes estimates of change point</th>
<th>Bayes estimates under General Entropy Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$q_1$ $q_3$ $m^<em>_L$ $m^</em>_E$</td>
<td>$\beta^<em>_2$ $\beta^</em>_1$ $\theta^<em>_1$ $\theta^</em>_2$</td>
<td></td>
</tr>
<tr>
<td>Informative</td>
<td>0.8 0.8 12 12</td>
<td>0.83 0.13 0.13 0.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.2 1.2 12 12</td>
<td>0.82 0.12 0.12 0.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.5 1.5 12 12</td>
<td>0.81 0.11 0.11 0.91</td>
<td></td>
</tr>
<tr>
<td>Non informative</td>
<td>$m^<em>_{L}$ $m^</em><em>{E}$ $\beta^*</em>{2E}$ $\beta^<em>_{1E}$ $\theta^</em><em>{1E}$ $\theta^*</em>{2E}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.8 0.8 13 13</td>
<td>0.86 0.16 0.68 1.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.2 1.2 13 13</td>
<td>0.85 0.15 0.66 1.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.5 1.5 13 13</td>
<td>0.83 0.13 0.65 1.2</td>
<td></td>
</tr>
</tbody>
</table>

We compute the Bayes estimates $m^*_L, m^*_E$ of $m, \theta^*_1, \theta^*_2$ and $\beta^*_1, \beta^*_2$ of $\beta_1$ and $\beta_2$ respectively for the data given in table-4.1. For a comparative purpose, Bayes estimates under the non informative prior and asymmetric loss functions are also computed and results are shown in table-4.3 and shows that $m^*_L, m^*_{GE}, \beta^*_1, \beta^*_2$ and $\theta^*_1, \theta^*_2$ are robust with respect to the change in the shape parameter of GE loss function.

4.6.2 Application of AR (1) model to Total Population of India (Census data):

We now apply the above method to the total population of India from 1901 to 2001, census of India. D. Chabrabarg and Kamal Dutta introduced exponential law of population growth to total population of India.
People are the most important and valuable resources of any nation. The first research on total population was started by Thomas R. Malthus (1798) using a differential equation based on the assumption of exponential population growth. A population cannot grow exponentially forever [Pearl (1925)]. Feldman and Roughgarden (1975), Prajneshu (1980, 1983), Chakraborty and Baruah (1991) and others have developed some models to describe the total population of a region as a function of time. We fitted AR(1) model described in section 2 to the total population of India. These total population data are given in Table 4.

**Table 4.4: Actual total population of India.**

<table>
<thead>
<tr>
<th>(i)</th>
<th>Year</th>
<th>Total Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1901</td>
<td>238,396,337</td>
</tr>
<tr>
<td>2</td>
<td>1911</td>
<td>252,093,390</td>
</tr>
<tr>
<td>3</td>
<td>1921</td>
<td>251,321,213</td>
</tr>
<tr>
<td>4</td>
<td>1931</td>
<td>278,977,238</td>
</tr>
<tr>
<td>5</td>
<td>1941</td>
<td>318,660,580</td>
</tr>
<tr>
<td>6</td>
<td>1951</td>
<td>361,088,090</td>
</tr>
<tr>
<td>7</td>
<td>1961</td>
<td>439,234,771</td>
</tr>
<tr>
<td>8</td>
<td>1971</td>
<td>548,159,652</td>
</tr>
<tr>
<td>9</td>
<td>1981</td>
<td>683,329,097</td>
</tr>
<tr>
<td>10</td>
<td>1991</td>
<td>576,302,688</td>
</tr>
<tr>
<td>11</td>
<td>2001</td>
<td>1027,015,247</td>
</tr>
</tbody>
</table>

Sources: Census of India – 1991 to 2001
We have drawn line chart of the Total population data given in table 4.4.

Figure 4.1 shows line chart of the Total population data. The figure 4.1 shows that there is structural change in the data. Hence, we choose Total population data for application of AR (1) change point model explained in section 4.2 and equation (4.2).

**Figure 4.1 Line chart of Total population**

![Line Chart of Total Population](image)

From the figure 4.1 we can find structural change in the data. We have estimated AR(1) model given in (4.4) for Total population data. The prior for $\beta_1$ and $\beta_2$ were assumed to be beta with prior means $\mu_1 = .70$, $\mu_2 = .44$ and common standard deviation $\sigma = 0.00279$ respectively, resulting in $a_1 = 52$, $b_1 = 22$, $a_2 = 33$, $b_2 = 42$. We have calculated posterior means of $m$, $\beta_1$, $\beta_2$ and $\theta_1$, $\theta_2$ and the posterior median of $m$. Posterior mode appears to be a bad estimate of $m$. For a comparative purpose point estimates under the non informative prior are also calculated. The results are shown in Table 4.5.
Table 4.5: The values of Bayes estimates of change point and autoregressive coefficient under SEL for the population data.

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>Bayes estimates of change point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Posterior Mean</td>
</tr>
<tr>
<td>Informative</td>
<td>8.03</td>
</tr>
<tr>
<td>Non Informative</td>
<td>8.51</td>
</tr>
</tbody>
</table>

Hence, estimated AR(1) model to population data given in Table- 4.4 is,

\[ X_1 = \begin{cases} 
0.75X_{i-1} + \varepsilon_i, & i = 1, 2, \ldots, 7 \\
0.46X_{i-1} + \varepsilon_i, & i = 8, 9, 10, 11.
\]  

(4.62)

We obtained residuals and they are given in table 4.6. Table 4.7 presents the residuals correspond to Total Population AR(1) change-point model (4.62).

We justify the AR (1) change point model (4.2) for the Total population data for this purpose. Table 4.6 shows that first-7 residuals follow exponential with variance 4 and last-6 residuals follow exponential with variance 1.
Table 4.6: Residuals of the Total population data

<table>
<thead>
<tr>
<th>YEAR</th>
<th>TOTAL POPULATION</th>
<th>RESIDUALS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1901</td>
<td>238396337</td>
<td>22944621.89</td>
</tr>
<tr>
<td>1911</td>
<td>252093390</td>
<td>-7615387.15</td>
</tr>
<tr>
<td>1921</td>
<td>251321213</td>
<td>-31380236.67</td>
</tr>
<tr>
<td>1931</td>
<td>278977238</td>
<td>-19364014.46</td>
</tr>
<tr>
<td>1941</td>
<td>318660580</td>
<td>-9766915.46</td>
</tr>
<tr>
<td>1951</td>
<td>361088090</td>
<td>-3537604.31</td>
</tr>
<tr>
<td>1961</td>
<td>439234771</td>
<td>37016366.19</td>
</tr>
</tbody>
</table>

Table 4.7: Descriptive Statistics of Residuals of the Total population data

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Minimum</th>
<th>Maximum</th>
<th>25th</th>
<th>50th (Median)</th>
<th>75th</th>
</tr>
</thead>
<tbody>
<tr>
<td>First 7 residuals</td>
<td>7</td>
<td>-1.6719E6</td>
<td>2.38142E7</td>
<td>-31380236.67</td>
<td>37016366.19</td>
<td>-1.9364E7</td>
<td>-7.6154E6</td>
<td>2.2945E7</td>
</tr>
<tr>
<td>Last 4 variables</td>
<td>4</td>
<td>2.0464E6</td>
<td>1.23525E7</td>
<td>-13240134.80</td>
<td>14732087.40</td>
<td>-1.0460E7</td>
<td>3.3468E6</td>
<td>1.3252E7</td>
</tr>
</tbody>
</table>

We have presented descriptive statistics and K-S test of residuals. The results are shown in Table 4.7 and 4.8 Table 4.6 shows that first 7-observation (residuals) has 4 variance while last-4 observation has 1 variance.
Table 4.8: One-Sample Kolmogorov-Smirnov Test

<table>
<thead>
<tr>
<th></th>
<th>First 7 residuals</th>
<th>Last 4 variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>Exponential parameter.(^{a,b})</td>
<td>Mean</td>
<td>2.9980E7</td>
</tr>
<tr>
<td>Most Extreme Differences</td>
<td>Absolute</td>
<td>.535</td>
</tr>
<tr>
<td></td>
<td>Positive</td>
<td>.291</td>
</tr>
<tr>
<td></td>
<td>Negative</td>
<td>-.535</td>
</tr>
<tr>
<td>Kolmogorov-Smirnov Z</td>
<td>.756</td>
<td>.745</td>
</tr>
<tr>
<td>Asymp. Sig. (2-tailed)</td>
<td>.617</td>
<td>.635</td>
</tr>
</tbody>
</table>

a. Test Distribution is Exponential.
b. Calculated from data.

Table 4.9: The Bayes estimates using Asymmetric Loss function

for the population data.

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>Shape Parameter</th>
<th>Bayes estimates of m</th>
<th>Bayes estimates of Autoregressive coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>q(_1)</td>
<td>q(_3)</td>
<td>(m^*_L)</td>
</tr>
<tr>
<td>Informative</td>
<td>-2</td>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>-1</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>Non Informative</td>
<td>-2</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>
We obtained the Bayes estimates of \((m, \beta_1, \beta_2, \theta_1, \theta_2, \) for data given in Table 4.4, using different values of shape parameter of GE loss and Linex loss and results are shown in table 4.9 which lead to the conclusion that \(m^*_{L}, m^*_{GE}, \beta^*_{1E}, \beta^*_{2E}\) are robust with respect to the change in the shape parameter of GE loss function.

**4.7 SENSITIVITY OF BAYES ESTIMATES**

In this section we study the sensitivity of the Bayes estimates, obtained in section-2 and section-3 with respect to change in the prior of the parameter. The means \(\mu_1, \mu_2\) and standard deviation \(\sigma\) of beta prior have been used as prior information in computing the parameters \(a_1, b_1, a_2, b_2\) of the prior. Following Calabria and Pulcini (1996), we also assume the prior information to be correct if the true value of \(\beta_1 (\beta_2\) is close to prior mean \(\mu_1(\mu_2)\) and is assumed to be wrong if \(\beta_1 (\beta_2\) is far from \(\mu_1(\mu_2)\). We have computed \(m^*\) and \(m^*_{1E}\) using (4.46) and (4.52) for the data given in Table-4.1 and Table-4.4 with common value of \(\sigma =0.1\) and \(\sigma =0.00279\) respectively, for \(q3 =0.9\), considering different values of \((\mu_1, \mu_2)\) and result are shown in Table-4.10 and Table-4.11 respectively.

The results shown in Table-4.10 and 4.11 lead to conclusion that \(m^*\) and \(m^*_{1E}\) are robust with respect to the correct choice of the prior density of \(\beta_1 (\beta_2\) and a wrong choice of the prior density of \(\beta_1 (\beta_2\). Moreover, they are also robust with respect to the change in the shape parameter of GE loss function.
Table-4.10: Posterior mean m* for the data given in Table-4.1.

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>m*</th>
<th>$m^*_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.6</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.07</td>
<td>0.8</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>11</td>
<td>10</td>
</tr>
</tbody>
</table>

Table-4.11: The Bayes estimates of change point for the population data.

<table>
<thead>
<tr>
<th>Values of prior Means</th>
<th>Bayes estimate under (SEL)</th>
<th>$q_3 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>m*</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>----</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4</td>
<td>7</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5</td>
<td>7</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6</td>
<td>7</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>7</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4</td>
<td>7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4</td>
<td>7</td>
</tr>
</tbody>
</table>
4.8 SIMULATION STUDY

In section 4.4, we have obtained Bayes estimates of \( m \) on the basis of the generated data given in Table-4.1 for given values of parameters. In section 4.5, we have found the values of Bayes estimators for the data given in Table 4.1. Now in this section, we have done all the calculations for 10,000 generated samples. To justify the results obtained in section 4.5, we have generated 10,000 different random samples with \( m=12, n=20, \theta_1=1.0, \theta_2=2.0, \beta_1=0.4, \beta_2=0.6 \) and obtained the frequency distributions of posterior mean, median of \( m \), \( m_L^* \), \( m_E^* \) and for reliability functions with the correct prior consideration. We also obtained the frequency distributions of Bayes estimates of autoregressive coefficients given in section 4.4 with the correct prior consideration. The result is shown in Table 4.9. The value of shape parameter of the general entropy loss and Linex loss used in simulation study for change point is taken as 0.1. We have also simulated several samples from AR (1) model explained section 4.2 with \( m=10, n=20, \theta_1=1.0, 0.6, 0.7; \theta_2=2.0, 0.8, 0.9 \) and \( \beta_1=0.2, 0.4, 0.5; \beta_2=0.4, 0.6, 0.7 \). For each \( \theta_1, \theta_2, \beta_1 \) and \( \beta_2 \), 1000 pseudo random samples with \( m=12 \) and \( n=20 \) have been simulated and Bayes estimators of change point \( m \) and autoregressive coefficients \( \beta_1 \) and \( \beta_2 \) using \( q_2 = 0.9 \) has been computed for same value of \( a_1, a_2, b_1, b_2 \) and for different prior means \( \mu_1 \) and \( \mu_2 \). We also obtained the frequency distributions of Bayes estimates of autoregressive coefficients given in section 4.4 with the correct prior consideration for generated 10,000 different random samples. The results are shown in Table 4.12 to 4.14.
Table 4.12: Frequency distributions of the Bayes estimates of the change point

<table>
<thead>
<tr>
<th>Bayes estimate</th>
<th>% Frequency for</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>01-09</td>
</tr>
<tr>
<td>Posterior mean</td>
<td>18</td>
</tr>
<tr>
<td>Posterior median</td>
<td>10</td>
</tr>
<tr>
<td>Posterior mode</td>
<td>12</td>
</tr>
<tr>
<td>mL*</td>
<td>13</td>
</tr>
<tr>
<td>mE*</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 4.13: Frequency distributions of the Bayes estimates of autoregressive coefficients $\beta_1$ and $\beta_2$ using General Entropy Loss function (informative)

<table>
<thead>
<tr>
<th>Bayes estimate</th>
<th>% Frequency for</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1-0.3</td>
</tr>
<tr>
<td>$\beta_{1E}$</td>
<td>13</td>
</tr>
<tr>
<td>$B_{2E}$</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 4.14: Frequency distributions of the Bayes estimates of (non-informative)

<table>
<thead>
<tr>
<th>Bayes estimate</th>
<th>% Frequency for</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1-0.3</td>
</tr>
<tr>
<td>$\beta_{1E}$</td>
<td>13</td>
</tr>
<tr>
<td>$B_{2E}$</td>
<td>11</td>
</tr>
</tbody>
</table>
4.9 CONCLUSIONS

Our numerical study showed that the Bayes estimators posterior mean of m, and $m^*_e$ are robust with respect to the correct choice of the prior specifications on $\beta_1$ ($\beta_2$) and wrong choice of the prior specifications on $\beta_2$ ($\beta_1$) and are sensitive in case prior specifications on both $\beta_1$, $\beta_2$ deviate simultaneously from the true values. Numerical study also showed that posterior mean of m is sensitive when prior specifications on both $\beta_1$, $\beta_2$ deviate simultaneously from the true values.