3 Shape changing (intensity redistribution) collisions of solitons in mixed coupled nonlinear Schrödinger equations

3.1 Introduction

It was suggested a long time ago that solitons can be used to carry data at very high bit rate in optical communication systems, because of their ability to overcome the dispersion limitation through a balance between the self-phase modulation and dispersion effects. In fact soliton pulses are known to have many other desirable properties, such as their robustness against small changes in the pulse shape or amplitude around the exact soliton profile which leads one to treat such changes only as small perturbations on soliton propagation [22, 23, 101]. Strictly speaking, the soliton properties can exit only in an ideal fiber. Indeed, in a standard telecommunication fiber, the propagation of light pulses gives rise to a host of perturbing effects which inhibit the desirable properties of solitons [43]. One of the strongly perturbing effects that comes inevitably into play is the linear attenuation of light along the fiber (which is of the order of 0.2dB/km at carrier wavelength 1.55µm), which does not permit to keep a constant balance between the self-phase modulation and the group-
velocity dispersion [43]. Although the fundamental soliton propagation cannot be obtained in standard fibers, pulse propagation over relatively long distances (and even transoceanic distances) can still be obtained through an appropriate combination of dispersion management and optical amplification (now mostly based on Er-doped fiber amplifiers and Raman amplifiers) [48, 102, 103].

All the existing amplification processes involve three major ingredients: The first one is a \textit{pump wave}, which serves as a photon reservoir. The second one is an \textit{amplification medium}, that is, a special material in which the pump wave is mixed with the signal. The third ingredient is a \textit{physical mechanism} that can cause a transfer of photons from the pump to the signal. Only three types of physical mechanisms have been exploited so far in optical amplifiers, namely the \textit{laser process} used in laser optical amplifiers (e.g. Er-doped fiber amplifiers, semi-conductor optical amplifiers) [104], the \textit{stimulated Raman scattering} (used in Raman amplifiers) [43, 103] and \textit{parametric wave mixing} (used in parametric amplifiers) [43, 103]. Such optical amplifiers do permit to fully compensate the fiber losses, but the amplification process is unavoidably accompanied by an undesirable effect of noise generation which is commonly referred to as the “amplified spontaneous emission” (ASE) [105–107]. Hence, one of the most important characteristic parameters of the optical amplifiers developed so far is the so-called “noise figure”, which serves as a measure of the amount of noise generated during the amplification process [108]. The ASE increases with the amplifier gain, and there exists an unavoidable amount of noise, known as the amplifier noise figure limit of 3 dB [108–110]. The ASE is one of the major effects that severely degrades the transmission quality of ultra-short light pulses over long distances [43, 102, 111]. To radically resolve the problem of ASE limitation in high-speed long-distance transmission systems, it is clear that the conceptual approach of optical amplification based on the three ingredients mentioned above needs to be partially or totally reformulated.

In this chapter, we review the shape changing (intensity redistribution) col-
3.1 Introduction

Collisions of vector solitons in mixed CNLS equations, and also the possibility of constructing a novel approach of signal amplification. The novelty lies in viewing the collision process of solitons as a fundamental physical mechanism for transferring energy from the pump to the signal. The collision involves two vector solitons. One of the two solitons, say $S_1$, is chosen, to be the signal, while the other soliton ($S_2$) serves as the energy reservoir (pump wave). The major virtue of this type of collision-based amplification process is that it does not induce any noise, as it does not make use of any external amplification medium.

The fundamental integrable mixed $N$-CNLS system has been discussed in the introductory chapter and it is given by Eq. (1.42). It is found to be completely integrable [75, 112, 113] and the corresponding Lax pair was obtained in [75]. In their pioneering works Makhankov et al. [75, 112] have shown that Eq. (1.42), for $N = 2$, admits particular bright-bright, bright-dark, dark-dark type one soliton solutions depending upon the asymptotic behaviour of the complex amplitudes $q_j$, $j = 1, 2$. Since then very few works have appeared in the literature to analyze the problem further [64, 68, 114, 115] (for a detailed review of existing results one can refer to [115]). Particularly, in a recent work [115], Kanna et al. have obtained stationary solutions of mixed CNLS equations with singularities by following an algebraic approach [116–118]. It was observed that despite the points of singularities the solutions behave smoothly in finite region of the temporal domain. In Ref. [69], it was shown that the bright solitons of regular type do exist, provided the soliton parameters satisfy certain conditions and that the underlying solitons undergo novel shape changing/intensity redistribution collisions. The singular solutions turn out to be special cases (with specific parametric choices) of the general soliton solutions. An important new feature in the collision process of regular solitons in the mixed CNLS case is that after collision a soliton can gain energy in all its components, while the opposite takes place in the other soliton.
Section 3.2 contains brief overview of Hirota’s bilinearization procedure for
the mixed N-CNLS equations to obtain soliton solutions. Though the solutions
discussed in this chapter admit both singular and non-singular behaviours, we
call them as soliton solutions ascribing to their soliton nature in some specific
region. We present the explicit forms of one-, two-, and three-soliton solu-
tions for further use in the thesis. Section 3.3 is devoted to a brief analysis of
shape changing (intensity redistribution) collisions exhibited by these soliton
solutions. The procedure to obtain one, two soliton solutions is extended to
multisoliton solutions in Gram determinant form and also for multicomponent
system in section. 3.4 and also the proof of multisoliton solution of multicom-
ponent system is given in the same section.

3.2 Hirota’s bilinearization of mixed N-CNLS equations

The set of equations (1.42) has been shown to be completely integrable [75,
113], admitting certain types of single soliton solutions [75, 112], for the
$N = 2$
case, as mentioned in Chapter 1. Here we are concerned with bright-bright
multisoliton solutions whose intensity profiles vanish asymptotically and with
the nature of soliton interactions.

By applying the bilinearizing transformation

$$q_j = \frac{g^{(j)}}{f}, \quad j = 1, 2, ..., N,$$

(3.1)

to Eq. (1.42) similar to the focusing case $\sigma_l = 1, \ l = 1, 2, ..., N$ [66]. This results
in the following set of bilinear equations,

$$(iD_z + D_t^2)g^{(j)}.f = 0, \quad j = 1, 2, ..., N,$$

(3.2a)

$$D_t^2 (f.f) = 2\mu \sum_{l=1}^N \sigma_l g^{(l)} g^{(l)*},$$

(3.2b)
where \( \sigma_l \) is given by Eq. (1.42b), \(*\) denotes the complex conjugate, \( g^{(j)}\)'s are complex functions, while \( f(z,t) \) is a real function and the Hirota’s bilinear operators \( D_z \) and \( D_t \) are defined by

\[
D^n_z D^m_t (a,b) = \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m a(z,t) b(z',t') \bigg|_{(z=z',t=t')}.
\] (3.2c)

The above set of equations can be solved by introducing the following power series expansions for \( g^{(j)}\)'s and \( f \):

\[
g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + ..., \quad j = 1, 2, ..., N, \quad (3.3a)
\]
\[
f = 1 + \chi^2 f_2 + \chi^4 f_4 + ..., \quad (3.3b)
\]

where \( \chi \) is the formal expansion parameter. The resulting set of equations, after collecting the terms with the same power in \( \chi \), can be solved recursively to obtain the forms of \( g^{(j)}\)'s and \( f \). The explicit forms of bright one-, two-, and three-soliton solution of mixed 2-CNLS equations are given below.

### 3.2.1 Bright-bright soliton solutions of mixed 2-CNLS system

Consider the following integrable mixed 2-CNLS equations,

\[
\begin{align*}
iq_{1,z} + q_{1,u} + 2 \left( |q_1|^2 - |q_2|^2 \right) q_1 &= 0, \quad (3.4a) \\
iq_{2,z} + q_{2,u} + 2 \left( |q_1|^2 - |q_2|^2 \right) q_2 &= 0. \quad (3.4b)
\end{align*}
\]

In this subsection, we review the soliton solutions of the above equation following the work of Kanna et al [69] then in the next section, we discuss the shape changing collision scenario of the mixed 2-CNLS system.
3.2 Hirota’s bilinearization of mixed N-CNLS equations

3.2.1.A One soliton (bright-bright) solution of mixed 2-CNLS equations

The one soliton solution of the above equation (3.4) can be written as [69]

$$
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} =
\begin{pmatrix}
\alpha_1^{(1)} \\
\alpha_1^{(2)}
\end{pmatrix}
\frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}} =
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}
k_{1R} \text{sech} \left( \eta_{1R} + \frac{R}{2} \right) e^{i\eta_1},
$$

where

$$
\eta_1 = k_1(t + ik_1z) = \eta_{1R} + i\eta_{1I},
$$

$$
A_j = \frac{\alpha_1^{(j)}}{\left( |\alpha_1^{(1)}|^2 - |\alpha_1^{(2)}|^2 \right)^{1/2}}, \quad j = 1, 2,
$$

$$
e^R = \frac{\left( |\alpha_1^{(1)}|^2 - |\alpha_1^{(2)}|^2 \right)}{(k_1 + k_1^*)^2}.
$$

Note that this one soliton solution is characterized by three arbitrary complex parameters $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, and $k_1 = k_{1R} + ik_{1I}$, where the suffices $R$ and $I$ represent the real and imaginary parts, respectively. The quantities $k_{1R}A_1$ and $k_{1R}A_2$, give the amplitude of the soliton in components $q_1$ and $q_2$, respectively, and the soliton velocity in each component is given by $2k_{1I}$. The position of the soliton is found to be

$$
\frac{R}{2k_{1R}} = \frac{1}{2k_{1R}} \ln \left[ \frac{\left( |\alpha_1^{(1)}|^2 - |\alpha_1^{(2)}|^2 \right)}{(k_1 + k_1^*)^2} \right].
$$

From Eq. (3.5b), it is clear that singular solutions start occurring when $|\alpha_1^{(1)}| = |\alpha_1^{(2)}|$. In this case, one can easily observe from Eq. (3.5e) that the quantity $e^R$
becomes 0, and one gets the solution

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} e^{\eta}$$

(3.6)

which is unbounded. Such an unbounded solution is depicted in Fig. 2.1(a) for $k_1 = 1 + i$, and $\alpha_1^{(1)} = \alpha_1^{(2)} = 1$. When $|\alpha_1^{(1)}| < |\alpha_1^{(2)}|$, $e^{R}$ becomes negative (so $R$ becomes complex). In this case, singularity occurs, whenever

$$1 - |e^{R}|e^{2\eta_R} = 0,$$

(3.7a)

or

$$\eta_R = \frac{1}{2} \ln \left( \frac{1}{|e^{R}|} \right).$$

(3.7b)

Again a singular solution in this case is plotted in Fig. 2.1(b) for $k_1 = 1 + i$, $\alpha_1^{(1)} = 0.8$, $\alpha_1^{(2)} = 1$. However the bright soliton solution is always regular as long as the condition $|\alpha_1^{(1)}| > |\alpha_1^{(2)}|$ is valid in which case $e^{R}$ is always real and positive,
as the denominator \((1 + e^{\eta_1 + \eta_1^* + R})\) in Eq. (3.5a) is always positive definite (as \(\eta_{1R}\) is real) for this choice. This regular one soliton solution is shown in Fig. 3.2 for \(k_1 = 1 + i, \alpha_1^{(1)} = 1, \text{and} \alpha_1^{(2)} = 0.2, \text{and} \mu = 1\).

It is also interesting to note here that the polarization vector evolves in a hyperboloid defined by the surface \(|A_1|^2 - |A_2|^2 = \frac{1}{\mu}|75\rangle\), whereas in the Manakov case it is a sphere (that is \(|A_1|^2 + |A_2|^2 = \frac{1}{\mu}|66\rangle\)). This allows Eq. (1.42) to admit a rich variety of singular and non-singular solutions and makes significant difference in the collision scenario of bright solitons arising in the two systems as we will see in the following sections.

### 3.2.1.B Two soliton (bright-bright) solution of mixed 2-CNLS equations

The two soliton solution of the mixed 2-CNLS equation (3.4) is obtained as [69]

\[
q_j = \frac{\alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + e^{\eta_1 + \eta_2 + \delta_3} + e^{\eta_1 + \eta_2 + \eta_1^* + \delta_j}}{D}, \quad j = 1, 2, \tag{3.8a}
\]
where

\[
D = 1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + R_1} + e^{\eta_1 + \eta_2 + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2} + e^{\eta_1 + \eta_2 + \eta_2^* + R_3}.
\]  

(3.8b)

Various quantities found in Eq. (3.8), are defined as below:

\[
\begin{align*}
\eta_i &= k_i(t + ik_i z), \quad \delta_0 = \eta_1^* \kappa_1 + \eta_2^* \kappa_2, \quad \delta_1 = \kappa_1^2 \kappa_2^2 - |\kappa_{12}|^2, \quad \delta_2 = \kappa_1^2 \kappa_2^2 - |\kappa_{12}|^2, \\
\kappa_{ij} &= \left( \alpha_i^{(1)} \alpha_i^{(1)*} - \alpha_i^{(2)} \alpha_i^{(2)*} \right) / (k_i + k_j^*), \quad i, j = 1, 2.
\end{align*}
\]

(3.9a)

and

\[
\kappa_{ij} = \left( \alpha_i^{(1)} \alpha_i^{(1)*} - \alpha_i^{(2)} \alpha_i^{(2)*} \right) / (k_i + k_j^*), \quad i, j = 1, 2.
\]

(3.9b)

The above solution is characterized by six arbitrary complex parameters \( \alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_2^{(1)}, \alpha_2^{(2)}, k_1, \) and \( k_2. \) Although the above solution features both singular and nonsingular solutions in the following we consider only the nonsingular soliton solution which results for the choice

\[
\begin{align*}
\kappa_{11} &\geq 0, \quad \kappa_{22} \geq 0, \quad \kappa_{11} \kappa_{22} - |\kappa_{12}|^2 > 0, \\
\frac{1}{2} \sqrt{\kappa_{11} \kappa_{22}} + \frac{|k_1 - k_2|}{2 |k_1 + k_2^*|} \sqrt{\kappa_{11} \kappa_{22} - |\kappa_{12}|^2} &> \frac{|\kappa_{12}|}{|k_1 + k_2^*|}.
\end{align*}
\]

(3.10a)
### 3.2.1.C Three soliton solution

The explicit form of three soliton solution can be written as

\[
q_j = \frac{\alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + \alpha_3^{(j)} e^{\eta_3} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{1j}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{2j}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{3j}}}{D} + e^{\eta_2 + \eta_3 + \delta_{1j}} + e^{\eta_3 + \eta_2 + \delta_{1j}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{2j}} + e^{\eta_1 + \eta_2 + \eta_3 + \delta_{3j}} + e^{\eta_2 + \eta_3 + \eta_1 + \tau_{1j}} + e^{\eta_1 + \eta_2 + \eta_3 + \eta_1 + \tau_{2j}} + e^{\eta_2 + \eta_3 + \eta_1 + \tau_{3j}} \frac{D}{D}, \quad j = 1, 2, \tag{3.11a}
\]

where

\[
D = 1 + e^{\eta_1 + \eta_2 + R_1} + e^{\eta_1 + \eta_2 + R_2} + e^{\eta_3 + \eta_2 + R_3} + e^{\eta_1 + \eta_2 + \delta_{10}} + e^{\eta_1 + \eta_2 + \delta_{20}} + e^{\eta_1 + \eta_2 + \delta_{30}} + e^{\eta_1 + \eta_2 + \eta_2 + \delta_{20}} + e^{\eta_1 + \eta_2 + \eta_3 + \eta_3 + R_5} + e^{\eta_2 + \eta_2 + \eta_3 + \eta_3 + R_6} + e^{\eta_1 + \eta_1 + \eta_2 + \eta_2 + \tau_{10}} + e^{\eta_1 + \eta_1 + \eta_2 + \eta_2 + \tau_{20}} + e^{\eta_1 + \eta_1 + \eta_2 + \eta_2 + \tau_{30}} + e^{\eta_1 + \eta_2 + \eta_2 + \eta_3 + \eta_3 + R_7}. \tag{3.11b}
\]

All the quantities defined in the above expressions are given in the previous chapter by redefining \(\kappa_{ij}\) as

\[
\kappa_{ij} = \mu \sum_{l=1}^{2} \sigma_1 \alpha_i^{(l)} \alpha_j^{(l)*} \frac{(k_i + k_j^*)}{(k_i + k_j^*)}, \quad i, j = 1, 2, 3, \tag{3.11c}
\]

where \(\sigma_1 = 1\) and \(\sigma_2 = -1\). The solution (3.11) also features singular and non-singular behaviours, as in the case of one and two soliton solutions depending upon the values of the soliton parameters. Though the denominator \(D\) in the solution (3.11) is cumbersome, possible non-singular conditions can be
obtained with some effort. Eq. (3.11b) can be rewritten as

\[ D = 2e^{\eta_{1R} + \eta_{2R} + \eta_{3R}} \left\{ e^{(R_1 + R_6)/2} \cosh (\eta_{1R} - \eta_{2R} - \eta_{3R} + (R_1 - R_6)/2) 
+ e^{(R_2 + R_5)/2} \cosh (\eta_{2R} - \eta_{1R} - \eta_{3R} + (R_2 - R_5)/2) 
+ e^{(R_3 + R_4)/2} \cosh (\eta_{3R} - \eta_{1R} - \eta_{2R} + (R_3 - R_4)/2) 
+ 2e^{(\delta_{10R} + \tau_{30R})/2} (\cosh (X_1 \cos (Y_1) \cos (Z_1)) - \sinh (X_1 \sin (Y_1) \sin (Z_1))) 
+ 2e^{(\delta_{20R} + \tau_{20R})/2} (\cosh (X_2 \cos (Y_2) \cos (Z_2)) - \sinh (X_2 \sin (Y_2) \sin (Z_2))) 
+ 2e^{(\delta_{30R} + \tau_{10R})/2} (\cosh (X_3 \cos (Y_3) \cos (Z_3)) - \sinh (X_3 \sin (Y_3) \sin (Z_3))) 
+ e^{R_7/2} \cosh (\eta_{1R} + \eta_{2R} + \eta_{3R} + R_7/2) \right\}, \] (3.12a)

where

\[
\begin{align*}
X_1 &= -\eta_{3R} + \frac{(\delta_{10R} - \tau_{30R})}{2}, & X_2 &= -\eta_{2R} + \frac{(\delta_{20R} - \tau_{20R})}{2}, \\
X_3 &= -\eta_{1R} + \frac{(\delta_{30R} - \tau_{10R})}{2}, & Y_1 &= \eta_{1I} - \eta_{2I} + \frac{(\delta_{10I} + \tau_{30I})}{2}, \\
Y_2 &= \eta_{1I} - \eta_{3I} + \frac{(\delta_{20I} + \tau_{20I})}{2}, & Y_3 &= \eta_{2I} - \eta_{3I} + \frac{(\delta_{30I} + \tau_{10I})}{2}, \\
Z_1 &= \frac{(\delta_{10I} - \tau_{30I})}{2}, & Z_2 &= \frac{(\delta_{20I} - \tau_{20I})}{2}, & Z_3 &= \frac{(\delta_{30I} - \tau_{10I})}{2}. \quad (3.12b)
\end{align*}
\]

Here the suffices \( R \) and \( I \) denote the real and imaginary parts, respectively. As in the case of two soliton solution here also we find the following conditions need to be satisfied for the solution to be regular:

\[
e^{R_i} > 0, \ i = 1, 2, ..., 7, \] (3.13a)

\[
e^{(R_1 + R_6)/2}, e^{(R_2 + R_5)/2}, e^{(R_3 + R_4)/2}, e^{R_7/2} > 4 \max \left\{ e^{\delta_{10R} + \tau_{30R}}, e^{\delta_{20R} + \tau_{20R}}, e^{\delta_{30R} + \tau_{10R}} \right\}. \] (3.13b)

Note that, the conditions given in (3.13a) are necessary as the falsity of any of them always results in singular solution and the last condition (3.13b) is sufficient to ensure that the given solution is regular. In fact these conditions can
3.3 Shape changing (intensity redistribution) collisions of solitons

![Figure 3.3: Shape changing (intensity redistribution) collision of two solitons in the mixed CNLS system for $N = 2$ case.](image)

also be expressed in terms of soliton parameters, but due to their cumbersome nature we do not present them here. The appropriate choice of parameters can be made by carefully looking at the explicit forms of $e^{R_i}$, $e^{b_i^0}$, and $e^{r_i^0}$, $i = 1, \ldots, 7$, and $j = 1, 2, 3$.

3.3 Shape changing (intensity redistribution) collisions of solitons

Now it is of interest to understand the collision behaviour, shown in Fig. 3.3, of the regular two soliton solution. Figure 3.3 shows the interaction of two solitons $S_1$ and $S_2$ which are well separated before and after collision, in the $q_1$ and $q_2$ components. This figure shows that after collision, the first soliton $S_1$ in the component $q_1$ gets enhanced in its amplitude while the soliton $S_2$ is suppressed. Interestingly, the same kind of changes are observed in the second component $q_2$ as well [69]. This collision scenario is entirely different from the one observed in the Manakov system where one soliton gets suppressed in
one component and is enhanced in the other component with commensurate changes in the other soliton.

On the other hand, conceptually, the collision scenario shown in Fig. 3.3 may be viewed as an amplification process in which the soliton $S_1$ represents a signal (or data carrier) while the soliton $S_2$ represents an energy reservoir (pump). The main virtue of this amplification process is that it does not require any external amplification medium and therefore the amplification of $S_1$ does not induce any noise [69].

The understanding of this fascinating collision process can be facilitated by making an asymptotic analysis of the two soliton solution as in the Manakov case [62, 63, 66]. We perform the analysis for the choice $k_{1R}, k_{2R} > 0$ and $k_{1I} > k_{2I}$. For any other choice the analysis is similar. The study shows that due to collision, the amplitudes of the colliding solitons $S_1$ and $S_2$ change from $(A_1^1, A_1^2)$ to $(A_1^{1+}, A_1^{2+})$, and $(A_2^1, A_2^2)$ to $(A_2^{1-}, A_2^{2-})$, respectively [69]. Here the superscripts in $A_i^j$'s denote the solitons (number(1,2)), the subscripts represent the components (number(1,2)) and '±' signs stand for '$z \to \pm \infty$'. They are defined as

\[
\begin{align*}
\begin{pmatrix} A_1^1 \ A_1^2 \end{pmatrix} &= \begin{pmatrix} \alpha_1^{(1)} \\
\alpha_1^{(2)} \end{pmatrix} \frac{e^{-R_1/2}}{(k_1 + k_1^*)}, \\
\begin{pmatrix} A_2^1 \ A_2^2 \end{pmatrix} &= \begin{pmatrix} e^{\delta_{11}} \\
\alpha_1^{(1)} \alpha_1^{(2)} \end{pmatrix} \frac{e^{-(R_1+R_3)/2}}{(k_2 + k_2^*)}, \\
\begin{pmatrix} A_1^1 \ A_1^2 \end{pmatrix} &= \begin{pmatrix} \alpha_1^{(1)} \\
\alpha_1^{(2)} \end{pmatrix} \frac{e^{-(R_2+R_3)/2}}{(k_1 + k_1^*)}, \\
\begin{pmatrix} A_2^1 \ A_2^2 \end{pmatrix} &= \begin{pmatrix} \alpha_2^{(1)} \\
\alpha_2^{(2)} \end{pmatrix} \frac{e^{-R_2/2}}{(k_2 + k_2^*)}.
\end{align*}
\]

All the quantities in the above expressions are given in Eq. (3.8). The analysis reveals the fact that, for the non-singular two soliton solution, the colliding soli-
Shape changing (intensity redistribution) collisions of solitons

Solitons change their amplitudes in each component according to the conservation equation

\[ |A_j^-|^2 - |A_j^2|^2 = |A_j^+|^2 - |A_j^1|^2 = \frac{1}{\mu}, \quad j = 1, 2. \tag{3.15} \]

This can be easily verified from the actual expressions given in Eq. (3.14).

This condition allows the given soliton to experience the same effect in each component during collision, which may find potential applications in some physical situations like noiseless amplification of a pulse. It can be easily observed from the conservation relation (3.15) that each component of a given soliton experiences the same kind of energy switching during collision process. The other soliton (say \( S_2 \)) experiences an opposite kind of energy switching due to the conservation law

\[ \int_{-\infty}^{\infty} |q_j|^2 \, dt = \text{constant}, \quad j = 1, 2, \tag{3.16} \]

as required from Eq. (1.42).

The asymptotic analysis also results in the following expression relating the intensities of solitons \( S_1 \) and \( S_2 \) in \( q_1 \) and \( q_2 \) components before and after interaction (see Eq. (3.14)),

\[ |A_j^{l+}|^2 = |T_j^l|^2 |A_j^{l-}|^2, \quad j, l = 1, 2, \tag{3.17} \]

where the superscripts \( l \pm \) represent the solitons designated as \( S_1 \) and \( S_2 \) at \( z \to \pm \infty \). The transition intensities are defined as

\[ |T_j^1|^2 = \frac{|1 - \lambda_2 (\alpha_{2j}^{(j)}) / \alpha_{1j}^{(j)}|^2}{|1 - \lambda_1 \lambda_2|}, \tag{3.18a} \]

\[ |T_j^2|^2 = \frac{|1 - \lambda_1 (\alpha_{1j}^{(j)}) / \alpha_{2j}^{(j)}|^2}{|1 - \lambda_1 \lambda_2|}, \quad j = 1, 2, \tag{3.18b} \]

\[ \lambda_1 = \frac{\kappa_{21}}{\kappa_{11}}, \quad \lambda_2 = \frac{\kappa_{12}}{\kappa_{22}}. \tag{3.18c} \]
For the standard elastic collision property ascribed to the scalar solitons to occur here we need the magnitudes of the transition intensities to be unity which is possible for the specific choice

\[
\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}.
\] (3.19)

This is shown in Fig. 3.4. The other quantities characterizing this collision process, along with this energy redistribution, are the amplitude dependent phase shifts and change in relative separation distances. Their explicit forms can be obtained as in the case of the Manakov model [62, 66]. Explicit expressions for the phase shifts \(\Phi_1\) and \(\Phi_2\) of solitons \(S_1\) and \(S_2\), respectively, during the collision are obtained from the asymptotic analysis as

\[
\Phi_1 = -\Phi_2 = \frac{(R_3 - R_1 - R_2)}{2},
\] (3.20)

where \(R_1\), \(R_2\), and \(R_3\) are defined in Eq. (3.8).

Figure 3.4: Elastic collision of two solitons in the mixed CNLS system for \(N = 2\) case.
Then, the change in relative separation distance between the solitons can be expressed as

$$\Delta t_{12} = t_{12}^+ - t_{12}^- = \frac{(k_{1R} + k_{2R})}{k_{1R}k_{2R}} \Phi_1,$$

(3.21)

where $t_{12}^\pm$ is the position of $S_2$ (at $z \to \pm \infty$) minus position of $S_1$ (at $z \to \pm \infty$).

A non-singular solution representing the shape changing (intensity redistribution) collision of three solitons [69] $S_1$, $S_2$, and $S_3$ in the two components $q_1$ and $q_2$ is shown in Fig. 3.5 for the parameter choice $k_1 = 1 + i$, $k_2 = 1.2 - 0.5i$, $k_3 = 1 - i$, $\mu = 1$, $\alpha_1^{(1)} = \cosh(\theta_1)e^{i\phi_1}$, $\alpha_2^{(1)} = \cosh(\theta_2)e^{i\phi_1}$, $\alpha_3^{(1)} = \cosh(\theta_3)e^{i\phi_1}$, $\alpha_1^{(2)} = \sinh(\theta_1)e^{i\phi_2}$, $\alpha_2^{(2)} = \sinh(\theta_2)e^{i\phi_2}$, $\alpha_3^{(2)} = \sinh(\theta_3)e^{i\phi_2}$, where $\theta_1 = 0.8$, $\theta_2 = 0.4$, $\theta_3 = 0.2$, $\phi_1 = 0.5$, and $\phi_2 = 1.0$. From the figure it has been shown that after collision, solitons $S_1$ and $S_2$ are enhanced in their intensities while there occurs suppression of intensity for soliton $S_3$ in both the components $q_1$ and $q_2$. It can be verified that before and after collision the conservation relation

$$|A_j^{1-}|^2 - |A_j^{2-}|^2 = |A_j^{1+}|^2 - |A_j^{2+}|^2 = \frac{1}{\mu}, \quad j = 1, 2, 3,$$

(3.22)
3.4 Generalization of the results to multisoliton solutions and multicomponent case

Having discussed the nature of two- and three-soliton collision in the two-component case ($N = 2$), we now wish to study multisoliton solution for the multicomponent cases.

Figure 3.6: Elastic collision of three solitons in the mixed CNLS system for $N = 2$ case.
3.4 Generalization of the results to multisoliton solutions and multicomponent case

3.4.1 Multicomponent case with \( N > 2 \)

The next step is to generalize the above results for the \( N = 2 \) case to arbitrary \( N \) with \( N > 2 \). To do this we follow the previous chapter on the focusing type CNLS equations with all \( \sigma_l = 1, \ l = 1, 2, \ldots, N \). This study shows that the solutions of mixed CNLS equations with \( N = 2 \) case can be generalized to arbitrary \( N \) case just by allowing the number of components to run from 2 to \( N \) and redefining \( \kappa_{ij} \)'s suitably [69].

The procedure can be well understood by considering the example of writing down the soliton solutions of Eq. (1.42) for the case \( N = 3 \).

3.4.1.A One soliton solution

The one soliton solution of mixed 3-CNLS equations obtained by Hirota’s method can be written as

\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} = \begin{pmatrix}
\alpha_1^{(1)} \\
\alpha_1^{(2)} \\
\alpha_1^{(3)}
\end{pmatrix} \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}},
\]

where

\[
\eta_1 = k_1(t + ik_1 z), \quad e^R = \frac{\kappa_{11}}{(k_1 + k_1^*)},
\]

in which \( \kappa_{11} = \frac{\mu(\sigma_1|\alpha_1^{(1)}|^2 + \sigma_2|\alpha_1^{(2)}|^2 + \sigma_3|\alpha_1^{(3)}|^2)}{(k_1 + k_1^*)} \) and without loss of generality we assume either \( \sigma_1 = 1, \sigma_2 = \sigma_3 = -1 \) or \( \sigma_1 = \sigma_2 = 1, \sigma_3 = -1 \). As in the case of \( N = 2 \), the solution is singular if \( \sigma_1|\alpha_1^{(1)}|^2 + \sigma_2|\alpha_1^{(2)}|^2 + \sigma_3|\alpha_1^{(3)}|^2 \leq 0 \). Otherwise the solution is regular. It can be noticed that for any other combination of \( \sigma_i \)'s also the above solution satisfies Eq. (1.42), for \( N = 3 \).
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3.4.1.B Two soliton solution

The two soliton solution for the \( N = 3 \) case is found to possess the same form of Eq. (3.8), with \( j = 1, 2, 3 \), and \( \kappa_{ij} \) is given by

\[
\kappa_{ij} = \frac{\mu \left( \sigma_1 \alpha_i^{(1)} \alpha_j^{(1)*} + \sigma_2 \alpha_i^{(2)} \alpha_j^{(2)*} + \sigma_3 \alpha_i^{(3)} \alpha_j^{(3)*} \right)}{(k_i + k_j^*)}, \quad i, j = 1, 2, \tag{3.24}
\]

where \( \sigma_l \)'s, \( l = 1, 2, 3 \) can take the value either +1 or −1. Here also the non-singular solution exists for the conditions (3.10a) and (3.10b) with the redefinition of \( \kappa_{ij} \)'s as in Eq. (3.24).

3.4.1.C Three and multisoliton solutions

A similar analysis can be done for the multisoliton solutions of the multicomponent case with arbitrary \( N \). Particularly the three soliton solution of the mixed 3-CNLS equations , Eq. (1.42) with \( N = 3 \), can be identified to have the form of three soliton solution for the \( N = 2 \) case with \( j \) running from 1 to 3 (that is, now we have three components \( q_1, q_2, \) and \( q_3 \)) and here \( \kappa_{ij} \) is redefined as

\[
\kappa_{ij} = \frac{\mu \left( \sigma_1 \alpha_i^{(1)} \alpha_j^{(1)*} + \sigma_2 \alpha_i^{(2)} \alpha_j^{(2)*} + \sigma_3 \alpha_i^{(3)} \alpha_j^{(3)*} \right)}{(k_i + k_j^*)}, \quad i, j = 1, 2, 3, \tag{3.25}
\]

where \( \sigma_l \)'s, \( l = 1, 2, 3 \), can take the value either +1 or −1.

This procedure can be generalized further to obtain multisoliton solutions of the multi-component case with arbitrary \( N \). For completeness we present the Gram determinant form of the \( n \)-soliton solution of \( N \)-component case. Also we present the proof of multisoliton solution of multicomponent case.

To write down the multicomponent multisoliton solutions in a formal way we define the following \((1 \times n)\) row matrix \( C_s \), \((N \times 1)\) column matrices \( \psi_j \), \( \phi \),
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$j, s = 1, 2, ..., N$, and the \((n \times n)\) matrix \(\sigma\):

\[
C_s = - \left( \alpha_1^{(s)}, \alpha_2^{(s)}, ..., \alpha_n^{(s)} \right), \quad \psi_j = \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \\ \vdots \\ \alpha_j^{(N)} \end{pmatrix}, \quad \phi = \begin{pmatrix} e^{\eta_1} \\ e^{\eta_2} \\ \vdots \\ e^{\eta_n} \end{pmatrix},
\]

\[
\sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}, \tag{3.26a}
\]

where \(\sigma_j, j = 1, 2, ..., N\), can take value either +1 or −1 and \(\alpha_j^{(s)}, s = 1, 2, ..., N, j = 1, 2, ..., n\). Then the \(n\)-soliton solution of \(N\)-CNLS system (1.42) with mixed signs of nonlinearities can be written as

\[
q_s = g^{(s)} \frac{D}{D}, \quad s = 1, 2, 3, ..., N, \tag{3.26b}
\]

where

\[
g^{(s)} = \begin{vmatrix} A & I & \phi \\ -I & B & 0 \\ 0 & C_s & 0 \end{vmatrix}, \quad D = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \tag{3.26c}
\]

in which \(s\) denotes the component. Here \(I\) is \((n \times n)\) unit matrix and the \((n \times n)\) matrices \(A\) and \(B\) are defined as

\[
A_{i,j} = \frac{e^{\eta_i + \eta_j^*}}{k_i + k_j}, \quad B_{i,j} = \kappa_{ji} = \frac{\mu (\psi_i \psi_j^* \sigma)}{k_i + k_j}, \quad i, j = 1, 2, ..., n, \tag{3.26d}
\]

where \(\eta_i = k_i(t + ik_i z)\), \(k_i\) is complex, \(\dagger\) represents the transpose conjugate. Here we remark that though presenting the solutions in determinant form seems to be compact, one has to explicitly write down the solutions for a complete
characterization and analysis of the solution. This way of expressing the solutions explicitly is also useful to identify the particular parameter choice for which the singular stationary \( n \)-soliton solution of \( N \)-component case results from the general solutions.

### 3.4.2 Proof of multisoliton solutions

By applying the transformation (3.1) to the Eq. (1.42) with the choices of \( \sigma_l \) as in Eq. (1.42b), we arrive at a following bilinear equations,

\[
(iD_z + D^2_t) (g^{(s)} \cdot f) = 0, \quad s = 1, 2, \ldots, N, \tag{3.27a}
\]
\[
D^2_t (f \cdot f) = 2 \sum_{s=1}^{N} \sigma_s g^{(s)} g^{(s)*}. \tag{3.27b}
\]

In this subsection \( \sigma_s \) is defined as in Eq. (1.42b). The bright-bright multisoliton solution of the mixed N-CNLS equations is given by \( q_s = g^{(s)} f \). Here, the \( \sigma \) matrix in \( B_{ij} \) is redefined as \( \sigma_l = 1 \) for \( l = 1, 2, \ldots, m \) and \( \sigma_l = -1 \) for \( l = m+1, m+2, \ldots, N \). However, the bright soliton solution is regular as long as the condition \( B_{ij} > 0 \) is valid. The derivatives \( g_t^{(s)}, f_t, g_z^{(s)}, g_{tt}^{(s)} \) and \( f_z \) can be derived as below (see also [100]):

\[
g_t^{(s)} = \begin{vmatrix}
A & I & \phi & \phi_t \\
-I & B & 0^T & 0^T \\
0 & C_s & 0 & 0 \\
0 & 0 & -1 & 0
\end{vmatrix}, \quad f_t = \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0
\end{vmatrix}, \tag{3.28a}
\]

\[
f_{tt} = \begin{vmatrix}
A & I & \phi_t \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0
\end{vmatrix} + \begin{vmatrix}
A & I & \phi \\
-I & B & 0^T \\
-\phi^\dagger & 0 & 0
\end{vmatrix}, \tag{3.28b}
\]
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$$g_z^{(s)} = i \begin{vmatrix} A & I & \phi & \phi_t \\ -I & B & 0^T & 0^T \\ 0 & C_s & 0 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix} - i \begin{vmatrix} A & I & \phi & \phi_t \\ -I & B & 0^T & 0^T \\ 0 & C_s & 0 & 0 \\ \phi^\dagger & 0 & 0 & 0 \end{vmatrix},$$ (3.28c)

$$g_{tt}^{(s)} = \begin{vmatrix} A & I & \phi & \phi_t \\ -I & B & 0^T & 0^T \\ 0 & C_s & 0 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} A & I & \phi & \phi_t \\ -I & B & 0^T & 0^T \\ 0 & C_s & 0 & 0 \\ \phi^\dagger & 0 & 0 & 0 \end{vmatrix},$$ (3.28d)

and

$$f_z = i \begin{vmatrix} A & I & \phi_t \\ -I & B & 0^T \\ -\phi^\dagger & 0 & 0 \end{vmatrix} - i \begin{vmatrix} A & I & \phi \\ -I & B & 0^T \\ -\phi^\dagger & 0 & 0 \end{vmatrix}. $$ (3.28e)

The conjugate of $g^{(s)}$ can be written as

$$g^{(s)*} = - \begin{vmatrix} A & I & 0^T \\ -I & B & -C_s^\dagger \\ -\phi^\dagger & 0 & 0 \end{vmatrix}. $$ (3.28f)

As the forms of $g^{(s)}$ and $f$ for focusing and mixed case are the same except for the $B_{ij}$’s, the first bilinear equation (3.27a) is indeed satisfied. Substituting for $g_z^{(s)}$, $g_t^{(s)}$, $g_{tt}^{(s)}$, $f_z$, $f_t$, and $f_{tt}$ in equation (3.27a), we get
This is nothing but a Jacobian identity and hence $g^{(s)}$ and $f$ satisfy the first bilinear equation (3.27a). Considering the second bilinear equation we note that $f_t$ and $f_{tt}$ can be written for the mixed case as

$$f_t = \sum_{s=1}^{N} \sigma_s \begin{vmatrix} A & I & \phi & 0^T \\ -I & B & 0^T & 0^T \\ 0 & C_s & 0 & 0 \\ \phi^T & 0 & 0 & 0 \end{vmatrix},$$

(3.29b)

$$f_{tt} = \sum_{s=1}^{N} \sigma_s \begin{vmatrix} A & I & \phi & 0^T \\ -I & B & 0^T & -C_s^T \\ 0 & C_s & 0 & 0 \\ -\phi^T & 0 & 0 & 0 \end{vmatrix},$$

(3.29c)

The second bilinear equation (3.27b) then gives the following Jacobian identities,
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$$\sum_{s=1}^{N} \sigma_s$$

$$\begin{bmatrix} A & I & \phi & 0^T \\ -I & B & 0^T & -C_s^\dagger \\ -\phi^\dagger & 0 & 0 & 0 \\ 0 & C_s & 0 & 0 \end{bmatrix} = \sum_{s=1}^{N} \sigma_s$$

$$\begin{bmatrix} A & I & 0^T \\ -I & B & -C_s^\dagger \\ 0 & C_s & 0 \\ -\phi^\dagger & 0 & 0 \end{bmatrix}$$

By regrouping the terms corresponding to $$\sigma_s = 1$$ and $$\sigma_s = -1$$, respectively, we identify that the regrouped expressions are Jacobian identities separately. One can deduce the previously discussed one, two, three soliton solutions by specializing the Gram determinant form as was done in the previous chapter for the case of focusing CNLS equations.

- One soliton solution:
  Specializing to the case of $$n = 1$$ in Eq. (3.26) so that the Gram determinants take the form

  $$g^{(s)} = \begin{vmatrix} A_{11} & 1 & e^{\eta} \\ -1 & B_{11} & 0 \\ 0 & -\alpha_1^{(s)} & 0 \end{vmatrix}, \quad f = \begin{vmatrix} A_{11} & 1 \\ -1 & B_{11} \end{vmatrix}, \quad (3.31)$$

  where $$A_{11} = \frac{e^{\eta_1} + \eta_1}{k_1 + \kappa_1}$$, and $$B_{11} = \kappa_{11} = \frac{\left(\sum_{s=1}^{N} \sigma_s|\alpha_1^{(s)}|^2\right)}{k_1 + \kappa_1^s}$$. Here $$\sigma_s$$'s, $$s = 1, 2, \ldots, N$$ can take the value either +1 or -1.

- Two-soliton solution:
  To obtain the two soliton solution, we take $$n = 2$$ in Eq. (3.26) and deduce
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the Gram determinant forms as

\[
g^{(s)} = \begin{vmatrix}
A_{11} & A_{12} & 1 & 0 & e^{\eta^1} \\
A_{21} & A_{22} & 0 & 1 & e^{\eta^2} \\
-1 & 0 & B_{11} & B_{12} & 0 \\
0 & -1 & B_{21} & B_{22} & 0 \\
0 & 0 & -\alpha_{1}^{(s)} & -\alpha_{2}^{(s)} & 0
\end{vmatrix}, \quad f = \begin{vmatrix}
A_{11} & A_{12} & 1 & 0 \\
A_{21} & A_{22} & 0 & 1 \\
-1 & 0 & B_{11} & B_{12} \\
0 & -1 & B_{21} & B_{22} \\
\end{vmatrix},
\] (3.32)

where \( A_{ij} = \frac{e^{\eta^i + \eta^j}}{k_i + k_j} \), and \( B_{ij} = \kappa_{ji} = \frac{\sum_{s=1}^{N} \sigma_s \alpha_{j}^{(s)}, \alpha_{i}^{(s)^*}}{(k_j + k_i^*)} \), \( i, j = 1, 2 \). And \( \sigma_s \)'s, \( s = 1, 2, \ldots, N \) can take the value either +1 or -1.

- Three-soliton solution:

Similarly for three soliton solution, we put \( n = 3 \) in Eq. (3.26) and the Gram determinants take the forms as

\[
g^{(s)} = \begin{vmatrix}
A_{11} & A_{12} & A_{13} & 1 & 0 & 0 & e^{\eta^1} \\
A_{21} & A_{22} & A_{23} & 0 & 1 & 0 & e^{\eta^2} \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 1 & e^{\eta^3} \\
-1 & 0 & 0 & B_{11} & B_{12} & B_{13} & 0 \\
0 & -1 & 0 & B_{21} & B_{22} & B_{23} & 0 \\
0 & 0 & -1 & B_{31} & B_{32} & B_{33} & 0 \\
0 & 0 & 0 & -\alpha_{1}^{(s)} & -\alpha_{2}^{(s)} & -\alpha_{3}^{(s)} & 0
\end{vmatrix}, \quad (3.33a)
\]

\[
f = \begin{vmatrix}
A_{11} & A_{12} & A_{13} & 1 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 1 & 0 \\
A_{31} & A_{32} & A_{33} & 0 & 0 & 1 \\
-1 & 0 & 0 & B_{11} & B_{12} & B_{13} \\
0 & -1 & 0 & B_{21} & B_{22} & B_{22} \\
0 & 0 & -1 & B_{31} & B_{32} & B_{32}
\end{vmatrix}, \quad (3.33b)
\]
where $A_{ij} = \frac{e^{\eta_{ij}}}{k_i + k_j^*}$, and $B_{ij} = \kappa_{ji} = \frac{\left(\sum_{s=1}^{N} \sigma_s \alpha_j^{(s)} \alpha_i^{(s)*}\right)}{(k_j + k_i^*)}$, $i, j = 1, 2, 3$. Here $\sigma_s$'s, $s = 1, 2, \ldots, N$ can take the value either +1 or -1.