CHAPTER 2

k-idempotent matrices

A k-idempotent matrix is defined and some of its basic characterizations are derived (see [33]) in this chapter. It is shown that if $A$ is a $k$-idempotent matrix then it is quadripotent (i.e., $A^4 = A$). Necessary and sufficient condition for the sum of two $k$-idempotent matrices to be $k$-idempotent, is determined and then it is generalized for the sum of ‘$n’$ $k$-idempotent matrices. A condition for the product of two $k$-idempotent matrices to be $k$-idempotent is also determined and then it is generalized for the product of ‘$n’$ $k$-idempotent matrices. Relations between power hermitian matrices ($A^n = A^*$) and $k$-idempotent matrices are investigated (cf. [32]). It is proved that a $k$-idempotent matrix $A$ reduces to an idempotent matrix when it commutes with the associated permutation matrix $K$ (i.e., $AK = KA$).
2.1 Characterizations of $k$-idempotent matrices

A $k$-idempotent matrix is defined and its characterizations are discussed in this section.

**Definition 2.1.1**

For a fixed product of disjoint transpositions $k \in S_n$, a matrix $A = \{a_{ij}\} \in \mathbb{C}^{n \times n}$ is said to be $k$-idempotent if $\sum_{t=1}^{n} a_{k(i)t} a_{t k(j)} = a_{ij}$. This is equivalent to $KA^2K = A$, where $K$ is the associated permutation matrix of ‘$k$’.

**Example 2.1.2**

If $A = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & -2 & -\sqrt{3} & 0 \\
0 & \sqrt{3} & 1 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}$

Then $A^2 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 1 & \sqrt{3} & 0 \\
0 & -\sqrt{3} & -2 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}$

Here $A$ is a $k$-idempotent matrix with $k = \langle 1,4 \rangle \langle 2,3 \rangle$. The associated permutation matrix $K$ is a matrix with ones on its southwest – northeast diagonal and zeros everywhere else. That is,

$$K = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

It can be easily verified that $KA^2K = A$.

**Note 2.1.3**

In particular if $k(i) = i$ then the associated permutation matrix $K$ reduces to identity matrix and $k$-idempotent matrix reduces to idempotent matrix.
Remark 2.1.4

$KA^2K = A$ implies that $KAK = A^2$. The following relations can also be obtained which would be useful in computational aspects

$$KA = A^2K \quad \text{or} \quad KA^2 = AK$$

$$KA^3 = A^3K \quad \text{or} \quad KA^3K = A^3$$

$$A^3 = (KA)^2 \quad \text{or} \quad (AK)^2$$

Theorem 2.1.5

Let $A$ be a $k$-idempotent matrix. Then $I - A$ is $k$-idempotent if and only if $A$ is idempotent.

Proof

$$I - A = K(I - A)^2K$$

$$= K(I - 2A + A^2)K$$

$$= I - 2A^2 + A$$

Hence $2(A - A^2) = 0$, which implies that $A$ is idempotent.

Conversely, if $A$ is idempotent then $A$ commutes with the permutation matrix $K$ (cf. lemma 2.2.5)

$$K(I - A)^2K = K(I - 2A + A^2)K$$

$$= K(I - A)K$$

$$= I - A$$

Hence $I - A$ is $k$-idempotent.

Remark 2.1.6

Consider $k = (1,2)$. Then the associated permutation matrix $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. With reference to this $k$,

(i) $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$ is $k$-idempotent as well as idempotent.
\[ I - A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ is } k\text{-idempotent.} \]

(ii) \[ B = \begin{pmatrix} \frac{-1+i\sqrt{7}}{2} & -i \\ i & \frac{-1-i\sqrt{7}}{2} \end{pmatrix} \text{ is } k\text{-idempotent but not idempotent.} \]

\[ I - B = \begin{pmatrix} \frac{3-i\sqrt{7}}{2} & i \\ -i & \frac{3+i\sqrt{7}}{2} \end{pmatrix} \text{ is not } k\text{-idempotent.} \]

**Theorem 2.1.7**

Let \( A \) be a \( k\)-idempotent matrix. Then

(a) \( A^*, \overline{A}, A^T \) and \( A^{-1} \) (when it exists) are also \( k\)-idempotent.

(b) \( A^n \) is \( k\)-idempotent for all positive integers ‘\( n \)’.

(c) \( A \) is quadripotent when \( A \) is not an idempotent. Further, If \( A \) is non-singular then \( A^3 = I \) and \( A^2 = A^{-1} \).

(d) \( A^3 \) is idempotent.

(e) \( KA \) and \( AK \) are tripotent matrices.

**Proof**

(a) \( A^* = (KA^2K)^* \)

\[ = KA^*A^*K \]

\[ = K(A^*)^2K \]

A similar proof may be given for the remaining matrices.

(b) \( A^n = (KA^2K)^n \)

\[ = KA^2KKA^2K \ldots n \text{- times} \]

\[ = KA^{2n}K \]

\[ = K(A^n)^2K \]

(c) \( A^4 = A^2A^2 \)

\[ = KAKKAK \)
Since $A^2 \neq A$, we have $A$ is quadripotent.
If $A$ is non-singular then $A^3 = I$ and $A^2 = A^{-1}$ are immediate consequences of $A^4 = A$.

(d) \[(A^3)^2 = [(KA)^2]^2 \]
\[= KAKAKAKA \]
\[= KAA^2AKA \]
\[= KAKA \]
\[= A^3 \]

(e) \[(KA)^3 = KAKAKA \]
\[= KAA^2A \]
\[= KA \]

The proof is similar for $AK$.

Example 2.1.8
Consider the matrix $B$ in (ii) of remark 2.1.6. The following observations can be made.

(i) $B$ is (1,2)-idempotent.

(ii) $B^* = \begin{pmatrix} \frac{-1-i\sqrt{7}}{2} & -i \\ i & \frac{-1+i\sqrt{7}}{2} \end{pmatrix}$ is also (1,2)-idempotent.

(iii) It can be seen that $B^4 = B$.

(iv) Clearly, $B^3 (= I)$ is idempotent.

(v) $KB = \begin{pmatrix} i & \frac{-1-i\sqrt{7}}{2} \\ \frac{-1+i\sqrt{7}}{2} & -i \end{pmatrix}$ and $BK = \begin{pmatrix} -i & \frac{-1+i\sqrt{7}}{2} \\ \frac{-1-i\sqrt{7}}{2} & i \end{pmatrix}$ are tripotent matrices.

Theorem 2.1.9
If $A$ is a $k$-idempotent matrix then $G = \{ A, A^2, A^3, KA, AK, KA^3 \}$ is a group under matrix multiplication.
Proof

Using the remark 2.1.4, the following matrix multiplication table can be found.

<table>
<thead>
<tr>
<th>.</th>
<th>A</th>
<th>A^2</th>
<th>A^3</th>
<th>KA</th>
<th>AK</th>
<th>KA^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A^2</td>
<td>A^3</td>
<td>A</td>
<td>KA^3</td>
<td>KA</td>
<td>AK</td>
</tr>
<tr>
<td>A^2</td>
<td>A^3</td>
<td>A</td>
<td>A^2</td>
<td>AK</td>
<td>KA^3</td>
<td>KA</td>
</tr>
<tr>
<td>A^3</td>
<td>A</td>
<td>A^2</td>
<td>A^3</td>
<td>KA</td>
<td>AK</td>
<td>KA^3</td>
</tr>
<tr>
<td>KA</td>
<td>AK</td>
<td>KA^3</td>
<td>KA</td>
<td>A^3</td>
<td>A</td>
<td>A^2</td>
</tr>
<tr>
<td>AK</td>
<td>KA^3</td>
<td>KA</td>
<td>AK</td>
<td>A^2</td>
<td>A^3</td>
<td>A</td>
</tr>
<tr>
<td>KA^3</td>
<td>KA</td>
<td>AK</td>
<td>KA^3</td>
<td>A</td>
<td>A^2</td>
<td>A^3</td>
</tr>
</tbody>
</table>

Table 1.3

From the above table, we see that \( G \) is closed under matrix multiplication.

It is obvious that matrix multiplication is associative.

We observe that \( A^3 \) acts as an identity element in \( G \). Besides, it is the only element in \( G \) having this property. The inverse for each elements are given by,

\[
A^{-1} = A^2 \quad ; \quad (A^2)^{-1} = A \quad ; \quad (A^3)^{-1} = A^3 \quad ; \quad (KA)^{-1} = KA
\]

\[
(AK)^{-1} = AK \quad \text{and} \quad (KA^3)^{-1} = KA^3
\]

Hence \( G \) is a group under matrix multiplication. \( \blacksquare \)

**Remark 2.1.10**

(i) If \( H = \{ A, A^2, A^3 \} \) then \( H \) is a cyclic subgroup of \( G \).

<table>
<thead>
<tr>
<th>.</th>
<th>A</th>
<th>A^2</th>
<th>A^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A^2</td>
<td>A^3</td>
<td>A</td>
</tr>
<tr>
<td>A^2</td>
<td>A^3</td>
<td>A</td>
<td>A^2</td>
</tr>
<tr>
<td>A^3</td>
<td>A</td>
<td>A^2</td>
<td>A^3</td>
</tr>
</tbody>
</table>

Table 1.4

(ii) If \( A \) is non-singular then by theorem 2.1.7 (c), the group \( G \) becomes

\( G = \{ A, A^2, I, KA, AK, K \} \).
2.2 Sums and Products of $k$-idempotent matrices

In this section, the sums and products of $k$-idempotent matrices are discussed and some related results are obtained.

**Theorem 2.2.1**

Let $A$ and $B$ be two $k$-idempotent matrices. Then $A + B$ is $k$-idempotent if and only if $AB = -BA$.

**Proof**

\[
A + B = KA^2K + KB^2K = K(A^2 + B^2)K = K(A + B)^2K \text{ iff } AB = -BA
\]

**Generalization:**

Let $A_1, A_2, ..., A_n$ be $k$-idempotent matrices. Then $\sum_{i=1}^{n} A_i$ is $k$-idempotent if and only if $\sum_{i \neq j} A_iA_j = 0$ for $i$ and $j$ in \{1, 2, ..., $n$\}.

**Proof**

\[
K \left( \sum_{i=1}^{n} A_i \right)^2 = K \left( \sum_{i=1}^{n} \sum_{j=1}^{n} A_iA_j \right)K = K \left( \sum_{i=1}^{n} A_i^2 \right)K + K \left( \sum_{i \neq j} A_iA_j \right)K
\]

\[
= \sum_{i=1}^{n} KA_i^2K + K \left( \sum_{i \neq j} A_iA_j \right)K
\]

Since $A_i$’s are $k$-idempotent matrices, we have

\[(2.1) \quad K \left( \sum_{i=1}^{n} A_i \right)^2 = \sum_{i=1}^{n} A_i + K \left( \sum_{i \neq j} A_iA_j \right)\]

Here $i, j \in \{1, 2, ..., n\}$
If $\sum_{i=1}^{n} A_i$ is $k$-idempotent then from (2.1),

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i + K \left( \sum_{i \neq j} A_iA_j \right) K$$

Hence, it follows that $\sum_{i \neq j} A_iA_j = 0$.

Conversely, if we assume that $\sum_{i \neq j} A_iA_j = 0$ then from (2.1)

$$K \left( \sum_{i=1}^{n} A_i \right)^2 = \sum_{i=1}^{n} A_i$$

Hence $\sum_{i=1}^{n} A_i$ is $k$-idempotent.

Remark 2.2.2

Theorem 2.2.1 fails when we relax the condition that $A$ and $B$ any commute. For example,

Let $A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$ and $B = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

Clearly, $A$ and $B$ are (1,2)-idempotent matrices.

But $AB = \begin{pmatrix} -1+\sqrt{3} & 1-\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$ and $BA = \begin{pmatrix} -1-\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1-\sqrt{3} \end{pmatrix}$

i.e., $AB \neq -BA$.

Also, $A + B$ is not a (1,2)-idempotent matrix.
**Theorem 2.2.3**

Let $A$ and $B$ be $k$-idempotent matrices. If $AB = BA$ then $AB$ is also a $k$-idempotent matrix.

**Proof**

\[
AB = KA^2KKB^2K
\]

\[
= KAA^2B^2K
\]

\[
= KAABBK
\]

\[
= K(AB)^2K \quad \text{[by } AB = BA]\]

Hence the matrix $AB$ is $k$-idempotent.

**Generalization:**

If $A_1, A_2, \ldots, A_n$ be $k$-idempotent matrices belonging to a commuting family of matrices then $\prod_{i=1}^{n} A_i$ is a $k$-idempotent matrix.

**Proof**

\[
K \left( \prod_{i=1}^{n} A_i \right)^2 K = K(A_1A_2 \ldots A_nA_1A_2 \ldots A_n)K
\]

\[
= KA_1^2 A_2^2 \ldots A_n^2 K
\]

\[
= KA_1^2 KKA_2^2 K \ldots KKA_n^2 K
\]

\[
= A_1A_2 \ldots A_n
\]

\[
= \prod_{i=1}^{n} A_i
\]

Hence the matrix $\prod_{i=1}^{n} A_i$ is $k$-idempotent.

**Remark 2.2.4**

If we relax the condition of commutability of matrices $A$ and $B$ in *Theorem 2.2.3* then the product need not be $k$-idempotent. For example the matrices $A$ and $B$ in *Remark 2.2.2* can be considered.

It can be seen that $AB \neq BA$. Also the product $AB$ is not a $k$-idempotent matrix.
If \([A, B]\) denotes the commutator of the matrices \(A\) and \(B\) (see definition 1.2.19), theorems 2.2.1 and 2.2.3 can be restated as

If \(A\) and \(B\) are two \(k\)-idempotent matrices then

\[ A + B \text{ is } k\text{-idempotent if and only if } [A, B] = 2AB. \]

\[ AB \text{ is } k\text{-idempotent if } [A, B] = 0. \]

By theorem 1.2.1, the generalization of theorem 2.2.3 can also be restated as

‘For \(k\)-idempotent matrices \(A_1, A_2, \ldots, A_n\), if there is a unique matrix \(S\) such that \(S^{-1}A_i S = \text{diag}(A_i)\) then the product \(\prod_{i=1}^{n} A_i\) is \(k\)-idempotent’.

**Lemma 2.2.5**

Let \(A\) be a \(k\)-idempotent matrix. Then \(A\) is idempotent if and only if \(AK = KA\), where \(K\) is the associated permutation matrix of ‘\(k\)’.

**Proof**

Assume that \(AK = KA\).

Pre multiplying by \(K\), we have \(KAK = A\)

But \(A^2 = A\) \hspace{1cm} \([A \text{ is } k\text{-idempotent}]\)

Hence \(A\) is idempotent. By retracing the above steps the converse follows. \(\blacksquare\)

**Example 2.2.6**

\[
A = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

is a \((1,2)\)-idempotent matrix and it also commutes with the associated permutation matrix \(K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), that is \(AK = KA\). We see that \(A\) is idempotent. Lemma 2.2.5 fails if we relax the condition of commutability of matrices \(A\) and \(K\). Examples 2.1.2 and 2.1.8 are not idempotents. Note that \(AK \neq KA\) in such cases.

**Theorem 2.2.7**

If \(A\) and \(B\) are \(k\)-idempotent matrices then \(A(A + B)B\) commutes with the permutation matrix \(K\).
Proof

\[ A(A + B)B = A^2B + AB^2 \]
\[ = KAKB + AKBK \]
\[ = KAB^2K + KA^2BK \]
\[ = K(AB^2 + A^2B)K \]
\[ = K(A^2B + AB^2)K \]
\[ = KA(A + B)BK \]

Hence \( KA(A + B)B = A(A + B)BK \) \( \blacksquare \)

**Theorem 2.2.8**

Let \( A \) and \( B \) are two commuting \( k \)-idempotent matrices. The \( k \)-idempotency of \( A(A + B)B \) necessarily implies that it is a null matrix.

Proof

For any two \( k \)-idempotent matrices \( A \) and \( B \) we have \( A(A + B)B \) commutes with the permutation matrix \( K \) by *theorem* 2.2.7.

If \( A(A + B)B \) is \( k \)-idempotent then by *lemma* 2.2.5, it reduces to an idempotent matrix.

i.e., \( [A(A + B)B]^2 = A(A + B)B \)

(2.2) \[ (A^2B)^2 + (AB^2)^2 + A^2BAB^2 + AB^2A^2B = A^2B + AB^2 \]

Since \( A \) and \( B \) are \( k \)-idempotent matrices, we have \( A^4 = A \) and \( B^4 = B \) by *theorem* 2.1.7 (c). Substituting in (2.2),

\[ AB^2 + A^2B + A^3B^3 + A^3B^3 = A^2B + AB^2 \]

\[ 2A^3B^3 = 0 \]
\[ A^3B^3 = 0 \]

(2.3) \[ (AB)^3 = 0 \] \[ \text{[ by } AB = BA\text{]} \]

Since \( A \) and \( B \) are commuting \( k \)-idempotent matrices, \( AB \) is also \( k \)-idempotent by *theorem* 2.2.3. Hence \( (AB)^4 = AB \). \[ \text{[ by *theorem* 2.1.7 (c) ]} \]
Pre multiplying (2.3) by $AB$, we have $AB = 0$.

It follows that $A(A + B)B = 0$. ■

**Remark 2.2.9**

For example the matrices $A$ and $B$ in *remark 2.2.2* can be considered.

$$A(A + B)B = A^2B + AB^2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$ Clearly the matrix $A(A + B)B$ commutes with the permutation matrix $K$. It can be easily verified that $A(A + B)B$ is not a $k$-idempotent matrix.
2.3 \( k \)-idempotency of power hermitian matrices

In this section, conditions for power hermitian matrices to be \( k \)-idempotent are derived and some related results are given.

**Theorem 2.3.1**

Any two of the following imply the other one. If \( A \in \mathbb{C}^{n \times n} \) then

(a) \( A \) is \( k \)-idempotent
(b) \( A \) is \( k \)-hermitian
(c) \( A \) is square hermitian

**Proof**

(a) and (b) \( \Rightarrow \) (c):

\[
KA^2K = A \quad \text{and} \quad KA^*K = A
\]

\[
KA^2K = KA^*K
\]

\[A^2 = A^*.\] Hence \( A \) is square hermitian.

(b) and (c) \( \Rightarrow \) (a):

Substituting \( A^2 = A^* \) in \( KA^*K = A \),

We have \( KA^2K = A \). Hence \( A \) is \( k \)-idempotent.

(c) and (a) \( \Rightarrow \) (b):

Substituting \( A^2 = A^* \) in \( KA^2K = A \),

We have \( KA^*K = A \). Hence \( A \) is \( k \)-hermitian.

**Corollary 2.3.2**

Let \( A \) be \( k \)-hermitian \( k \)-idempotent matrix. If \( A \) is non-singular then \( A \) is unitary.

**Proof**

Since \( A \) is \( k \)-hermitian \( k \)-idempotent, \( A \) is square hermitian \([\text{by theorem 2.3.1}]\)

If \( A \) is non-singular then \( A^2 = A^{-1} \) \([\text{by theorem 2.1.7 (c)}]\)

Therefore \( A^* = A^{-1} \) and hence \( A \) is unitary.
Example 2.3.3

Let \( A = \begin{pmatrix} \frac{1+i\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1-i\sqrt{2}}{2} \end{pmatrix} \)

(i) \( A \) is \((1,2)\)-idempotent.
(ii) \( A \) is square Hermitian.
(iii) \( A \) is \((1,2)\)-Hermitian.
(iv) \( A \) is unitary

Theorem 2.3.4

Let \( A \) be a \( k \)-idempotent matrix. If \( A \) is cube Hermitian then it reduces to an orthogonal projector.

Proof

Since \( A \) is a \( k \)-idempotent matrix, we have

\[(2.4) \quad KA^3K = A^3 \quad \text{[by remark 2.1.4]}\]

If \( A \) is cube Hermitian then

\[(2.5) \quad A^3 = A^* \]

Pre and post multiplying (2.5) by \( K \), we have \( KA^3K = KA^*K \)

\[A^3 = KA^*K \quad \text{[by (2.4)]}\]

\[A^* = KA^*K \quad \text{[by (2.5)]}\]

\[A = KAK\]

\[A = A^2\]

Substituting in (2.5), we have \( A = A^* \). Hence \( A \) is an orthogonal projector. \( \blacksquare \)
Corollary 2.3.5

Let $A$ be a $k$-idempotent matrix. If $A$ is $k$-cube hermitian then it reduces to an orthogonal projector.

Proof
If $A$ is $k$-cube hermitian then $KA^*K = A^3$

$$A^* = KA^3K$$

$$A^* = A^3 \quad \text{[by remark 2.1.4]}$$

Hence $A$ reduces to cube hermitian matrix. By theorem 2.3.4, the matrix $A$ is an orthogonal projector.

Note 2.3.6

Let $A$ be a $k$-idempotent matrix. If $A^4 = A^*$ then by theorem 2.1.7 (c) we have $A = A^*$ (i.e., $A$ reduces to hermitian matrix).

Theorem 2.3.7

Let $A$ be a $k$-idempotent matrix. Then the following are equivalent.

(i) $KA$ is cube hermitian.

(ii) $KA$ is hermitian.

(iii) $A$ is square hermitian.

Proof

(i)$\Rightarrow$ (ii):
If $KA$ is cube hermitian then $(KA)^3 = (KA)^*$

$$KA = (KA)^* \quad \text{[by theorem 2.1.7 (e)]}$$

Hence $KA$ is hermitian.

(ii)$\Rightarrow$ (iii):
If $KA$ is hermitian then $(KA)^* = KA$

$$A^*K = A^2K$$

$$A^* = A^2 \quad \text{. Hence } A \text{ is square hermitian.}$$
(iii)⇒ (ii):

If $A$ is square hermitian then $A^2 = A^*$

But $(KA)^3 = KA$

$= A^2 K$

$= A^* K$

$= (KA)^*$

Hence $KA$ is cube hermitian.

\[\square\]

**Theorem 2.3.8**

Let $A$ be a $k$-idempotent matrix. Then the necessary and sufficient condition for the matrix $KA$ to be square hermitian is

(i) $A$ is idempotent.

(ii) $A = A^* K$

**Proof**

Assume that $KA$ is square hermitian.

\[(KA)^2 = (KA)^*\]

\[(2.6) \quad A^3 = A^* K\]

Pre and post multiplying (2.6) by $K$ respectively,

we have $KA^3 = KA^* K$ and $A^3 K = A^*$

since $KA^3 = A^3 K$, we have $KA^* K = A^*$

$KAK = A$

$A^2 = A$.

Hence $A$ is idempotent and substituting this in (2.6), we have $A = A^* K$.

Conversely, assume that $A$ is idempotent with $A = A^* K$. 
\[(KA)^2 = A^3\]
\[= A\]
\[= A^*K\]
\[= (KA)^*\]

Hence \(KA\) is square hermitian.

**Corollary 2.3.9**

Let \(A\) be a \(k\)-idempotent matrix. Then the necessary and sufficient condition for \(KA\) to be \(k\)-square hermitian is

(i) \(A\) is idempotent

(ii) \(A = A^*K\)

**Proof**

Assume that \(KA\) is \(k\)-square hermitian.

\[\iff K(KA)^2K = (KA)^*\]
\[\iff KA^3K = (KA)^*\]
\[\iff A^3 = (KA)^*\]
\[\iff (KA)^2 = (KA)^*\]

\[\iff KA\] is square hermitian.

By theorem 2.3.8, this is equivalent to \(A\) is idempotent and \(A = A^*K\)

**Theorem 2.3.10**

Let \(A\) be a \(k\)-idempotent matrix. Then the following are equivalent.

(i) \(KA\) is \(k\)-cube hermitian

(ii) \(A\) is hermitian

(iii) \(A\) is \(k\)-square hermitian

(iv) \(KA\) is \(k\)-hermitian
\textbf{Proof}

(i) $\Rightarrow$ (ii):

If $KA$ is $k$-cube hermitian then $K(KA)^3K = (KA)^*$

$$KKAK = A^*K \quad \text{[by theorem 2.1.7 (e)]}$$

$$A = A^*.$$  

Hence $A$ is hermitian.

(ii) $\Rightarrow$ (iii):

If $A$ is hermitian then $A^* = A$

$$A^* = KA^2K.$$  

Hence $A$ is $k$-square hermitian.

(iii) $\Rightarrow$ (iv):

If $A$ is $k$-square hermitian then $KA^2K = A^*$

$$KA^2 = A^*K$$

$$KA^2 = (KA)^*$$

Pre and post multiplying by $K$, we have

$$A^2K = K(KA)^*K$$

$$KA = K(KA)^*K.$$  

Hence $KA$ is $k$-hermitian.

(iv) $\Rightarrow$ (i):

If $KA$ is $k$-hermitian then $KA = K(KA)^*K$

But $(KA)^3 = KA$  \hspace{1cm} \text{[by theorem 2.1.7 (e)]}

$(KA)^3 = K(KA)^*K.$

Hence $KA$ is $k$-cube hermitian.  \hfill $\blacksquare$
Note 2.3.11

The following theorem can be considered to be a converse of the above theorem 2.3.10.

Theorem 2.3.12

If $A$ is hermitian and $k$-square hermitian then $A$ is $k$-idempotent.

Proof

Assume that $A^* = A$ and $KA^2K = A^*$

Combining the above two relations, we have $KA^2K = A$.

Hence $A$ is $k$-idempotent. \hfill \blacksquare