CHAPTER 5

$k$-idempotency of linear combinations of an idempotent matrix and a tripotent matrix

In this chapter, it is proved that $k$-idempotent matrices are $\{3\}$-group periodic. A set of necessary and sufficient conditions for a linear combinations $C = c_1A + c_2B$ of two commutative idempotent matrices $A$ and $B$ to be $k$-idempotent, is listed analogous to theorem 1.2.11. Then it is generalized to the problem of characterizing all situations in which the linear combination $C = c_1A + c_2B$ (where $A$ is an idempotent matrix and $B$ is a tripotent matrix) to be $k$-idempotent, is thoroughly studied analogous to theorem 1.2.10.
5.1 \{3\}-group periodicity of \(k\)-idempotent matrices

In this section, first we define a \(\{k\}\)-group periodic matrices (cf.[13]) and then it is to be proved that every \(k\)-idempotent matrices are \(\{3\}\)-group periodic.

**Definition 5.1.1**

Let \(A^\#\) be the group inverse of a matrix \(A \in \mathbb{C}^{n \times n}\). If \(A^\# = A^{k-1}\) for \(k = 2,3,\ldots\), then \(A\) is called \(\{k\}\)-group periodic matrix.

**Example 5.1.2**

\[
A = \begin{pmatrix}
-1 & i & 1 \\
0 & 1 & 0 \\
-1 & 2i & 0
\end{pmatrix}
\]

is a \(\{3\}\)-group periodic matrix.

It can be verified that \(A^\# = A^2 = \begin{pmatrix} 0 & 2i & -1 \\ 0 & 1 & 0 \\ 1 & i & -1 \end{pmatrix}\)

**Note 5.1.3**

If \(A\) is a \(\{k\}\)-group periodic matrix then we see that this is equivalent to \(A^{k+1} = A\) for \(k = 1,2,\ldots\).

Hence we denote the set of all \(\{k\}\)-group periodic matrices by \(\mathcal{S}^\#(k)\) and is defined as

\[
\mathcal{S}^\#(k) = \{ A \in \mathbb{C}^{n \times n} : A^{k+1} = A \} \text{ for } k = 1,2,\ldots
\]

**Remark 5.1.4**

If \(A\) is a \(\{k\}\)-group periodic matrix then \(A^{k+1} = A\) (by **note 5.1.3**). Hence the eigen values of \(A\) are \(\{0\} \cup \Omega_k\), where \(\Omega_k\) is the set of roots of unity of order \(k\). For example, the eigen values of \(A\) in example 5.1.2 are \(1, \omega\) and \(\omega^2\).

**Lemma 5.1.5**

If \(A\) is a \(k\)-idempotent matrix then \(A\) is \(\{3\}\)-group periodic.

**Proof**

\[
\text{rank}(A^2) = \text{rank}(KAK)
\]

\[= \text{rank}(A) \quad \text{[by } K \text{ is non-singular]}\]

Therefore group inverse exists and since \(A\) is quadripotent [by theorem 2.1.7 (c)], we have
\[ AA^2A = A^4 = A \]
\[ A^2AA^2 = A^5 = A^2 \]

Since \( AA^2 = A^2A \), we have \( A^\# = A^2 \)

That is \( A^\# = A^{3-1} \). Hence \( A \) is \( \{3\} \)-group periodic by definition 5.1.1.

**Note 5.1.6**

By *lemma 5.1.5*, we see that the group inverse of \( k \)-idempotent matrix found to be \( A^2 \). The group inverse of remaining matrices \( A^2, A^3, KA, AK, KA^3 \) of the group \( G \) (see *theorem 2.1.9*) are found in section 6.2.

**Remark 5.1.7**

The geometrical and topological aspects of \( S^n(3) \)—the set of all \( \{3\} \)-group periodic matrices as a special case of \( \{k\} \)-group periodic matrices (cf.[13]) can be seen that \( S^n(1) \) is a closed subset of \( S^n(3) \).

Here \( S^n(3) = \{ M \in \mathbb{C}^{n \times n} : M^4 = M \} \) and \( S^n(1) = \{ A \in \mathbb{C}^{n \times n} : A^2 = A \} \)

By defining the function as \( f : S^n(3) \rightarrow \mathbb{C}^{n \times n} \)

\[ f(M) = M^2 - M \]

It can be seen that \( f \) is continuous. Since \( S^n(1) = f^{-1}\{0\} \), we have \( S^n(1) \) is a closed subset of \( S^n(3) \). It is well known that the class of idempotent matrices \( S^n(1) \) is the intersection of class of quadripotent matrices \( S^n(3) \) and the class of tripotent matrices \( S^n(2) \).

**Theorem 5.1.8**

Let \( A \) be a \( k \)-idempotent matrix. Then

(i) \( A^2 \) is \( \{3\} \)-group periodic

(ii) \( A^3 \) is \( \{2\} \)-group periodic

(iii) \( KA \) is \( \{2\} \)-group periodic
(iv) $AK$ is $\{2\}$-group periodic

(v) $KA^3$ is $\{2\}$-group periodic

**Proof**

By *theorem 6.2.6*, we see that (i) to (v) can be easily proved as a consequence of definition of $\{k\}$-group periodic matrix.
5.2 \(k\)-idempotency of linear combinations of two commutative idempotent matrices

In this section, a complete solution is established to the problem of characterizing all situations in which a linear combination \(C = c_1A + c_2B\) is \(k\)-idempotent when \(A\) and \(B\) are two commutative idempotent matrices. A set of necessary and sufficient conditions for the matrix \(C\) to be \(k\)-idempotent is derived in theorem 5.2.3.

The following theorem 5.2.1 exhibits all the possibilities of non-zero complex numbers \(c_1\) and \(c_2\) such that \(c_1A + c_2B\) to be \(\{3\}\)-group periodic as a special case of \(\{k\}\)-group periodic matrices (compare [13]).

**Lemma 5.2.1**

Let \(A\) and \(B\) be two non-zero idempotent matrices of order \(n\) such that \(AB = BA\) and \(A \neq B\). Then for non-zero complex numbers \(c_1\) and \(c_2\) the matrix \(c_1A + c_2B\) is \(\{3\}\)-group periodic if and only if any one of the following possibilities hold.

1. \(c_1 \in \Omega_3\), \(c_1 + c_2 = 0\) and \(AB = B\)
2. \(c_1 \in \Omega_3\), \(c_1 + c_2 \in \Omega_3\) and \(AB = B\)
3. \(c_1 \in \Omega_3\), \(c_2 \in \Omega_3\) and \(AB = 0\)
4. \(c_2 \in \Omega_3\), \(c_1 + c_2 = 0\) and \(AB = A\)
5. \(c_2 \in \Omega_3\), \(c_1 + c_2 \in \Omega_3\) and \(AB = A\)

**Proof**

Since \(A\) and \(B\) are idempotent matrices, they are diagonalizable.

Since \(AB = BA\), by theorem 1.2.1. There exists a non-singular matrix \(S\) and two diagonal matrices \((D_A\) and \(D_B))\) such that

\[
D_A = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)
\]

\[
D_B = \text{diag}(\mu_1, \mu_2, ..., \mu_n)
\]

Here \(\lambda_i\) and \(\mu_i\) \((i = 1,2, ..., n)\) are the eigen values of \(A\) and \(B\) respectively.

Also, \(A = SD_A S^{-1}\) and \(B = SD_B S^{-1}\)

Since \(A\) and \(B\) are idempotents, we have \(\lambda_i\), \(\mu_i \in \{0,1\}\)

Assuming that \(c_1A + c_2B\) is \(\{3\}\)-group periodic.
(5.2) \[(c_1A + c_2B)^4 = c_1A + c_2B\]

But \[c_1A + c_2B = c_1SD_A S^{-1} + c_2SD_B S^{-1}\]

(5.3) \[c_1A + c_2B = S(c_1D_A + c_2D_B)S^{-1}\]

\[(c_1A + c_2B)^4 = S(c_1D_A + c_2D_B)^4S^{-1}\]

\[c_1A + c_2B = S(c_1D_A + c_2D_B)^4S^{-1}\] \[\text{[ by (5.2)]}\]

\[S(c_1D_A + c_2D_B)S^{-1} = S(c_1D_A + c_2D_B)^4S^{-1}\] \[\text{[ by (5.3)]}\]

\[(c_1D_A + c_2D_B)^4 = (c_1D_A + c_2D_B)\]

Therefore we have

\[(c_1\lambda_i + c_2\mu_i)^4 = (c_1\lambda_i + c_2\mu_i)\] for \(i = 1,2,\ldots,n\)

Hence \(c_1\lambda_i + c_2\mu_i = 0\) or \(c_1\lambda_i + c_2\mu_i \in \Omega_3\), That is

(5.4) \[c_1\lambda_i + c_2\mu_i \in \{0\} \cup \Omega_3\]

If \(\lambda_i = \mu_i\) for all \(i\) then \(D_A = D_B\)

\[SD_A S^{-1} = SD_B S^{-1}\]

\[A = B\]

—a contradiction. Hence there exists \(r \in \{1,2,\ldots,n\}\) such that \(\lambda_r \neq \mu_r\).

Since \(\lambda_r, \mu_r \in \{0,1\}\), we have the following possibilities

(i) \(\lambda_r = 1, \mu_r = 0\)

(ii) \(\lambda_r = 0, \mu_r = 1\)

(i) \(\lambda_r = 1, \mu_r = 0\):

By (5.4), we have \(c_1 \in \{0\} \cup \Omega_3\)

Since \(c_1 \neq 0\), we have \(c_1 \in \Omega_3\).
If \( \mu_i = 0 \) for all \( i \) then \( D_B = 0 \), which implies that \( B = 0 \). – a contradiction.

Hence we can find one \( s \in \{1, 2, \ldots, n\} \) such that \( \mu_s \neq 0 \) (i.e., \( \mu_s = 1 \)).

Correspondingly we have the following possibilities for \( \lambda_s \)

\[
\lambda_s = 1 \text{ or } \lambda_s = 0
\]

\( \lambda_s = 1 \) implies that \( c_1 + c_2 \in \{0\} \cup \Omega_3 \) [ by (5.4)]

\( \lambda_s = 0 \) implies that \( c_2 \in \{0\} \cup \Omega_3 \) [ by (5.4)]

Since \( c_2 \neq 0 \), we have \( c_2 \in \Omega_3 \).

The first three cases of the theorem is obtained.

(ii) \( \lambda_r = 0 \), \( \mu_r = 1 \):

By (5.4), we have \( c_2 \in \{0\} \cup \Omega_3 \)

Since \( c_2 \neq 0 \), we have \( c_2 \in \Omega_3 \).

If \( \lambda_i = 0 \) for all \( i \) then \( D_A = 0 \), which implies that \( A = 0 \). – a contradiction.

Hence we can find one \( s \in \{1, 2, \ldots, n\} \) such that \( \lambda_s \neq 0 \) (i.e., \( \lambda_s = 1 \)).

Correspondingly we have the following possibilities for \( \mu_s \)

\[
\mu_s = 1 \text{ or } \mu_s = 0
\]

\( \mu_s = 1 \) implies that \( c_1 + c_2 \in \{0\} \cup \Omega_3 \) [ by (5.4)]

\( \mu_s = 0 \) implies that \( c_1 \in \{0\} \cup \Omega_3 \) [ by (5.4)]

Since \( c_1 \neq 0 \), we have \( c_1 \in \Omega_3 \).

The remaining two cases of theorem is now obtained.

Now, we define \( Q = (c_1 A + c_2 B)^4 - c_1 A + c_2 B \)

Clearly the matrix \( C = c_1 A + c_2 B \) is \( \{3\} \)-group periodic if and only if \( Q = 0 \).
Hence \( c_1^4 A + c_2^4 B + [(c_1 + c_2)^4 - c_1^4 - c_2^4] AB - c_1 A - c_2 B = 0. \) That is

\begin{equation}
(5.5) \quad c_1 (c_1^3 - 1) A + c_2 (c_2^3 - 1) B + [(c_1 + c_2)^4 - c_1^4 - c_2^4] AB = 0
\end{equation}

**Case 1.** \( c_1 \in \Omega_3 \), \( c_1 + c_2 = 0 \)

From (5.5),

\[-c_1 (-1 - 1) B + (0 - c_1 - c_1) AB = 0\]

\[2c_1 B = 2c_1 AB\]

\[B = AB\] \hspace{1cm} [by \( c_1 \neq 0 \)]

**Case 2.** \( c_1 \in \Omega_3 \), \( c_1 + c_2 \in \Omega_3 \)

From (5.5),

\[c_2 (c_2^3 - 1) B + [c_1 + c_2 - c_1 - c_2^4] AB = 0\]

\[(c_2^3 - 1)(B - AB) = 0\] \hspace{1cm} [by \( c_2 \neq 0 \)]

Hence \( c_2 \in \Omega_3 \) or \( B - AB = 0 \).

The possibility of \( c_2 \in \Omega_3 \) is neglected by remark 1.2.14.

Hence \( AB = B \)

**Case 3.** \( c_1 \in \Omega_3 \), \( c_2 \in \Omega_3 \)

From (5.5),

\[(c_1 + c_2) [((c_1 + c_2)^3 - 1] AB = 0\]

It is clear that \( c_1 + c_2 \neq 0 \) and the possibility of \( c_1 + c_2 \in \Omega_3 \) is neglected by remark 1.2.14 as before. Therefore \( AB = 0 \).
**Case 4.** $c_2 \in \Omega_3, c_1 + c_2 = 0$

From (5.5),

$$-c_2(-1 - 1)A + (0 - c_2 - c_2)AB = 0$$

$$2c_2A = 2c_2AB$$

$$A = AB$$

[by $c_2 \neq 0$]

**Case 5.** $c_2 \in \Omega_3, c_1 + c_2 \in \Omega_3$

From (5.5),

$$c_1(c_1^3 - 1)A + [c_1 + c_2 - c_1^4 - c_2]AB = 0$$

$$(c_1^3 - 1)(A - AB) = 0$$

[by $c_1 \neq 0$]

Hence $c_1 \in \Omega_3$ or $A - AB = 0$.

The possibility of $c_1 \in \Omega_3$ is neglected by remark 1.2.14

Hence $A = AB$.

Conversely, if we substitute the corresponding subcases in equation (5.5), we see that $Q = 0$ and hence $C = c_1A + c_2B$ is $\{3\}$-group periodic.

**Lemma 5.2.2**

Let $P$ and $Q$ be two non-zero idempotent matrices such that $P = \alpha Q$ for some complex number $\alpha$ then $\alpha = 1$.

**Proof**

Since $P = \alpha Q$, we have $P^2 = (\alpha Q)^2$

$$P = \alpha(\alpha Q)$$

[using $P^2 = P, Q^2 = Q$ and $P = \alpha Q$]

$$(1 - \alpha)P = 0$$

Since $P \neq 0$, we have $\alpha = 1$. 

83
**Theorem 5.2.3**

If $A$ and $B$ are two non-zero commutative idempotent matrices then the linear combination $C = c_1A + c_2B$ with non-zero scalars $c_1$ and $c_2$ in $\mathbb{C}$, is $k$-idempotent if and only if one of the following conditions holds.

(1) $A - B = 0$ holds along with either one of the following sets of conditions,

(i) $c_1 + c_2 = 0$

(ii) $c_1 + c_2 = 1$ and $AK = KA$

(2) $A - B \neq 0$ holds along with either one of the following sets of conditions

(i) $c_1 = 1, c_2 = -1$, $AB = B$ and $K(A - B)K = A - B$

(ii) $c_1 = \omega, c_2 = \omega^2 - \omega$, $AB = B$ and $K(A - B)K = B$

(iii) $c_1 = \omega^2, c_2 = \omega - \omega^2$, $AB = B$ and $K(A - B)K = B$

(iv) $c_1 = 1, c_2 = 1$, $AB = 0$ and $K(A + B)K = A + B$

(v) $c_1 = \omega$ or $\omega^2, c_2 = \omega^2$ or $\omega$, $AB = 0$ and $KAK = B$ or $KBK = A$

(vi) $c_1 = -1, c_2 = 1$, $AB = A$ and $K(B - A)K = B - A$

(vii) $c_1 = \omega^2 - \omega, c_2 = \omega$, $AB = A$ and $K(B - A)K = A$

(viii) $c_1 = \omega - \omega^2, c_2 = \omega^2$, $AB = A$ and $K(B - A)K = A$

**Proof**

(1) If $A - B = 0$ then $C = (c_1 + c_2)A$ and it must satisfy the following relation
to be $k$-idempotent.

$$K[(c_1 + c_2)A]^2K = (c_1 + c_2)A$$

$$(c_1 + c_2)^2KAK = (c_1 + c_2)A$$  \[\text{[by } A \text{ is idempotent]}\]

If $c_1 + c_2 = 0$, then situation (i) follows.

If $c_1 + c_2 \neq 0$, then

(5.6) $$(c_1 + c_2)KAK = A$$
Since $A$ and $KAK$ are two non-zero idempotents, we have $c_1 + c_2 = 1$ by lemma 5.2.2. Substituting this in (5.6), we have $KA = AK$, which is situation (ii).

Conversely.

If $A - B = 0$ holds along with $c_1 + c_2 = 0$ then we have $C = 0$. Hence we have $C$ is $k$-idempotent trivially.

If $A - B = 0$ holds along with $c_1 + c_2 = 1$ then we have $C = A$.

Hence $KC^2K = KA^2K$

$$= KAK$$

$$= A$$

[by $AK = KA$]

$$= C$$

Hence $C$ is $k$-idempotent

(2). If $A - B \neq 0$, the matrix $C = c_1A + c_2B$ must satisfy the following relation to be $k$-idempotent.

$$K(c_1A + c_2B)^2K = c_1A + c_2B$$

(5.7)

$$K(c_1^2A + c_2^2B + 2c_1c_2AB)K = c_1A + c_2B$$

By lemma 5.1.6, $C = c_1A + c_2B$ is $\{3\}$-group periodic. Hence the choice of $c_1$ and $c_2$ must be necessarily one among the following cases by lemma 5.2.1.

Case 1. $c_1 \in \Omega_3 , c_1 + c_2 = 0$ and $AB = B$

It follows from (5.7) that

$$c_1K(A - B)K = A - B$$

[by $c_1 \neq 0$]

(5.8)

Since $A - B$ and $K(A - B)K$ are two non-zero idempotents, we have $c_1 = 1$ by lemma 5.2.2. Substituting in (5.8), we have $K(A - B)K = A - B$. Hence the situation (i) follows.
**Case 2.** $c_1 \in \Omega_3$, $c_1 + c_2 \in \Omega_3$ and $AB = B$.

Post multiplying (5.7) by $KB$,

$$ (c_1 + c_2)^2 KB = c_1 AKB + c_2 BKB \tag{5.9} $$

Pre multiplying (5.7) by $BK$,

$$ (c_1 + c_2)^2 BK = c_1 BKA + c_2 BKB \tag{5.10} $$

Post multiplying (5.10) by $A$ and $B$ leads respectively to

$$ (c_1 + c_2)^2 BKA = c_1 BKA + c_2 BKB \tag{5.11} $$

$$ (c_1 + c_2)^2 BKB = c_1 BKB + c_2 BKB $$

That is $(c_1 + c_2)^2 BKB = (c_1 + c_2)BKB$

$$(c_1 + c_2)BKB = BKB \quad \text{[by } c_1 + c_2 \neq 0 \text{]}$$

It follows that $c_1 + c_2 = 1$ or $BKB = 0$

(a) Assume that $c_1 + c_2 = 1$

From (5.9),

$$ KB = c_1 AKB + c_2 BKB \tag{5.12} $$

Pre multiplying (5.12) by $A$,

$$ AKB = c_1 AKB + c_2 BKB $$

$$ AKB = BKB \quad \text{[by } (1 - c_1) = c_2 \neq 0 \text{]}$$

Substituting this in (5.12), we have

$$ KB = BKB \tag{5.13} $$

From (5.11), $(1 - c_1)BKA = c_2 BKB$

$$ BKA = BKB $$

Substituting this in (5.10), we have
(5.14) \[ BK = BKB \]

From (5.13) and (5.14), \( KB = BK \)

Substituting this in (5.7) implies that

\[ c_1^2 KAK + c_2^2 B + 2c_1 c_2 B = c_1 A + c_2 B \quad \text{[by } AB = B] \]

\[ c_1^2 (KAK - B) + B = c_1 A + c_2 B \quad \text{[by } c_1 + c_2 = 1] \]

\[ c_1^2 K(A - B)K = c_1 (A - B) \quad \text{[using } c_2 - 1 = -c_1 \text{ and } KB = BK] \]

Since \( c_1 \neq 0 \), we have \( c_1 K(A - B)K = A - B \)

Since \( A - B \) and \( K(A - B)K \) are two non-zero idempotents we have \( c_1 = 1 \) by lemma 5.2.2.

But this implies that \( c_2 = 0 \). –a contradiction. This sub case is not possible.

(b) Assume that \( BKB = 0 \)

It follows from (5.11) that \( [(c_1 + c_2)^2 - c_1]BKA = 0 \)

We have \( (c_1 + c_2)^2 = c_1 \) or \( BKA = 0 \)

If \( BKA = 0 \) then from (5.10), it follows that \( BK = 0 \) and then \( B = 0 \). –a contradiction.

Therefore \( (c_1 + c_2)^2 = c_1 \)

Squaring \( c_1 + c_2 = c_1^2 \) \quad \text{[by } c_1 + c_2 \in \Omega_3] \]

Substituting the value of \( c_2 \) in (5.7) implies that

\[ K[c_1^2 A - c_1^2 B + c_1 B]K = c_1 A + (c_2^2 - c_1)B \]

\[ c_1^2 K[A + (c_1^2 - 1)B]K = c_1 [A + (c_1 - 1)B] \quad \text{[by } c_1 \in \Omega_3] \]

Since \( c_1 \neq 0 \), we have

(5.15) \[ c_1 K[A + (c_1^2 - 1)B]K = A + (c_1 - 1)B \]

Cubing, \( KAK = A \) \quad \text{[by } AB = B \text{ and } c_1 \in \Omega_3 \text{ (i.e., } c_1^3 = 1)] \]

(5.15) implies that \( c_1 A + (1 - c_1)KBK = A + (c_1 - 1)B \)
\[(c_1 - 1)A = (c_1 - 1)(B + KBK)\]

This implies that, \(K(A - B)K = B\) \[\text{by } c_1 \neq 1 \text{ otherwise } c_2 = 0\]

Since \(c_1 \in \Omega_3\), we have \(c_1 = \omega\) or \(\omega^2\)

Hence the situation (ii) or (iii) follows.

**Case 3.** \(c_1 \in \Omega_3\), \(c_2 \in \Omega_3\) and \(AB = 0\)

It follows from (5.7) that

\[(5.16) \quad K[c_1^2 A + c_2^2 B]K = c_1 A + c_2 B\]

Cubing (5.16), we have

\[(5.17) \quad K(A + B)K = A + B\]

Pre and post multiplying (5.16) by \(AK\) and \(A\) respectively implies that

\[c_1^2 AKA = c_1 AKA\]

\[(c_1 - 1)AKA = 0\]  \[\text{by } c_1 \neq 0\]

\[(5.18) \quad c_1 = 1 \text{ or } AKA = 0\]

Pre and post multiplying (5.16) by \(BK\) and \(B\) respectively implies that

\[c_2^2 BKB = c_2 BKB\]

\[(c_2 - 1)BKB = 0\]  \[\text{by } c_2 \neq 0\]

\[(5.19) \quad c_2 = 1 \text{ or } BKB = 0\]

From (5.18) and (5.19), we have at least one of the following situation holds.

(a) \(c_1 = 1, c_2 = 1\)

(b) \(c_1 = 1, BKB = 0\)

(c) \(c_2 = 1, AKA = 0\)

(d) \(AKA = 0, BKB = 0\)
(a) It follows from (5.16) that $K(A + B)K = A + B$, which is the situation in (iv)

(b) Post multiplying (5.16) by $B$, we have

\[(5.20) \quad KAKB = c_2B\]

Post multiplying (5.17) by $B$, we have $KAKB = B$

Substituting this in (5.20), $(c_2 - 1)B = 0$

Since $B \neq 0$, we have $c_2 = 1$, which leads to the situation (iv) again along with $BKB = 0$.

(c) This situation is similar to the sub case (b) and it implies that $c_1 = 1$, which turns again to the situation (iv).

(d) Post multiplying (5.17) by $B$ and $A$ leads respectively to

\[(5.21) \quad KAKB = B\]

\[(5.22) \quad KBKA = A\]

Pre multiplying (5.17) by $BK$ and $AK$ leads respectively to

\[(5.23) \quad AK = AKB\]

\[(5.24) \quad BK = BKA\]

It follows from (5.21) and (5.22) by substituting (5.23) and (5.24) that $KAK = B$ and $KBK = A$

Post multiplying (5.16) by $B$, we have $c_1^2 KAKB = c_2B$

\[(c_1^2 - c_2)B = 0\] \quad [\text{by (5.21)}]

Since $B \neq 0$, we have

\[(5.25) \quad c_1^2 = c_2\]

By similar argument, it follows from (5.16) by post multiplying by $A$ that
(5.26) \[ c_2^2 = c_1 \]

From (5.25) and (5.26), we have the following three possibilities

\[ c_1 = c_2 = 1 \quad ; \quad c_1 = \omega, c_2 = \omega^2 \quad ; \quad c_1 = \omega^2, c_2 = \omega \quad \text{[by } c_1, c_2 \in \Omega_3]\]

Hence the situations in (iv) and (v) are obtained.

**Case 4.** \( c_2 \in \Omega_3 \), \( c_1 + c_2 = 0 \) and \( AB = A \)

This is analogous to the case 1 and hence the situation (vi) follows.

**Case 5.** \( c_2 \in \Omega_3 \), \( c_1 + c_2 \in \Omega_3 \) and \( AB = A \)

This is analogous to the case 2 and hence the situations in (vii) and (viii) are obtained by similar arguments.

Conversely, by substituting the corresponding sets of conditions (i) to (viii) in (5.7), the sufficiency can be settled.

**Note 5.2.4**

The following algorithm is helpful to understand the hypothesis (i.e., the different possibilities for a linear combination \( C = c_1 A + c_2 B \) where \( A \) and \( B \) are commutative idempotent matrices to be \( k \)-idempotent) of above theorem 5.2.3.
5.2.5. An algorithm investigating the \( k \)-idempotency of \( C = c_1A + c_2B \)
(where \( A \) and \( B \) are commutative idempotent matrices)

**Step 1.** If \( A - B \neq 0 \) then go to step 5.

**Step 2.** If \( c_1 + c_2 = 0 \) then \( C = 0 \), which is trivial. Go to step 25.

**Step 3.** If \( c_1 + c_2 = 1 \) then \( C = A \), if \( AK = KA \) then go to step 25.

**Step 4.** Otherwise go to step 24.

**Step 5.** If \( c_1 = 1 \) and \( c_2 = -1 \) then go to step 15.

**Step 6.** If \( c_1 = \omega \) and \( c_2 = \omega^2 - \omega \) then go to step 16.

**Step 7.** If \( c_1 = \omega^2 \) and \( c_2 = \omega - \omega^2 \) then go to step 16.

**Step 8.** If \( c_1 = 1 \) and \( c_2 = 1 \) then go to step 17.

**Step 9.** If \( c_1 = \omega \) and \( c_2 = \omega^2 \) then go to step 18.

**Step 10.** If \( c_1 = \omega^2 \) and \( c_2 = \omega \) then go to step 18.

**Step 11.** If \( c_1 = -1 \) and \( c_2 = 1 \) then go to step 19.

**Step 12.** If \( c_1 = \omega^2 - \omega \) and \( c_2 = \omega \) then go to step 20.

**Step 13.** If \( c_1 = \omega - \omega^2 \) and \( c_2 = \omega^2 \) then go to step 20.

**Step 14.** If \( c_1 \) and \( c_2 \) do not obey any possibilities from steps(5 – 13) then go to step 24.

**Step 15.** If \( K(A - B)K = A - B \) then go to step 21.

Otherwise, go to step 24.

**Step 16.** If \( K(A - B)K = B \) then go to step 21.

Otherwise, go to step 24.

**Step 17.** If \( K(A + B)K = A + B \) then go to step 22.

Otherwise, go to step 24.
**Step 18.** If $KAK = B$ then go to step 22.

Otherwise, go to step 24.

**Step 19.** If $K(B - A)K = B - A$ then go to step 23.

Otherwise, go to step 24.

**Step 20.** If $K(B - A)K = A$ then go to step 23.

Otherwise, go to step 24.

**Step 21.** If $AB = B$ then go to step 25.

Otherwise, go to step 24.

**Step 22.** If $AB = 0$ then go to step 25.

Otherwise, go to step 24.

**Step 23.** If $AB = A$ then go to step 25.

Otherwise, go to step 24.

**Step 24.** $C = c_1A + c_2B$ is not a $k$-idempotent matrix and the algorithm is complete.

**Step 25.** $C = c_1A + c_2B$ is a $k$-idempotent matrix and the algorithm is complete.
5.3 $k$-idempotency of linear combinations of an idempotent matrix and an essentially tripotent matrix that commute

In this section, the problem of characterizing situations for a linear combination $C = c_1 A + c_2 B$ of an idempotent matrix $A$ and an essentially tripotent matrix $B$ to be $k$-idempotent matrix, is studied. An essentially tripotent matrix $B$ is one for which $B^3 = B$ and $B^2 \neq B$. Here it is assumed that the matrices $A$ and $B$ are commutative.

It is well known that a tripotent matrix $B$ can uniquely be represented as a difference of two idempotent matrices $B_1$ and $B_2$ (i.e., $B = B_1 - B_2$) which are disjoint in the sense that $B_1 B_2 = 0 = B_2 B_1$ (cf. lemma 1.2.5). If $B_1$ and $B_2$ are non-zero then $B$ is clearly an essentially tripotent matrix. Otherwise, the matrix $B$ reduces to a scalar multiple of an idempotent matrix (i.e., $B = B_1$ or $B = -B_2$).

Example 5.3.1

$B = \begin{pmatrix} -1 & 8 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix}$ is an essentially tripotent matrix and it can uniquely be represented as a difference of two idempotent matrices $B_1$ and $B_2$.

That is

$B = B_1 - B_2$

$\begin{pmatrix} -1 & 8 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & -2 & 0 \end{pmatrix}$

Theorem 5.3.2

Let $B \in \mathbb{C}^{n \times n}$ be a tripotent matrix uniquely decomposed as $B = B_1 - B_2$ where $B_1$ and $B_2$ are idempotent matrices such that $B_1 B_2 = 0 = B_2 B_1$. Let $A \in \mathbb{C}^{n \times n}$. Then $AB = BA$ if and only if $AB_1 = B_1 A$ and $AB_2 = B_2 A$.

Proof

(5.27) $B = B_1 - B_2$

$B^2 = (B_1 - B_2)^2$

(5.28) $B^2 = B_1 + B_2$ [by $B_1 B_2 = 0 = B_2 B_1$]
From (5.27) and (5.28), solving for $B_1$ and $B_2$ we have

$$B_1 = \frac{B + B^2}{2} \quad \text{and} \quad B_2 = \frac{B^2 - B}{2}$$

Assume that $AB = BA$.

$$AB_1 = A \left( \frac{B + B^2}{2} \right)$$

$$= \frac{1}{2} (AB + AB^2)$$

$$= \frac{1}{2} (B + B^2)A$$

[by $AB = BA$]

$$AB_1 = B_1A$$

Similarly it can be proved that $AB_2 = B_2A$.

Conversely, if we assume that $AB_1 = B_1A$ and $AB_2 = B_2A$ then

$$AB = A(B_1 - B_2)$$

$$= AB_1 - AB_2$$

$$= B_1A - B_2A$$

$$= (B_1 - B_2)A$$

$$= BA$$

Hence the theorem is proved.

\[ \blacksquare \]

**Theorem 5.3.3**

Let $B \in \mathbb{C}^{n \times n}$ be an essentially tripotent matrix uniquely decomposed as $B = B_1 - B_2$ where $B_1$ and $B_2$ are non-zero idempotent matrices such that $B_1B_2 = 0 = B_2B_1$. Let $A \in \mathbb{C}^{n \times n}$ be a non-zero idempotent matrix such that $AB = BA$. If a linear combination $C = c_1A + c_2B$ with non-zero $c_1, c_2 \in \mathbb{C}$ is a $k$-idempotent matrix then at least one of the following sets of conditions necessarily hold.
(a₁) \( c₁ = \frac{1}{2}, \ c₂ = -\frac{1}{2}, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = B₂KB₂A \)

(a₂) \( c₁ = \frac{1}{4}, \ c₂ = -\frac{1}{4}, \ AB₁KB₁ = B₁KB₁A \text{ and } B₂KB₂A = 0 \)

(a₃) \( c₁ = 2, \ c₂ = -2, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = 0 \)

(a₄) \( c₁ + c₂ = 0, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = 0 = B₂KB₂A \)

(b₁) \( c₁ = c₂ = \frac{1}{2}, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = B₂KB₂A \)

(b₂) \( c₁ = 2, \ c₂ = -1, \ B₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = 0 = B₂KB₂A \)

(b₃) \( c₁ = \frac{5 \pm i\sqrt{7}}{8}, \ c₂ = \frac{3 \pm i\sqrt{7}}{8}, \ AB₁KB₁ = B₁KB₁A \text{ and } B₂KB₂A = 0 \)

(b₄) \( c₁ = 2 \mp \sqrt{2}, \ c₂ = -1 \pm \sqrt{2}, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = 0 \)

(b₅) \( c₁ + c₂ = 1, \ AB₁KB₁ = B₁KB₁A \text{ and } AB₂KB₂ = 0 \)

(c₁) \( c₁ = c₂ = \frac{1}{4}, \ B₁KB₁A = 0 \text{ and } AB₂KB₂ = B₂KB₂A \)

(c₂) \( c₁ = c₂ = 2, \ AB₁KB₁ = 0 \text{ and } AB₂KB₂ = B₂KB₂A \)

(c₃) \( c₁ - c₂ = 0, \ c₁ \notin \{\frac{1}{4}, 2\}, \ AB₁KB₁ = 0 = B₁KB₁A \text{ and } AB₂KB₂ = B₂KB₂A \)

(d₁) \( c₁ = 2, \ c₂ = 1, \ AB₁KB₁ = 0 = B₁KB₁A \text{ and } B₂KB₂ = B₂KB₂A \)

(d₂) \( c₁ = \frac{5 \pm i\sqrt{7}}{8}, \ c₂ = \frac{-3 \pm i\sqrt{7}}{8}, \ B₁KB₁A = 0 \text{ and } AB₂KB₂ = B₂KB₂A \)

(d₃) \( c₁ = 2 \pm \sqrt{2}, \ c₂ = 1 \pm \sqrt{2}, \ AB₁KB₁ = 0 \text{ and } AB₂KB₂ = B₂KB₂A \)

(d₄) \( c₁ - c₂ = 1, \ AB₁KB₁A = 0 \text{ and } AB₂KB₂ = B₂KB₂A \)

(e) The choice of \( c₁ \) and \( c₂ \) differ from all the above cases then

\( AB₁KB₁A = 0 = AB₂KB₂A \)

**Proof**

A matrix \( C = c₁A + c₂B₁ - c₂B₂ \) must satisfy the following to be \( k \)-idempotent.
\[ K(c_1 A + c_2 B_1 - c_2 B_2)^2 = c_1 A + c_2 B_1 - c_2 B_2 \]

(5.29) \[ K(c_1^2 A + c_2^2 B_1 + c_2^2 B_2 + 2c_1 c_2 AB_1 - 2c_1 c_2 AB_2)K = c_1 A + c_2 B_1 - c_2 B_2 \]

Pre and post multiplying (5.29) by \( B_1 K \) and \( B_1 \) respectively

(5.30) \[ (c_1^2 AB_1 + c_2^2 B_1 + 2c_1 c_2 AB_1)K B_1 = B_1 K(c_1 AB_1 + c_2 B_1) \]

Pre and post multiplying (5.30) by \( A \),

\[(c_1 + c_2)^2 AB_1 K B_1 A = (c_1 + c_2) AB_1 K B_1 A\]

Hence we have

(5.31) \[ c_1 + c_2 = 0 \] otherwise \( c_1 + c_2 = 1 \) or \( AB_1 K B_1 A = 0 \)

Pre and post multiplying (5.29) by \( B_2 K \) and \( B_2 \) respectively

(5.32) \[ (c_1^2 AB_2 + c_2^2 B_2 - 2c_1 c_2 AB_2)K B_2 = B_2 K(c_1 AB_2 + c_2 B_2) \]

It follows from (5.32) by pre and post multiplying \( A \) that

(5.33) \[ c_1 - c_2 = 0 \] otherwise \( c_1 - c_2 = 1 \) or \( AB_2 K B_2 A = 0 \)

While combining the above two sets of conditions (5.31) and (5.33) we have the following different possibilities.

(1) \( c_1 + c_2 = 0, c_1 - c_2 = 0 \) \hspace{1cm} (2) \( c_1 + c_2 = 0, c_1 - c_2 = 1 \)

(3) \( c_1 + c_2 = 0, AB_2 K B_2 A = 0 \) \hspace{1cm} (4) \( c_1 + c_2 = 1, c_1 - c_2 = 0 \)

(5) \( c_1 + c_2 = 1, c_1 - c_2 = 1 \) \hspace{1cm} (6) \( c_1 + c_2 = 1, AB_2 K B_2 A = 0 \)

(7) \( c_1 - c_2 = 0, AB_1 K B_1 A = 0 \) \hspace{1cm} (8) \( c_1 - c_2 = 1, AB_1 K B_1 A = 0 \)

(9) \( c_1 \) and \( c_2 \) differ from (1) to (8) and \( AB_1 K B_1 A = 0, AB_2 K B_2 A = 0 \)

**Case (1):** \( c_1 + c_2 = 0, c_1 - c_2 = 0 \)

This case is not possible since it leads to \( c_2 = 0 \) — a contradiction.

**Case (2):** \( c_1 + c_2 = 0, c_1 - c_2 = 1 \)

It follows immediately that \( c_1 = \frac{1}{2} \) and \( c_2 = -\frac{1}{2} \)
(5.30) implies that

\[
\frac{3}{4} B_1 KB_1 = \frac{1}{2} B_1 KB_1 A + \frac{1}{4} AB_1 KB_1
\]

Pre multiplying (5.34) by \( A \) we have \( AB_1 KB_1 = AB_1 KB_1 A \)

Post multiplying (5.34) by \( A \) we have \( B_1 KB_1 A = AB_1 KB_1 A \)

Hence \( AB_1 KB_1 = B_1 KB_1 A \). By similar arguments, it follows from (5.32) that \( AB_2 KB_2 = B_2 KB_2 A \), which is the situation \( (a_1) \).

**Case (3):** \( c_1 + c_2 = 0, AB_2 KB_2 A = 0 \)

(5.30) implies that

\[
c_1^2 (B_1 - AB_1) KB_1 = c_1 B_1 K (AB_1 - B_1)
\]

Since \( c_1 \neq 0 \), we have

\[
(c_1 + 1)B_1 KB_1 = B_1 KB_1 A + c_1 AB_1 KB_1
\]

Pre multiplying (5.35) by \( A \) we have \( AB_1 KB_1 = AB_1 KB_1 A \)

Post multiplying (5.35) by \( A \) we have \( B_1 KB_1 A = AB_1 KB_1 A \)

Hence \( AB_1 KB_1 = B_1 KB_1 A \).

(5.32) implies that \( c_1^2 (3AB_2 + B_2) KB_2 = c_1 B_2 K (AB_2 + B_2) \)

Since \( c_1 \neq 0 \), we have

\[
3c_1 AB_2 KB_2 + (c_1 - 1)B_2 KB_2 = B_2 KB_2 A
\]

Pre multiplying (5.36) by \( A \), we have \( (4c_1 - 1)AB_2 KB_2 = 0 \)

Post multiplying (5.36) by \( A \), we have \( (c_1 - 2)B_2 KB_2 A = 0 \)

If \( c_1 = \frac{1}{4} \) then \( c_2 = -\frac{1}{4} \) and \( B_2 KB_2 A = 0 \), which is \( (a_2) \).

If \( c_1 = 2 \) then \( c_2 = -2 \) and \( AB_2 KB_2 = 0 \), which is \( (a_3) \).

Otherwise \( AB_2 KB_2 = 0 = B_2 KB_2 A \), which is \( (a_4) \).
Case (4): $c_1 + c_2 = 1, c_1 - c_2 = 0$

This implies that $c_1 = c_2 = \frac{1}{2}$. It follows from (5.30) that

\[
\frac{3}{4} AB_1 KB_1 - \frac{1}{4} B_1 KB_1 = \frac{1}{2} B_1 KB_1 A
\]

Pre multiplying (5.37) by $A$, we have $AB_1 KB_1 = AB_1 KB_1 A$

Post multiplying (5.37) by $A$, we have $AB_1 KB_1 A = B_1 KB_1 A$

Hence $AB_1 KB_1 = B_1 KB_1 A$. By similar arguments, it follows from (5.32) that $AB_2 KB_2 = B_2 KB_2 A$, which is the situation $(b_1)$.

Case (5): $c_1 + c_2 = 1, c_1 - c_2 = 1$

This case is not possible since it leads to $c_2 = 0$. — a contradiction.

Case (6): $c_1 + c_2 = 1, AB_2 KB_2 A = 0$

(5.30) implies that

\[
(AB_1 + c_2^2 B_1 - c_2^2 AB_1)KB_1 = B_1 K(AB_1 + c_2 B_1 - c_2 AB_1)
\]

Pre multiplying (5.38) by $A$, we have $(1 - c_2) AB_1 KB_1 = (1 - c_2) AB_1 KB_1 A$.

Since $c_2 \neq 1$ (otherwise $c_1 = 0$), we have

(5.39) $AB_1 KB_1 = AB_1 KB_1 A$

Post multiplying (5.38) by $A$, we have $(1 - c_2^2) AB_1 KB_1 A = (1 - c_2^2) B_1 KB_1 A$

(5.40) $(1 - c_2^2) AB_1 KB_1 = (1 - c_2^2) B_1 KB_1 A$ [by (5.39)]

(5.32) implies that

\[
(3c_2^2 - 4c_2 + 1)AB_2 KB_2 + (c_2^2 + c_2)B_2 KB_2 = (1 - c_2) B_2 KB_2 A
\]

Pre multiplying (5.41) by $A$, we have

\[
(4c_2^2 - 3c_2 + 1)AB_2 KB_2 = 0
\]

Post multiplying (5.41) by $A$, we have
(5.43) \( (c_2^2 + 2c_2 - 1)B_2KB_2A = 0 \)

(i) Considering (5.40), if \( c_2 = 1 \) then \( c_1 = 0 \).—a contradiction.

If \( c_2 = -1 \) then \( c_1 = 2 \).

(5.30) implies \( B_1KB_1 = B_1KB_1A \). (5.32) implies that

(5.44) \( 4AB_2KB_2 = B_2KB_2A \)

Pre and post multiplying (5.44) by \( A \) respectively leads to

\( AB_2KB_2 = 0 \) and \( B_2KB_2A = 0 \), which is \( (b_2) \).

(ii) Considering (5.42), if \( c_2 = \frac{3\pm i\sqrt{7}}{8} \) then \( c_1 = \frac{5\pm i\sqrt{7}}{8} \)

From (5.40), \( AB_1KB_1 = B_1KB_1A \)

From (5.43), \( B_2KB_2A = 0 \), which is \( (b_3) \).

(iii) Considering (5.43), if \( c_2 = -1 \pm \sqrt{2} \) then \( c_1 = 2 \mp \sqrt{2} \).

From (5.40), \( AB_1KB_1 = B_1KB_1A \)

From (5.42), \( AB_2KB_2 = 0 \), which is \( (b_4) \).

(iv) If the choice of \( c_2 \) differ from all the above three sub cases then by (5.40), we have

\( AB_1KB_1 = B_1KB_1A \), which is \( (b_5) \).

**Case (7):** \( c_1 - c_2 = 0, \ AB_1KB_1A = 0 \)

(5.32) implies that

(5.45) \( c_2(B_2 - AB_2)KB_2 = B_2K(AB_2 - B_2) \)

Pre multiplying (5.45) by \( A \), we have \( AB_2KB_2A = AB_2KB_2 \)

Post multiplying (5.45) by \( A \), we have \( B_2KB_2A = AB_2KB_2A \)

Hence \( AB_2KB_2 = B_2KB_2A \). (5.30) implies that

(5.46) \( 3c_2 AB_1KB_1 + (c_2 - 1)B_1KB_1A = B_1KB_1A \)
Pre multiplying (5.46) by $A$, we have

\[(5.47) \quad (4c_2 - 1) AB_1 KB_1 = 0\]

Post multiplying (5.46) by $A$, we have

\[(5.48) \quad (c_2 - 2)B_1 KB_1 A = 0\]

It follows from (5.47) and (5.48) that

If $c_2 = \frac{1}{4}$ then $c_1 = \frac{1}{4}$ and $B_1 KB_1 A = 0$, which is $(c_1)$. If $c_2 = 2$ then $c_1 = 2$ and $AB_1 KB_1 = 0$, which is $(c_2)$. Otherwise $(c_3)$ follows.

**Case (8):** $c_1 - c_2 = 1$, $AB_1 KB_1 A = 0$

(5.32) implies that

\[(5.49) \quad (2 - c_1)AB_2 KB_2 + (c_1 - 1)B_2 KB_2 = B_2 KB_2 A \quad \text{[by}\ c_1 \neq 0]\]

Pre multiplying (5.49) by $A$, we have

\[(5.50) \quad AB_2 KB_2 = AB_2 KB_2 A\]

Post multiplying (5.49) by $A$, we have $(c_1 - 2)B_2 KB_2 A = (c_1 - 2)AB_2 KB_2 A$

\[(5.51) \quad (c_1 - 2)B_2 KB_2 A = (c_1 - 2)AB_2 KB_2 \quad \text{[by}\ (5.50)]\]

(5.30) implies that

\[(5.52) \quad (c_1^2 - 3c_1 + 2)B_1 KB_1 + (3c_1^2 - 2c_1)AB_1 KB_1 = c_1 B_1 KB_1 A\]

Pre multiplying (5.52) by $A$, we have

\[(5.53) \quad (4c_1^2 - 5c_1 + 2)AB_1 KB_1 = 0\]

Post multiplying (5.52) by $A$, we have

\[(5.54) \quad (c_1^2 - 4c_1 + 2)B_1 KB_1 A = 0\]

(i) Considering (5.51), if $c_1 = 2$ then $c_2 = 1$

From (5.49), $B_2 KB_2 = B_2 KB_2 A$
From (5.53) and (5.54), \( AB_1KB_1 = 0 = B_1KB_1A \), which is \((d_1)\)

(ii) Considering (5.53), if \( c_1 = \frac{5\pm i\sqrt{7}}{8} \) then \( c_2 = \frac{-3\pm i\sqrt{7}}{8} \)

From (5.51), \( AB_2KB_2 = B_2KB_2A \)

From (5.54), \( B_1KB_1A = 0 \), which is \((d_2)\).

(iii) Considering (5.54), if \( c_1 = 2 \pm \sqrt{2} \) then \( c_2 = 1 \pm \sqrt{2} \)

From (5.51), \( AB_2KB_2 = B_2KB_2A \)

From (5.53), \( AB_1KB_1 = 0 \), which is \((d_3)\).

(iv) If the choice of \( c_1 \) differ from all the above three sub cases then by (5.51), \( AB_2KB_2 = B_2KB_2A \). This is \((d_4)\).

**Case (9):** If \( c_1 \) and \( c_2 \) do not obey any of the above cases (1) – (8) then the remaining possibility is \( AB_1KB_1A = 0 \) and \( AB_2KB_2A = 0 \), which is \((e)\).

**Note 5.3.4**

The following algorithm is helpful to understand the hypothesis (i.e., the different possibilities for a linear combination \( C = c_1A + c_2B \) of an idempotent matrix \( A \) and an essentially tripotent matrix \( B \) that commute, to be \( k \)-idempotent) of above theorem 5.3.3.
5.3.5. An algorithm investigating the $k$-idempotency of $C = c_1A + c_2B$
(where $A$ is an idempotent matrix and $B$ is an essentially tripotent
matrix that commute)

**Step 1.** If $c_1 = 2$ and $c_2 = -1$ then go to step 7.

**Step 2.** If $c_1 + c_2 = 0$ or 1 then go to step 6.

**Step 3.** If $c_1 = 2$ and $c_2 = 1$ then go to step 6.

**Step 4.** If $c_1 - c_2 = 0$ or 1 then go to step 7.

**Step 5.** $c_1$ and $c_2$ differ from all the possibilities from steps $(1 - 4)$.

if $AB_1KB_1A = 0 = AB_2KB_2A$ then go to step 16. Otherwise go to step 17.

**Step 6.** While $AB_1KB_1 = B_1KB_1A$. Otherwise go to step 17.

(i). If $c_1 = \frac{1}{2}$ and $c_2 = \pm \frac{1}{2}$ then go to step 7.

(ii). If $c_1 = \frac{1}{4}$ and $c_2 = -\frac{1}{4}$ then go to step 10.

(iii). If $c_1 = 2$ and $c_2 = -2$ then go to step 11.

(iv). If $c_1 + c_2 = 0$ then go to step 7.

(v). If $c_1 = \frac{5 \pm i\sqrt{7}}{8}$, $c_2 = \frac{3 \pm i\sqrt{7}}{8}$ then go to step 10.

(vi). If $c_1 = 2 \mp \sqrt{2}$, $c_2 = -1 \pm \sqrt{2}$ then go to step 11.

(vii). If $c_1 + c_2 = 1$ then go to step 15.

(viii). If $c_1 - c_2 = 0$ and $c_1 \notin \left\{\frac{1}{4}, 2\right\}$ then go to step 12.

(ix). If $c_1 = 2$ and $c_2 = 1$ then go to step 9.

**Step 7.** While $AB_2KB_2 = B_2KB_2A$. Otherwise go to step 17.

(i). If $c_1 = \frac{1}{2}$ and $c_2 = \pm \frac{1}{2}$ then go to step 16.
(ii). If $c_1 + c_2 = 0$ then go to step 10.

(iii). If $c_1 = c_2 = \frac{1}{4}$ then go to step 12.

(iv). If $c_1 = c_2 = 2$ then go to step 13.

(v). If $c_1 - c_2 = 0$ then go to step 6.

(vi). If $c_1 = \frac{5 \pm i\sqrt{7}}{8}$, $c_2 = \frac{-3 \pm i\sqrt{7}}{8}$ then go to step 12.

(vii). If $c_1 = 2 \pm \sqrt{2}, c_2 = 1 \pm \sqrt{2}$ then go to step 13.

(viii). If $c_1 - c_2 = 1$ then go to step 14.

(ix). If $c_1 = 2$ and $c_2 = -1$ then go to step 8.

**Step 8.** If $B_1KB_1 = B_1KB_1A$ then go to step 10. Otherwise go to step 17.

**Step 9.** If $B_2KB_2 = B_2KB_2A$ then go to step 12. Otherwise go to step 17.

**Step 10.** If $B_2KB_2A = 0$ then go to step 16. Otherwise go to step 17.

**Step 11.** If $AB_2KB_2 = 0$ then go to step 16. Otherwise go to step 17.

**Step 12.** If $B_1KB_1A = 0$ then go to step 16. Otherwise go to step 17.

**Step 13.** If $AB_1KB_1 = 0$ then go to step 16. Otherwise go to step 17.

**Step 14.** If $AB_1KB_1A = 0$ then go to step 16. Otherwise go to step 17.

**Step 15.** If $AB_2KB_2A = 0$ then go to step 16. Otherwise go to step 17.

**Step 16.** $C = c_1A + c_2B$ is a $k$-idempotent matrix and the algorithm is complete.

**Step 17.** $C = c_1A + c_2B$ is not a $k$-idempotent matrix and the algorithm is complete.