CHAPTER-I

A BRIEF INTRODUCTION OF FIXED POINT THEORY
Chapter 1

A Brief Introduction of Fixed Point Theorey

1.1 Preliminaries

The presence or absence of fixed point is an intrinsic property of a function. However many necessary and/or sufficient conditions for the existence of such points involve a mixture of algebraic order theoretic or topological properties of mapping or its domain. Fixed point theory concerns itself with a very simple and basic mathematical setting.

A point is often called fixed point when it remains invariant, irrespective of the type of transformation it undergoes. For a function $f$ that has a set $X$ as both domain and range, a fixed point is a point $x \in X$ for which $f(x) = x$. Two fundamental theorems concerning fixed points are those of Banach [7] and of Brouwer [13].

We begin our studies with the following known definitions;

Definition 1.1.1: Let $X$ be a non-empty set. A mapping $d : X \times X \to \mathbb{R}$ (the set of reals) is said to be a metric (or distance function) iff $d$ satisfies the following axioms:

(M-1) $d(x, y) \geq 0$ for all $x, y \in X$, 

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(M-2) \( d(x, y) = 0 \) iff \( x = y \),

(M-3) \( d(x, y) = d(y, x) \) for all \( x, y \in X \),

(M-4) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

If \( d \) is metric for \( X \), then the ordered pair \((X, d)\) is called a metric space and \( d(x, y) \) is called the distance between \( x \) and \( y \).

**Definition 1.1.2** : Let \((X, d)\) be a metric space and let \( A \) be a non-empty subset of \( X \). Then the diameter of \( A \), denoted by \( \delta(A) \), is defined by:

\[
\delta(A) = \sup \{ d(x, y) : x, y \in A \}
\]

that is, the diameter of \( A \) is the supremum of the set of all distances between points of \( A \).

**Definition 1.1.3** : The distance between a point \( p \in X \) and a subset \( A \) of metric space \( X \) is denoted and defined by \( d(p, A) = \inf \{ d(p, x) : x \in A \} \). It is evident that \( d(p, A) = 0 \) if \( p \in A \).

**Definition 1.1.4** : The distance between two non-empty subsets \( A \) and \( B \) of a metric space \( X \) is denoted and defined as:

\[
d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}
\]

**Definition 1.1.5** : Let \((X, d)\) be a metric space and \( A \) be any subset of \( X \). A point \( x \in X \) is an interior point of \( A \) if there exists \( r > 0 \), such that \( x \in S_r(x) \subseteq A \).

**Definition 1.1.6** : Let \((X, d)\) be a metric space and \( A \) be any subset of \( X \). A point \( x \in X \) is an exterior point of \( A \) if there exists an open sphere \( S_r(x) \), such that \( S_r(x) \subseteq A^c \), or \( S_r(x) \cap A = \emptyset \).

**Definition 1.1.7** : Let \( A \) be a nonempty subset of a metric space \((X, d)\). A point \( x \in X \) is said to be the boundary point of \( A \) if \( x \) is neither an interior point of \( A \) nor an exterior point of \( A \). The boundary of \( A \) will be denoted by \( \partial A \).
Definition 1.1.8: A sequence of elements $x_1, x_2, x_3 \ldots \quad x_n \ldots$ in a metric space $X$ is said to converge to an element $x \in X$ if the sequence of real numbers converges to zero as $n \to \infty$, i.e. $\lim_{n \to \infty} x_n = x$.

Definition 1.1.9: Let $(X, d)$ be a metric space and let $\{p_n\}$ be a sequence of points in $X$, then it is said to be a Cauchy sequence in $X$ if and only if for every $\epsilon > 0$ there exists a positive integer $n(\epsilon)$ such that, $m, n \geq n(\epsilon) \Rightarrow d(p_m, p_n) < \epsilon$.

It is clear that every convergent sequence in a metric space is a Cauchy sequence but the converse need not be true. A metric space $(X, d)$ is said to be complete if and only if every Cauchy sequence in $X$ converges to a point in $X$.

Definition 1.1.10: A self mapping $T$ of a metric space $(X, d)$ is said to be Lipschitzian if for all $x, y \in X$ and $\alpha \geq 0$

$$d(T(x), T(y)) \leq \alpha d(x, y). \quad (1.1.1)$$

$T$ is said to be contraction on $\alpha$ if $\alpha \in [0, 1)$ and nonexpansive if $\alpha = 1$. A contraction mapping is always continuous.

In 1922, S. Banach's contraction principle appeared and this was known for its simple and elegant proof by using the Picard's iteration in a complete metric space. Banach's fixed point theorem states;

Theorem 1.1.1 [75]: Let $X$ be a complete metric space with metric $d$ and $f : X \to X$ is required to be a contraction, that is there must exists $\alpha < 1$ such that,

$$d(f(x), f(y)) \leq \alpha d(x, y), \quad \forall \ x, y \in X, \quad (1.1.2)$$

the conclusion is that, $f$ has a fixed point, in fact exactly one.
Proof: Let \( x \in X \) be an arbitrary element. Starting from \( x \) we form the iterations,

\[
x_1 = fx, \quad x_2 = fx_1, \quad x_3 = fx_2, \ldots, \quad x_n = fx_{n-1}\ldots
\]

We verify that \( \{x_n\} \) is a Cauchy sequence. We have,

\[
d(x_1, x_2) = d(fx, fx_1) \leq \alpha d(x, x_1) = \alpha d(x, fx)
\]

\[
d(x_2, x_3) = d(fx_1, fx_2) \leq \alpha d(x_1, x_2) \leq \alpha^2 d(x, fx)
\]

\[
d(x_3, x_4) = d(fx_2, fx_3) \leq \alpha d(x_2, x_3) \leq \alpha^3 d(x, fx).
\]

In general, for any positive integer \( n \), \( d(x_n, x_{n+1}) \leq \alpha^n d(x, fx) \). Also, for any positive integer \( p \),

\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})
\]

\[
\leq \alpha^n d(x, fx) + \alpha^{n+1} d(x, fx) + \cdots + \alpha^{n+p-1} d(x, fx)
\]

\[
= (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+p-1}) d(x, fx)
\]

\[
= \frac{\alpha^n(1 - \alpha^{n+p})}{1 - \alpha} d(x, fx) < \frac{\alpha^n}{1 - \alpha} d(x, fx), \quad (\text{since } 0 < \alpha < 1).
\]

Since \( \alpha < 1 \), the above relation shows that \( d(x_n, x_{n+p}) \to 0 \) as \( n \to \infty \). Therefore \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, the sequence \( \{x_n\} \) converge to a point \( x_0 \) (say) in \( X \). Now we show that \( fx_0 = x_0 \), for this by triangle inequality we have,

\[
d(x_0, fx_0) \leq d(x_0, x_n) + d(x_n, fx_0)
\]

\[
= d(x_0, x_n) + d(fx_{n-1}, fx_0)
\]

\[
\leq d(x_0, x_n) + \alpha d(x_{n-1}, x_0) \to 0 \quad \text{as} \quad n \to \infty.
\]

So \( fx_0 = x_0 \). Therefore \( x_0 \) is a fixed point of \( f \). Uniqueness can be easily check using contradiction method. So \( f \) has a unique fixed point in \( X \). \( \Box \)
In 1844 A. L. Cauchy [16] proved the existence and uniqueness of the solution of the differential equation \( \frac{du}{dx} = T(x,y) \); \( y(x_0) = y_0 \), where \( T \) is a continuously differentiable function. Later in 1877, R. Lipschitz [76] simplified Cauchy’s proof, in 1890 G. Peano [112] attempted a deeper result of Cauchy’s theorem by supposing only the continuity of \( T \).

**Definition 1.1.11** [121]: A set \( X \) is said to have a fixed point property (FPP) if each continuous mapping \( f : X \to X \) of this set into itself has a fixed point.

In [121], it is shown that fixed point property is a topological property.

Historically the study of fixed point theory began in 1912 with a theorem given by famous Dutch mathematician L. E. Y. Brouwer [13](1881-1966). This is the most famous and important theorem on the topological fixed point property. It can be formulated as; **The closed unit ball** \( B^n \in \mathbb{R}^n \) **has the topological fixed point property.** He also proved the fixed point theorems for a square, a sphere and their \( n \)-dimensional counterparts. Brouwer’s theorem has many applications in analysis, differential equation and generally in proving all kinds of so-called existence theorems for many types of equations. Its discovery has had a tremendous influence in the development of several branches of mathematics, especially algebraic topology.

An important generalization of Brouwer’s theorem was discovered in 1930 by J. Schauder [122] it may be stated as follows: **Any non empty, compact convex subset** \( K \) **of a Banach space has the topological fixed point property.** The compactness condition on subset is a stronger one. In analysis, many problems do not have a compact setting. It is natural to modify the theorem by relaxing the condition of compactness. In 1930 Schauder [122] also proved a theorem for a compact map which is known as second form of above stated theorem. Before stating the theorem, we need the following definition.
Definition 1.1.12 A self mapping $T$ of a Banach space $X$ is called a compact mapping (or completely continuous) if $T$ is continuous and $T$ maps bounded set to precompact set.

Another fixed point theorem of Schauder [122] stated that; Every compact self mapping of a closed bounded convex subset of a Banach space has at least one fixed point. Whereas in 1935 Tychonoff [141] modified Brouwer's result to a compact convex subset of a locally convex topological vector space, it may be stated as follows; Any compact convex non-empty subset of a locally convex space has the fixed point property.

Definition 1.1.13: Suppose $X$ is a topological space, $x_0$ is a point in $X$ and $f : X \to \mathbb{R} \cup (-\infty, \infty)$ is an extended real-valued function. $f$ is called upper semi-continuous at $x_0$ if for every $\epsilon > 0$ there exists a neighbourhood $U$ of $x_0$ such that $f(x) \leq f(x_0) + \epsilon$ for all $x \in U$. Equivalently this can be expressed as

$$\limsup_{x \to x_0} f(x) \leq f(x_0),$$

where 'lim sup' is the limit superior (of the function $f$ at point $x_0$)

The function $f$ is called upper semi-continuous if it is upper semi-continuous at every point of its domain.

Definition 1.1.14: Suppose $X$ is a topological space, $x_0$ is a point in $X$ and $f : X \to \mathbb{R} \cup (-\infty, \infty)$ is an extended real-valued function. $f$ is called lower semi-continuous at $x_0$ if for every $\epsilon > 0$ there exists a neighbourhood $U$ of $x_0$ such that $f(x) \geq f(x_0) - \epsilon$ for all $x \in U$. Equivalently this can be expressed as

$$\liminf_{x \to x_0} f(x) \geq f(x_0),$$

where 'lim inf' is the limit inferior (of the function $f$ at point $x_0$)
The function $f$ is called lower semi-continuous if it is lower semi-continuous at every point of its domain.

A primary early example of an extension of Banach’s principle is a theorem of Caccioppoli [14] which asserts that the Picard iterates of a mapping $T$ converge in a complete metric space $X$, provided that for each $n \geq 1$ there exists a constant $C_n$ such that

$$d(T^n(x), T^n(y)) \leq C_n d(x, y), \forall x, y \in X \quad \text{where} \quad \sum C_n < \infty. \quad (1.1.3)$$

In 1961 Boyd and Wong [142] obtained a more general result. In their theorem it is assumed that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous from the right that is $r_n \downarrow r \geq 0 \Rightarrow \lim_{n \to \infty} \sup \psi(r_n) \leq \psi(r)$. Boyd and Wong also showed in [142] that if the space $X$ is metrically convex, then the upper semi-continuity assumption on $\psi$ can be dropped. Matkowski has extended this fact even further in [83] by showing that it suffices to assume that $\psi$ is continuous at 0 and that there exists a sequence $t_n \downarrow 0$ for which $\psi(t_n) < t_n$.

Subsequently in 1962, Rakotch [116] generalized Banach contraction principle in the following way; “Let $X$ be a complete metric space and suppose $f : X \to X$ satisfies $d(f(x), f(y)) \leq \alpha d(x, y).d(x, y)$, for all $x, y \in X$, where $\alpha : \mathbb{R}^+ \to [0, 1)$ is monotonically decreasing. Then $f$ has a unique fixed point $\bar{x}$ and $\{f^n(x)\}$ converges to $\bar{x}$ for each $x \in X$”.

Edelstien [31] generalized Banach’s contraction mapping principle in more general case, while Chu & Diaz [24] and Sehgal [126] also extended the concept of contraction mapping. Sehgal obtained the fixed point of a continuous function $T$ of a complete metric space $(X, d)$ into itself for which there exists positive integer $n = n(x)$ satisfying
the following inequality,
\[ d(T^n x, T^n y) \leq \beta d(x, y), \quad \forall x, y \in X \text{ and } 0 < \beta < 1 \quad (1.1.4) \]

Sehgal’s result was further extended and generalized by Guseman [46], Iseki [54] & Sharma [128]. Further Rakotch [116], Boyd and Wong [142] generalized, Banach’s contraction principle by replacing Lipschitz constant \( k \) by some real valued functions, whose values are less than unity.

In the sequel a natural question arises that does there exist a contractive mapping, which does not enforce a mapping to be continuous? In 1968, it was Kannan [66] who affirmatively find the answer and obtained fixed point theorem for a self mapping \( T \) on complete metric space \( X \) satisfying the condition: for all \( x, y \in X \) there exists \( \gamma \in [0, 1/2) \) such that
\[ d(Tx, Ty) \leq \gamma [d(x, Tx) + d(y, Ty)] \]

After that a spate of results appeared containing a variety of contractive conditions such as Kannan [67] and Fukushima [38].

Now, we give the examples from Lahri [75] and compare the Banach’s and Kannan’s contraction conditions, as follows;

**Example 1.1.1** Let \( X = [0, 1] \) and \( T \) be the self mapping of non empty set \( X \) such that
\[ Tx = \frac{x}{4} \text{ for } x \in [0, 1/2), \quad Tx = \frac{x}{5} \text{ for } x \in [1/2, 1]. \]
In this example \( T \) is discontinuous at \( x = 1/2 \) therefore Banach’s contraction condition is not satisfied but for \( \gamma = 4/9 \) it satisfied Kannan’s contractive contractive condition.

**Example 1.1.2** Let \( X = [0, 1] \) and \( T \) be the self mapping of non empty set \( X \) such that; \( Tx = x/3 \) for \( x \in [0, 1] \).
In this example Banach's contraction condition is satisfied but for \( x = 1/3 \) and \( \gamma = 0 \) it does not satisfy Kannan's contractive condition.

In 1969, Meir and Keeler [85] have given a different approach which also received substantial attention. Following is the formulation:

Let \((X, d)\) be a complete metric space and suppose \( f : X \rightarrow X \) satisfies the condition; given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that,

\[
\varepsilon < d(x, y) \leq \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon,
\]

then \( f \) has a unique fixed point \( \bar{x} \) and \( \lim_{n \to \infty} f^n(x) = \bar{x}, \forall \ x \in X. \)

In 1997, Jachymski [55] gave a detailed comparison of the relationship between the results of Boyd and Wong [12] and Matakowski [83], as well as several related contractive conditions. Many other fixed point theorems of Meir-Keeler type are given by Assad [5], Park-Bae [107], Jungck [61], Pant [100]. In 2000, Chen [19], has given a variant of the Meir-Keeler theorem in ordered Banach spaces. Also in the spirit of the Boyd and Wong approach, Lim [77] has characterized the Meir-Keeler condition in terms of a function \( \psi. \)

Let \((X, d)\) be a metric space. Then, following Nadler [93], we recall

\[(N-1) \ CB(X) = \{A \in 2^X : A \text{ is nonempty closed and bounded set}\},\]

\[(N-2) \ C(X) = \{A \in 2^X : A \text{ is nonempty compact set}\}.\]

\[(N-3) \text{ For nonempty subsets } A, B \text{ of } X \text{ and } x \in X\]

\[
d(x, A) = \inf\{d(x, a) : a \in A\},
\]

\[
d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},
\]

\[
\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \text{ and }
\]
\[ H(A, B) = \max\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\} \].

Here we note that \( d(A, B) \leq H(A, B) \leq \delta(A, B) \). Also

(N-4) \( \delta(A, B) = 0 \) if and only if \( A = B = \{a\} \)

(N-5) \( \delta(A, B) = \delta(B, A) \)

(N-6) \( \delta(A, B) \leq \delta(A, C) + \delta(C, B) \).

**Definition 1.1.15:** Let \( K \) be a nonempty closed subset of metric space \((X, d)\). A mapping \( T : K \rightarrow CB(X) \) is said to be continuous at \( x_0 \in K \) if for every \( \epsilon > 0 \), there exists a \( \eta(\epsilon) > 0 \) such that \( H(Tx, Tx_0) < \epsilon \), whenever \( d(x, x_0) < \eta(\epsilon) \). If \( T \) is continuous at every point of \( K \), then \( T \) is continuous on \( K \).

**Lemma 1.1.1:** Let \( A, B \in CB(X) \) and \( k > 1 \). Then for each \( a \in A \), there exists a point \( b \in B \) such that \( d(a, b) \leq k H(A, B) \).

**Lemma 1.1.2:** Let \( A, B \in CB(X) \) and \( a \in A \), then for any positive number \( q < 1 \) there exists \( b = b(a) \) in \( B \) such that, \( q d(a, b) \leq H(A, B) \).

Alternately, if \( a \in A \) and \( \epsilon > 0 \) then using above lemma, one can always find \( b \in B \) such that \( d(a, b) \leq H(A, B) + \epsilon \).

**Definition 1.1.16:** Let \( X \) and \( Y \) be two non-empty sets. A multi-valued mapping \( T : X \rightarrow Y \) is a subset of \( F \subseteq X \times Y \). The inverse of multi-valued mapping \( T^{-1} : Y \rightarrow X \) is defined by \((y, x) \in T^{-1} \) if and only if \((x, y) \in A \).

**Definition 1.1.17:** Let \( X \) be a metric space with usual metric \( d \) and \( CB(X) \) the family of nonempty bounded closed subsets of \( X \). A multi-valued mapping \( T : X \rightarrow CB(X) \) is said to have a fixed point if the point belongs to its image set (i.e. \( z \in Tz \) for some \( z \in X \)).
Definition 1.1.18: Let \( X \) be a metric space with usual metric \( d \) and \( CB(X) \) the family of nonempty bounded closed subsets of \( X \). A multi-valued mapping \( T : X \to CB(X) \) is called a Lipschitz mapping with Lipschitz constant \( \alpha \geq 0 \), if \( H(Tx,Ty) \leq \alpha d(x,y) \), for all \( x, y \in X \). \( T \) is called nonexpansive if \( \alpha = 1 \) and a set-valued contraction if \( \alpha < 1 \).

Using Hausdorff metric, Banach contraction principle was extended nicely to set valued mappings by Markin [81] & Nadler [93]. The key idea is the following;

If \( A \) and \( B \) are non empty closed bounded subsets of a metric space and if \( x \in A \) then given \( \varepsilon > 0 \), there exists a point \( y \in B \) such that \( d(x,y) \leq H(A,B) + \varepsilon \), where \( H(A,B) \) denotes the Hausdorff distance between \( A \) and \( B \).

Nadler’s theorem may be stated as; **Every multi-valued contraction on a complete metric space has a fixed point.** Nadler’s multi-valued contraction theorem has led to a good fixed point theory for multi-valued operators in non-linear analysis.

Subsequently number of generalizations of the multi-valued contraction principle (which states that a multi-valued contraction mapping on a complete metric space having values in the set of all closed and bounded subsets of the metric space possesses a fixed point, [93]) were obtained, among others, are Ciric [25], Khan [69] and Smithson [138]. In [5], Assad and Kirk gave sufficient conditions for non-self mappings to ensure the fixed point by proving a result on multi-valued contraction in complete metrically convex metric spaces.

Definition 1.1.19 [5]: A metric space \((X,d)\) is said to be metrically convex if for any \( x, y \in X \) (with \( x \neq y \)) there exists a point \( z \) in \( X \) (\( x \neq y \neq z \)) such that

\[
d(x,z) + d(z,y) = d(x,y)
\]

Definition 1.1.20 [5]: Let \( K \) be any non-empty closed subset of a complete metrically convex metric space.
convex metric space \((X, d)\). Then for any \(x \in K\) and \(y\) not in \(K\) there exists a point \(z \in \partial K\) (the boundary of \(K\)) such that

\[
d(x, z) + d(z, y) = d(x, y)
\]

**Lemma 1.1.3** [5]: Let \(X\) be a complete and metrically convex metric space, \(K\) be a nonempty closed subset of \(X\) and \(T\) be a contraction mapping from \(K\) into \(CB(X)\). If \(Tx \subset K\) for each \(x \in \partial K\) then there exists \(z \in K\) such that \(z \in T(z)\).

The work of Rhoades [117] covers the period up through the mid seventies. In this connection, we can also see a paper by Hegedus [49], a similar survey by Park and Rhoades [106] and an analysis of work of Rhoades [117] by Callaco and Silva [27].

In 1963, Gahaler [39] introduced 2-metric spaces and after that there was a spate of papers dealing with this generalized spaces.

The notion of 2-metric, introduced by Gahlar[39] is as follows:

Let \(d : X \times X \times X \to \mathbb{R}\) be a function satisfying the following conditions:

(d1) For every distinct pair \(x, y \in X\), there is a \(z \in X\) such that \(d(x, y, z) \neq 0\).

(d2) \(d(x, y, z) = 0\) if any two of the triplet \(x, y, z\) are equal.

(d3) \(d(x, y, z) = d(y, z, x) = ...\) (symmetry)

(d4) \(d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)\) for \(x, y, z, a \in X\).

Then the function \(d\) is called a 2-metric on \(X\) and together with a 2-metric, \(X\) is called a 2-metric space.

In 1992 Dhage[28] introduced the notion of D-metric space called generalized metric space and proved some fixed point theorems. Let \(D : X \times X \times X \to \mathbb{R}\) be a function satisfying the following conditions:
Then the function $D$ is called a generalized metric or more specifically a $D$-metric space and in the analogous way of ordinary metric spaces we denote it by $(X, D)$. Then usual ordinary metric is generally called the distance function, similarly we may call a $D$-metric to be diameter function of the points of $X$. Dealing with $D$-metric space, Ahmed et al [2], Dhage [28, 29], Rhoades [119], Singh and Sharma [134] and others made a significant contribution in fixed point theory. Recently (2004 and 2005) Naidu et al [94] have observed that almost all theorems in $D$-metric spaces are not valid. In, [125] Sedghi et al have introduced $D^*$-metric space which is probable modification of the definition of $D$-metric introduced by Dhage [28, 29]. Following is the definition of $D^*$-metric space:

**Definition 1.1.21** [125] Let $X$ be a nonempty set. A generalized metric (or $D^*$-metric) on $X$ is a function, $D^* : X^3 \rightarrow [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:

1. $D^*(x, y, z) \geq 0$
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$,
3. $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where $p$ is a permutation function,
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair $(X, D^*)$ is called a generalized metric ($D^*$-metric) space.
In an attempt to respond to classical concern about imprecision in the natural world, Menger [86, 87] introduced the concept of a probabilistic metric space. These are spaces in which the distance between points is a probability distribution on $\mathbb{R}^+$ rather than a real number, i.e. to say given any two points $p$ and $q$ of a metric space, consider a single non-negative real number $F(p,q)$ as a measure of the distance between $p$ and $q$. A distribution function $F_{p,q}(x)$ is introduced which gives the probabilistic interpretation as the distance between $p$ and $q$ and is less than $x$ ($x > 0$).

**Definition 1.1.22** [86]: A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution if it is non-decreasing left continuous with $\inf \{F(t) : t \in \mathbb{R}\} = 0$ and $\sup \{F(t) : t \in \mathbb{R}\} = 1$. We shall denote by $L$ the set of all distribution functions defined by,

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

**Definition 1.1.23** [192]: A probabilistic metric space (PM-space) is an ordered pair $(X,F)$ where $X$ is an abstract set of elements and $F : X \times X \to L$ is defined by $(p,q) \in F_{p,q}$ where $L$ is the set of all distribution function i.e. $L = \{F_{p,q} : p,q \in X\}$ where the function $F_{p,q}$ satisfy:

1. **(FM-1)** $F_{p,q}(x) = 1$ for all $x > 0$ iff $p = q$,
2. **(FM-2)** $F_{p,q}(0) = 0$;
3. **(FM-3)** $F_{p,q} = F_{q,p}$;
4. **(FM-4)** If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x+y) = 1$,

where $x,y \in \mathbb{R}$ the set of real numbers.

**Definition 1.1.24** [123]: A mapping $t : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies following conditions:

1. $t(a,1) = a; t(0,0) = 0$
(2) \( t(a, b) = t(b, a) \);

(3) \( t(c, d) \leq t(a, b) \) whenever \( c \leq a \) and \( d \leq b \);

(4) \( t(t(a, b), c) = t(a, t(b, c)) \).

**Definition 1.1.25** [132]: A Menger space is a triplet \((X, F, t)\) where \((X, F)\) is PM-space and \( t \) is a t-norm such that for all \( p, q, r \in X \) and for all \( x, y > 0 \):

\[
F_{p,q}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y))
\]

**Proposition 1.1.1** [132]: If \((X, d)\) is a metric space then the metric \( d \) induces a mapping \( F: X \times X \to L \), defined by \( F_{p,q}(x) = H(x - d(p,q)), p, q \in X \) and \( x \in \mathbb{R} \). Further, if the t-norm \( t: [0,1] \times [0,1] \to [0,1] \) is defined by \( t(a, b) = \min(a, b) \), then \((X, F, t)\) is a Menger space. It is complete if \((X, d)\) is complete.

The space \((X, F, t)\) so obtained is called the induced Menger space.

In 1983, Hicks [51] considered another notion of probabilistic contraction mappings which is called C-contraction and is comparable with probabilistic contraction. Many fixed point theorems for mappings in Menger spaces have been proved by several authors, eg.- Bharucha-Raid [9], Chang [17], Hadzić [48]. For a detailed discussion of Menger spaces and their applications the works of Schweizer and Skaler [123] is noteworthy. The theory of Menger space is of fundamental importance in probabilistic functional analysis.

The theory of fuzzy sets was initiated by Zadeh [144], and it is now a vigorous area of research with manifold applications ranging from engineering and computer science to medical diagnosis and social behaviour, to name just a few. We can also see Yager [143] for a recent collection of research papers on the applications of several investigations to fuzzify a number of important mathematical structures, such as topological spaces, algebraic structure, fuzzy graphs, fuzzy linear programming etc.
Definition 1.1.26 [144]: Let $X$ be a non-empty set. A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0, 1]$.

Now we give some notations and notions from Kaleva and Seikkala [64];

Definition 1.1.27: A fuzzy number is a fuzzy set on the real axis, i.e. a mapping $x : \mathbb{R} \rightarrow [0, 1]$ associated with each real number $t$ its grade of membership $x(t)$. For $\alpha \in [0, 1]$ and a fuzzy number $x$, the set $[x]_{\alpha} = \{u \in \mathbb{R} : x(u) \geq \alpha\}$ is called a $\alpha$-level set of $x$.

A fuzzy number $x$ is called to be a convex if $r, s, t \in \mathbb{R}, r \leq s \leq t$, implies $\min\{x(r), x(t)\} \leq x(s)$.

A fuzzy number $x$ is said to be normal if there exists a point $u \in \mathbb{R}$ such that $x(u) = 1$.

A fuzzy number $x$ is said to be non-negative if $x(u) > 0$ for all $u < 0$. The fuzzy number $0$ is defined by $0(u) = 1$ for $u = 0$ and $0(u) = 0$ for $u \neq 0$.

Kaleva and Seikkala [64] introduced the notion of a fuzzy metric space in the following way;

Definition 1.1.28: Let $X$ be a non-empty set, $d : X \times X \rightarrow G$, where $G$ be a set of all nonnegative upper semi-continuous normal convex fuzzy numbers and let the mappings $L, R : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, nondecreasing in both arguments such that $L(0, 0) = 0$ and $R(1, 1) = 1$ and denote the $\alpha$-level set of $d(x, y)$ by

$$[d(x, y)]_{\alpha} = [\lambda_{\alpha}(x, y), \rho_{\alpha}(x, y)]$$

for all $x, y \in X, \alpha \in (0, 1)$, where $[d(x, y)]_{\alpha}$ is the $\alpha$-level set of fuzzy number $d(x, y) \in G$, which actually is a closed interval of $\mathbb{R}$ and $\lambda_{\alpha}(x, y), \rho_{\alpha}(x, y)$ are the left and right
extreme points of the closed interval $[d(x, y)]_a$, respectively. The quadruple $(X, d, L, R)$ is called a **fuzzy metric space** and $d$ a fuzzy metric, if

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) for all $x, y \in X$

\[ (a) \quad d(x, y)(s+t) \geq L(d(x, z)(s), d(z, y)(t)) \quad \text{whenever} \quad s \leq \lambda_1(x, z), t \leq \lambda_1(z, y) \quad \text{and} \quad s + t \leq \lambda_1(x, y) \]

\[ (b) \quad d(x, y)(s + t) \leq R(d(x, z)(s), d(z, y)(t)) \quad \text{whenever} \quad s \geq \lambda_1(x, z), t \geq \lambda_1(z, y) \quad \text{and} \quad s + t \geq \lambda_1(x, y). \]

**Definition 1.1.29** [64]: A sequence $\{x_n\}$ in a fuzzy metric space $(X, d, L, R)$ is said to converge to $x$ if $\lim_{n \to \infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ in a fuzzy metric space $(X, d, L, R)$ is said to be Cauchy if $\lim_{m,n \to \infty} d(x_m, x_n) = 0$. A fuzzy metric space $(X, d, L, R)$ is said to be complete if every Cauchy sequence in $X$ converges.

**Remark 1.1.1** (Theorem 3.2) [64]: If $(X, d, L, R)$ is a fuzzy metric space with $\lim_{a \to 0} R(a, a) = 0$, then there exists a topology $\tau$ on $X$ such that $(X, \tau)$ is a Hausdorff topological space and $v(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0, \alpha \in (0, 1]\}$, $x \in X$, is a basis of neighbourhoods of the point $x$ for the topology $\tau$, where $U_x(\epsilon, \alpha) = \{y \in X : \rho_\alpha(x, y) < \epsilon\}$, and $\rho_\alpha(x, y)$ is the right end point of $[d(x, y)]_\alpha$.

Recently many authors observed that the various contraction mappings in metric spaces may be exactly translated in to probabilistic metric spaces or fuzzy metric spaces.
endowed with special $t$-norm, such as min-norm. In both settings it is possible to introduce *metric* concepts and ask about the corresponding fixed point theorem for contraction mappings as well as their applications. Kramosil and Michalek [72] defined fuzzy metric space as follows;

**Definition 1.1.30** [72]: The 3-triple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary non-empty set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

1. **(FM-1)** $M(x, y, 0) = 0$,
2. **(FM-2)** $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
3. **(FM-3)** $M(x, y, t) = M(y, x, t)$,
4. **(FM-4)** $M(x, y, t) * M(y, z, t) \geq M(x, z, t + s)$,
5. **(FM-5)** $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous
6. **(FM-6)** $\lim_{t \to 0} M(x, y, t) = 1$.

In the definition of George and Veeramani [44], $M$ is a fuzzy set on $X^2 \times (0, \infty)$ and (FM-1), (FM-2), (FM-5) are replaced, respectively, by (GV-1), (GV-2), (GV-5) given below (the axiom (GV-2) is reformulated as in [44], Remark 1):

1. **(GV-1)** $M(x, y, 0) > 0 \ \forall \ t > 0$.
2. **(GV-2)** $M(x, x, t) = 1 \ \forall \ t > 0$ and $x \neq y \Rightarrow M(x, y, t) < 1 \ \forall \ t > 0$
3. **(GV-5)** $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous $\forall \ x, y \in X$.

**Example 1.1.3** [44]: Let $X = \mathbb{R}$. Define $a * b = ab$ and

$$M(x, y, t) = \left[ \exp\left(\frac{|x - y|}{t}\right) \right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a fuzzy metric space.
Example 1.1.4 [44]: Let $(X, d)$ be a metric space. Define $a \ast b = ab$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, \quad k, m, n \in \mathbb{R}^+.$$ 

Then $(X, M, \ast)$ is a fuzzy metric space.

Note that this example holds even with the t-norm $a \ast b = \min(a, b)$, also if we take $k = m = n = 1$, we get $M(x, y, t) = \frac{1}{t+d(x,y)}$. We call this fuzzy metric $M$, induced by the metric $d$ the standard fuzzy metric.

Definition 1.1.31 [44]: Let $(X, M, .)$ be a fuzzy metric space. One can define open ball $B(x, r, t)$ as well as closed ball $B[x, r, t]$ with center $x \in X$ and radius $r, 0 < r < 1, t > 0$ as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},$$

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}.$$ 

It is easy to see that every open ball is an open set and every closed ball is a closed set. Continuing in this direction George and Veeramani [44] proved that every fuzzy metric spaces is Hausdorff and the topology induced by the ordinary metric $d$ and the topology induced by the fuzzy metric $M$ are the same. Also they define convergent sequence, Cauchy sequence in fuzzy metric space and conclude that the definition of Cauchy sequence given by Grabiac [43] is not correct.

The fixed point theorems in probabilistic metric spaces and fuzzy metric spaces for some contraction mappings were investigated by many authors [88, 89, 90], [105], [115], [120], [124], [126]. Sehgal & Bharucha-Reid [126] obtained a generalization of Banach contraction principle in a complete Menger space. In 1998, Grabiec [43] extended the well known fixed point theorems of Banach and Edelstein [31] to fuzzy metric spaces in the sense of Kramosil and Michalek [72]. Moreover, it appears that the study of fuzzy metric spaces done by Kramosil and Michalek pave a way for developing a
soothing machinery in the field of fixed point theorems and in particular for the study of contractive type maps.

Fang [34] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize, unify and extend some main results of ([33], [64]). Following Grabiec, Kramosil and Michalek [72] and Mishra et al [92] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric spaces, which generalize, extend and fuzzyfy several fixed point theorems for contractive type maps on metric spaces and other spaces. Recently (in 2000) Mihet [88] obtained a generalization of Banach contraction principle in a complete fuzzy metric space. In 2003, Song [139] has modified the definition and the proof of existence of Cauchy sequence in fuzzy metric spaces defined by George and Veeramani [44] and also gave some interesting fixed point theorems in fuzzy metric spaces.

It is never possible to give a complete description of a wide subject like fixed point theory in a few pages. However, for a comprehensive study of fixed point theory and its related results the books by Goebel and Kirk [42] and Granas and Dugundji [44] are of special use.