Chapter 6

ANALYSIS OF STEADY FLOW IN SLENDER TUBES

6.1 Introduction

This chapter is a contribution to the study of steady flow in slender tubes. A tube of circular cross section whose radius is a function of a slow variable \( Z = \frac{1}{R} z \), where \( z \) is the co-ordinate in the axial direction and \( R \) is a large stream wise Reynold number, is termed as slender tube. The case of slender tube is slightly more difficult because of the extra complexity of the equation of motion.

The flow in slender channels has been intensively studied over the past few years. Frankel[61,62] in his study described that, for a particular class of slender channels with very small curvature of the walls, the solution can be expanded systematically in such a way that the appropriate local Jeffery-Hamel profile is the first approximation at each solution. Blottner[63] used a marching finite difference scheme to calculate non-self similar solutions for various channels. Daniels and Eagles[64] developed a theory for steady axisymmetric viscous flow in tubes of a slowly varying radius \( H(Z) = \exp(aZ) \) and showed that the flow was governed by an ordinary differential equation. Eagles[65], generalised this work to tubes which are locally exponential.

In this chapter, we study the problem of flow in slender tubes studied by Eagles and Muwezwa[66] with the aid of computer extended perturbation
series. Van Dyke[67] used long series analysis in laminar flow through a heated horizontal pipe. Recently, in the analysis of internal laminar flow separation, Bujurke, Pedley and Tutty[68] have used series analysis satisfactorily. The aim of the present work is to extend the direct series expansion to estimate the higher order perturbations for two different tubes given by \( H_1(Z) = 1 + \frac{1}{2} \tanh Z \) and \( H_2(Z) = 1 + \frac{1}{2} \tanh^2 Z \) and also to understand its general convergence properties and to use it to examine the analytic nature of separation. In the direct series expansion, the first 15 terms for different derived quantities are evaluated. The region of validity of the series in \( \lambda = \varepsilon R, \varepsilon \) is a small parameter and \( R \) is streamwise Reynolds number) increased by using Pade' approximants for summing them. It is possible to analyse the flow structure for fairly large values of \( \lambda \) compared with previous study (Eagles and Muwezwa[66]). Eagles et al [66] considered only three terms of the series for small \( \lambda \). However, as the problem has to be analysed for moderately large \( \lambda \), it requires the long series analysis and large number of terms in the expansion are required. As complex expressions involving elementary functions appear in the successive terms of the series it is possible to calculate only fifteen terms using MATHEMATICA.

6. 2. Formulation of the Problem

The full governing equations (Navier-stokes equations) with zero external forces are
Figure 6.1A: Sketch of slender tube 1: $H_1(Z) = 1 + \frac{1}{2} \tanh Z$

Figure 6.1B: Sketch of slender tube 2: $H_2(Z) = 1 + \frac{1}{2} \tanh^2 Z$
\[
\frac{\partial U}{\partial t} + U \cdot \nabla U = -\frac{1}{\rho} \nabla P + v \nabla^2 U
\]  
(6.2.1)

\[\nabla \cdot U = 0\]  
(6.2.2)

where \(U\), \(\rho\) and \(P\) are the velocity, density and pressure of the fluid respectively and \(v\) is the kinetic viscosity.

For axisymmetric flows with cylindrical polar co-ordinates \((r', \theta, z')\) and \(U=(u, 0, v)\), a stream function may be defined by

\[
u = \frac{1}{r'} \frac{\partial \psi'}{\partial z}, \quad v = \frac{1}{r'} \frac{\partial \psi'}{\partial r}
\]  
(6.2.3)

which satisfies the continuity equation (6.2.2). The variables are made dimensionless as follows

\[
z = \frac{z'}{L}, \quad r = \frac{r'}{L}, \quad \psi = \frac{\psi'}{L}, \quad t = \frac{M t'}{L^3}
\]  
(6.2.4)

where \(L\) is the radius of the tube at \(z' = 0\) and \(M\) is the volumetric flow rate.

The non-dimensional stream function satisfies the equation.

\[
\frac{1}{r} \frac{\partial}{\partial t} \frac{D^2 \psi}{r} + \frac{1}{r^2} \left( \psi, \frac{\partial}{\partial z} - \psi_z \frac{\partial}{\partial r} \right) D^2 \psi + \frac{1}{r^3} \left( 2 \psi_z \psi_{zz} + 3 \psi \psi_{rr} - \psi_z \psi_{r} \right) 
\]

\[-\frac{3}{r^2} \psi_z \psi_r = \frac{1}{R} \left( -D^4 \psi - 2 \frac{2}{r^2} \left( \psi_{zzz} + \psi_{r} \right) + \frac{3}{r^3} \psi_{rr} - \frac{3}{r^2} \psi \right)\]

(6.2.5)

where \(R = \frac{M}{\rho v}\) is the Reynold number and \(D^2 = \frac{\partial^2}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\).

The boundary conditions are
(i) \( \psi = O(r^2) \) as \( r \to 0 \)

(ii) \( \psi_r = 0 \) at the tube wall

(iii) \( \psi = \frac{1}{2\pi} \) at the tube wall

The total stream function \( \psi \) is considered to be made of two parts, steady state and the time dependent part. That is

\[
\psi(r, z, t) = \hat{\psi}(r, z) + \hat{\psi}_r(r, z, t)
\]

We have considered the steady state stream function equation [Muwezwa\[69\]] for flow in tubes of slowly varying radius by defining the boundary as \( r = H(z) \), where \( Z = \varepsilon z \) and \( \varepsilon \) is a small parameter. It is also useful to introduce the coordinate

\[
\eta = \frac{r}{H(z)}
\]

then \( \eta = 1 \) is the wall of the tube.

The steady state stream function \( \psi(\eta, Z) \) was expanded in the form

\[
\psi = \psi_0(\eta, Z) + \varepsilon^2 \psi_2(\eta, Z) + \ldots
\]

Substituting into steady state equation (3.2.8) and defining \( \psi_0 = F(\eta, Z) \), the \( O(1) \) equation for \( F \) becomes

\[
L\{F\} = \lambda \left[ 4 \frac{H'}{H} \left( \frac{1}{\eta^2} F_\eta^2 - \frac{1}{\eta} F_\eta F_{\eta\eta} \right) + F_\eta \left( \frac{1}{\eta} F_{\eta\eta} - \frac{1}{\eta^2} F_{\eta\eta} \right) 
+ F_\eta \left( - \frac{1}{\eta^2} F_{\eta\eta} + \frac{3}{\eta^3} F_{\eta\eta} - \frac{3}{\eta^4} F_\eta \right) \right]
\]

(6.2.11)
where $\lambda = \epsilon R$, $R$ is the stream wise Reynolds number

$$L = \frac{\partial^4}{\partial\eta^4} - \frac{2\partial^3}{\eta \partial\eta^3} + \frac{3\partial^2}{\eta^2 \partial\eta^2} - \frac{3\partial}{\eta^3 \partial\eta}$$

with boundary conditions

$$F = O(\eta^2) \quad \text{as } \eta \to 0 \quad (6.2.12a)$$
$$F = \frac{1}{2\pi} \quad \text{as } \eta = 1 \quad (6.2.12b)$$
$$F_\eta = 0 \quad \text{at } \eta = 1 \quad (6.2.12c)$$

The problem for $F$ is called the slender tube problem and it well constitutes our basic flow.

The dimensionless fluid velocities in the $r$ and $z$ directions respectively, within this approximations, are

$$u = \frac{1}{r} \frac{\partial\psi'}{\partial z} = \epsilon \left( -1 \frac{\partial E}{\eta H \partial Z} + H' \frac{\partial F}{H^2 \partial\eta} \right) \quad (6.2.13)$$

$$v = \frac{1}{r} \frac{\partial\psi'}{\partial r} = \frac{1}{\eta H^2} \frac{\partial F}{\partial\eta} \quad (6.2.14)$$

Equation (6.2.12a) ensures that the fluid velocity is finite at the centre of the tube while (6.2.12b) and (6.2.12b) ensure that it is zero on the wall $r = H(Z)$ or $\eta = 1$.

6.3. Method of solution (Extended perturbation solution)

A solution of (6.2.11) is attempted of the form

$$F(\eta, Z; \lambda) = F_0(\eta, Z) + \sum_{n=1}^{\infty} F_n(\eta, Z)\lambda^n \quad (6.3.1)$$
where $F_n$ is determined as functions of $\eta$ and $Z$ satisfy $F_n(1, Z) = 1$ for all $n \geq 1$ and $F_0(1, Z) = 1$. Substituting (6.3.1) into (6.2.11)-(6.2.12) and equating the like powers of $\lambda$ on both sides, the equations for higher order approximations ($n \geq 1$) in the series (6.3.1) can be written in terms of recurrence relations given by (for even and odd values of $n$)

$$L\{F_0\} = 0 \quad (6.3.2)$$

$$L\{F_1\} = 4 \frac{H'(z)}{H(z)} \left[ -\frac{1}{\eta} \left( \frac{F_0}{\eta} \right)^2 \right] + F_0 \left( \frac{1}{\eta} F_{0,\eta \eta} - \frac{1}{\eta^2} F_{0,\eta z} \right)$$

$$+ F_{0, z} \left( -\frac{1}{\eta} F_{0,\eta \eta \eta} - \frac{3}{\eta^2} F_{0,\eta \eta} - \frac{3}{\eta^3} F_{0,\eta} \right) \quad (6.3.3)$$

when $n$ is even, $n \geq 2$ (i.e. $n = 2, 4, 6, \ldots \ldots$)

$$L\{F_n\} = 4 \frac{H'(z)}{H(z)} \left\{ \frac{2}{n^2} \sum_{r=0}^{n-2} F_n F_{n-2-2r} \frac{F_{n+2r}}{2^2} - \frac{1}{n} \sum_{r=0}^{n-1} F_{n-1-r, \eta} F_{r, \eta} \right\}$$

$$+ \sum_{r=0}^{n-1} F_{n-r, \eta} \left[ \frac{1}{\eta} F_{r,\eta \eta \eta} - \frac{1}{\eta^2} F_{r,\eta \eta} \right] + F_{n-r, z} \left[ -\frac{1}{\eta} F_{r,\eta \eta \eta} + \frac{3}{\eta^2} F_{r,\eta \eta} - \frac{3}{\eta^3} F_{r,\eta} \right] \quad (6.3.4)$$

when $n$ is odd, for $n \geq 3$ (i.e for $n = 3, 5, \ldots \ldots$)

$$L\{F_n\} = 4 \frac{H'(z)}{H(z)} \left\{ \frac{1}{\eta^2} \left( \frac{F_{n-1}}{2^2} \right)^2 + \frac{2}{\eta^2} \sum_{r=0}^{n-3} F_n F_{n-3-2r} \frac{F_{n+2r}}{2^2} - \frac{1}{n} \sum_{r=0}^{n-1} F_{n-1-r, \eta} F_{n-r, \eta} \right\}$$

$$+ \sum_{r=0}^{n-1} F_{n-r, \eta} \left[ \frac{1}{\eta} F_{r,\eta \eta \eta} - \frac{1}{\eta^2} F_{r,\eta \eta} \right] + F_{n-r, z} \left[ -\frac{1}{\eta} F_{r,\eta \eta \eta} + \frac{3}{\eta^2} F_{r,\eta \eta} - \frac{3}{\eta^3} F_{r,\eta} \right] \quad (6.3.6)$$
with corresponding boundary conditions

\[
\begin{align*}
F_n &= O(\eta^2) \quad \text{as } \eta \to 0 \quad \text{for all } n \geq 0 \\
F_{n,\eta} &= 0 \quad \text{when } \eta = 1 \quad \text{for all } n \geq 0 \\
F_0 &= \frac{1}{2\pi} \quad \text{and} \quad F_n = 0 \quad \text{when } \eta = 1 \quad \text{for all } n \geq 1
\end{align*}
\]  

(6.3.7)

The solutions of the above equations up to the term \((\lambda^3)\) are (Eagles et al.[66])

\[
F_0(\eta) = \frac{1}{\pi} \left( \eta^2 - \frac{1}{2} \eta^4 \right) \tag{6.3.8a}
\]

which represents the local poiseuille -flow solution

\[
F_1(\eta, Z) = \frac{1}{\pi^2} \frac{H'(Z)}{H(Z)} \left[ \frac{2\eta^2}{9} - \frac{\eta^4}{2} + \frac{\eta^6}{3} - \frac{\eta^8}{18} \right] \tag{6.3.8b}
\]

\[
F_2(\eta, Z) = \frac{1}{\pi^3} \left\{ \begin{array}{l}
\left( \frac{H'(Z)}{H(Z)} \right)^2 \left[ \frac{37\eta^2}{270} - \frac{871\eta^4}{2160} + \frac{97\eta^6}{216} - \frac{35\eta^8}{144} + \frac{\eta^{10}}{15} - \frac{7\eta^{12}}{1080} \right] \\
+ \frac{H(Z) H'(Z)}{H(Z)^2} \left[ \frac{-13\eta^2}{900} + \frac{29\eta^4}{720} - \frac{\eta^6}{24} + \frac{\eta^8}{48} - \frac{\eta^{10}}{180} + \frac{\eta^{12}}{1800} \right]
\end{array} \right\} \tag{6.3.8c}
\]
\[
F_{3}(\eta, Z) = \frac{1}{\pi^3} \left\{ \frac{H'(Z)^3}{H(Z)^3} \right. \\
\left. \begin{aligned}
&\frac{323543 \eta^2}{3175200} - \frac{304071 \eta^4}{90720} + \frac{2009 \eta^6}{4320} + \frac{3241 \eta^8}{8640} + \frac{317 \eta^{10}}{1620} \\
&\left. - \frac{1349 \eta^{12}}{21600} + \frac{493 \eta^{14}}{45360} - \frac{949 \eta^{16}}{1270080} \right) \\
&+ \left[ \frac{H(Z)H'(Z)H''(Z)}{H(Z)^3} \right. \\
&\left. \begin{aligned}
&\frac{-273079 \eta^2}{9525600} + \frac{80149 \eta^4}{90720} - \frac{14449 \eta^6}{129600} \\
&\frac{12551 \eta^8}{155520} - \frac{1003 \eta^{10}}{25920} + \frac{503 \eta^{12}}{43200} - \frac{299 \eta^{14}}{151200} \\
&\left. + \frac{1727 \eta^{16}}{12700800} \right) \\
&+ \left[ \frac{H(Z)^2H''(Z)}{H(Z)^3} \right. \\
&\left. \begin{aligned}
&\frac{1459 \eta^2}{1587600} - \frac{31 \eta^4}{11200} + \frac{29 \eta^6}{8640} - \frac{119 \eta^8}{51840} + \frac{\eta^{10}}{960} \\
&\left. - \frac{13 \eta^{12}}{43200} + \frac{23 \eta^{14}}{453600} - \frac{\eta^{16}}{282240} \right) \\
&\right\}
\right.
\right. \\
(6.3.8)
\]

The calculation of higher order coefficients becomes too tedious beyond $h$ because of the complexity of the algebra involved.

6.4. Analysis of Series Solution

Eagles found the solutions up to $(\lambda^3)$. But have errors in first, second, and third approximations. Subsequent terms involve more functions of $Z$ and rapidly increasing number of powers of $\eta$, it is clearly impracticable to calculate them by hand. We also observe from (21) that, for each $n$, the different functions of $Z$ involve all possible combinations of $n$ functions multiplied together with $n$ derivatives (e.g
F₃ involves \( \frac{H^2H^*}{H^3}, \frac{HH'H^*}{H^3}, \text{and} \frac{H^3}{H^3} \).

F₄ involves \( \frac{H^3H^{IV}}{H^4}, \frac{H^2HH^*}{H^4}, \frac{H^2H^*2}{H^4}, \frac{HH^2H^*}{H^4}, \frac{H^4}{H^4} \) etc.

Thus the functions of Z in \( F_n \) span all members of the set

\[ H_{n j}(Z) = H^{a_n} H'^{a'_n} H^{\ldots} \ldots.. (n)^{a_n} \]  

(6.4.1)

where the \( a_m \) are non-negative integers satisfying the diaphantine equations

\[ \sum_{m=0}^{n} a_n = \sum_{m=1}^{n} ma_m = n \]  

(6.4.2)

Thus the number of such combinations (6.4.1) satisfying (6.4.2) is \( p(n) \), where \( p(n) \) is the number of partitions of \( n \). For any \( n \) partitions can be calculated systematically in variety of ways using the algorithm given by Gupta [70].

Similar analysis is also used by Bujurke et al [69] in the exhaustive study of internal laminar flow separation associated with flows in non-uniform channel.

Equations (6.3.2) – (6.3.6), with the boundary conditions (6.3.7) are solved using MATHEMATICA. Limitations imposed by the computer in this work have allowed the solution to be extended only through \( F_{15} \). The coefficients so generated are universal, i.e. valid for any smooth tube. We observe that the coefficients of \( \lambda^0 \) are monotonically decreasing with increasing \( n \). Therefore the series has the general nature of a convergent form.

In the present study, we take the slowly varying tubes (those given in figures (6.1A) and (6.2B)) given below.
Tube 1: \( H_1(Z) = 1 + \frac{1}{2} \tanh Z \) \hspace{1cm} (6.4.3)  
Tube 2: \( H_2(Z) = 1 + \frac{1}{2} \tanh^2 Z \) \hspace{1cm} (6.4.4)  

The expression for fluid velocity in the axial direction is of the form.

\[
G(\eta, z) = \frac{1}{\eta} \frac{\partial F}{\partial \eta} = \frac{1}{\eta} \sum_{n=1}^{\infty} \frac{\partial F}{\partial \eta} \lambda^n
\]  

(6.4.5)

In order to investigate the flow separation we need to compute the dimensionless shear rate (vorticity) as a function of position on the two walls. The wall shears \( \tau \) is computed as a function of \( Z \) for two different tubes. The expression for shear rate (vorticity)

\[
\tau = \frac{\partial v}{\partial t} = \frac{1}{H^3} \left[ \frac{1}{\eta^2} \frac{\partial F}{\partial \eta} + \frac{1}{\eta} \frac{\partial^2 F}{\partial \eta^2} \right]
\]  

(6.4.6)

Also second derivative of the velocity profile is given by

\[
G^*(\eta, z) = \frac{1}{\eta} F_{\eta \eta} - \frac{2}{\eta^2} F_{\eta} + \frac{2}{\eta} F_{\eta}
\]  

(6.4.7)

6.5 Results and Discussion:

The semi-analytical and semi-numerical technique is employed for studying the problem of steady flow in slender tubes. The main objective of this semi-analytical method is to see the possibility of enlarging the domain of validity of the series. Here, in the low Reynolds perturbation expansion large number of coefficients are generated using MATHEMATICA. The complex expressions involving elementsary functions appear in successive terms of the series and it is possible to calculate these up to 15 terms. To this order there are
\[ 4 \left( \sum_{m=1}^{15} mP_m \right) = 33,956 \text{ non-zero coefficients.} \text{ These coefficients in turn give universal polynomials} \ F_n(Z, \eta), \ n = 0, \ldots, 15. \text{ The series enables to predict the axial velocity profiles} \ G(\eta) = \frac{1}{\eta} \frac{\partial F}{\partial \eta} \text{ for different tubes} \ H_1(Z) \text{ and} \ H_2(Z) \text{ and for various} \ Z \text{ and} \ \lambda. \text{ The region of validity of the series}(6.4.5) \text{ is enhanced by the use of Pade' approximants from} \ \lambda = 10 \text{ (Eagles [66]) to} \ \lambda = 15. \text{ These are shown in figures (6.2)-(6.6) for different values of} \ Z. \text{ It is noticed from the figure that, there is flattening of the profiles in the convergent part of the tube and a sharpening in the divergent part. Figures (6.7)-(6.8) represent the second derivatives of the velocity profiles (the series 6.4.6) with respect to} \ \eta \text{ at the center of the tubes for different} \ \lambda. \text{ The coefficients of the series (6.4.7) representing dimensionless shear rate} \ \tau \text{ at} \ \eta = -1 \text{ are decreasing in magnitude but have no regular sign pattern. Domb–Sykes plot (Figures 6.9- 6.10) after extrapolation confirms the radius of convergence of the series to be( tube 1 )} \ \lambda = 8.20221, 7.94086, 25.207 \text{ for} \ Z = -1, 0, 6 \text{ respectively and (for tube 2) } \lambda = 35.977, 7.94263, 2.5234 \text{ for} \ Z = -4, 1, 2 \text{ respectively. The direct sum of the series for shear rate is valid only up to the radius of convergence. We use Pade approximants for summing the series for moderately large } \lambda (=10). \text{ The flow described by this experiences separation( a separation point that where the fluid breaks away from the wall(for Tube 1: )) from tube wall at the point} \ Z = 0 \text{ for } \lambda = 8.0. \text{ The shear stress at } \ \tau \text{ at} \ \eta = -1 \text{ decreases through zero at a point of separation and later converging(reattachment) flow is noticed(Figure 6.11).} \]
Figure 6.2: Velocity Profiles for Tube 1 at various values of \( Z \) and \( \gamma = 6 \).
Figure 6.2: Velocity Profiles for Tube 1 at various values of Z and $\lambda = 10$
Figure 6.4: Velocity Profiles for Tube 1 at various values of Z and $\lambda = 15$.
Figure 6.5: Velocity profiles for Tube 2 at various values of Z and η = 6
Figure 6.6: Velocity profiles for tube 2 at various values of z and θ = 10
Figure 6.7: Second derivative of velocity profile for Tube 1 & Tube 2 at \( n=0 \) and \( z=5 \).
Figure 6.8: Second derivative of velocity profile for Tube 1 at \( t = 0 \) and \( \gamma = 7 \)
Figure 6.9: Domb-Sykes for the series representing $t$ for the series (6.4.7) (for Tube-1)
Figure 6.10: Domb-Sykes for the series representing $x$ for the series (6.4.7) (for Tube-2)
Figure 6.11: Dimensionless wall shear magnitude as a function of $Z$ (for Tube)
Figure 6.12: Dimensionless wall shear magnitude as a function of Z (for Tube 2)