Chapter 4

ANALYSIS OF TUBE FLOW WITH VELOCITY SLIP

4.1 Introduction

In the previous chapter the problem of flow through porous channel with slip velocity is analyzed successfully using computer extended series method. Now we attempt to apply this method to the solution of problem of flow through porous pipe with slip coefficient.

The study of flow through a porous tube of uniform cross-section models biological and engineering systems as illustrated by many authors (Brady [43], Skalak and Wang [44]). For example in the separation of Uranium$^{235}$ from Uranium$^{238}$ by gaseous diffusion and also in transpiration cooling the walls of a pipe carrying a hot fluid are made of a porous material through which fluid is injected to form a protective layer of cooler fluid near the wall etc. Many investigators have analyzed the tube flow problems. Berman [30] assumed the flow to be axisymmetric and exploited the Heimenz. similarity form to reduce the steady Navier-Stokes equations to a fourth order ordinary differential equation. Sellars [45] extended Berman's work to high suction Reynolds number. Yuan and Finklestan [46] obtained analytic solution for the flow in porous circular pipe valid for large injection values and for small suction. The numerical work on the problem of flow in a porous pipe by White[47] revealed a multiplicity of solutions for certain ranges of suction velocity and established that at a certain critical
suction rate the velocity gradient at the wall becomes zero. The numerical results of Terrill et al.\[48,49\] shows that dual solution exist. He proved that dual solution exist every where except in the range $2.3 < R < 9.1$, where no solutions were found. In this respect, many other authors (Robinson \[37\], Brady\[11\], Zaturska et al.\[50\]) have developed and generalised this solution. In all the above analysis of fluid flow problems through porous pipes, they have not considered the slip boundary condition. But the experimental investigation reveals the existence of slip velocity at the porous bounding surface. Sparrow et al.\[51\] have presented the channel and tube flow problem with slip velocity. The details of the slip velocity been explained in detail in the previous chapter connected with channel flow.

In this chapter, we reinvestigate the problem of tubular membrane system studied by Sing and Laurence\[52\] and present some useful and interesting results based on series analysis. The present analysis is primarily concerned with the possible extensions of Singh and Laurence's\[52\] low wall Reynolds number $Re_w(=\frac{2v_r r_w}{\nu}$, $r_w$, tube radius, $v_w$, velocity of fluid through membrane) perturbation series by computer. With the nature of the few manually calculated functions in low wall Reynolds number expansion we can derive a recurrence relation for generating higher order polynomial functions. In this analysis, we systematically generate the universal polynomial functions by MATHEMATICA also this is more efficient and accurate compared to the procedure based on recurrence relation. Using these universal coefficient functions we obtain series
solution and calculate various physical parameter for different slip coefficient and Re\_w. The present series solution, which is expected to be limited in convergence by the presence of singularity, is extended to moderately high Reynolds number by analytic continuation.

4.2. Mathematical Formulation

Consider steady, incompressible laminar flow through a pipe. The flow is assumed to be fully developed at the tube entrance. Schematic diagram of the problem is presented in figure[4.1]. the sketch depicts a porous walled circular tube. The figure contains co-ordinates and other dimensional nomenclature relevant to the analysis. The equations of linear momentum and continuity in cylindrical co-ordinates are

$$\frac{u}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rv)}{\partial r} \right) + \frac{\partial^2 v}{\partial r^2} \right]$$

$$\frac{u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial x^2} \right]$$

and $$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial (rv)}{\partial r} = 0$$

The boundary conditions are

$$u(x, r_w) = -\frac{\sqrt{k}}{\alpha} \left( \frac{\partial u}{\partial r} \right)$$
Figure 4.1: Tubular Membrane
\begin{align*}
&\left( \frac{\partial u}{\partial r} \right)_{r=0} = 0 \tag{4.2.5} \\
v(x,0) = 0 \tag{4.2.6} \\
v(x,r_w) = v_w \tag{4.2.7}
\end{align*}

The boundary conditions (4.2.4) allows for a tangential component of fluid velocity along the porous boundary. Here the slip velocity at the membrane surface is proportional to the shear rate at the permeable boundary (Beavers and Joseph[39]), which has been discussed in the previous chapter. Boundary condition (4.2.7) denotes constant permeation flux along the length of the tube.

For a two dimensional incompressible flow a stream function $\psi(x,\eta)$ exists such that

\begin{align*}
&u(x,\eta) = \frac{2}{r_w^2} \frac{\partial \psi}{\partial \eta} \tag{4.2.8} \\
v(x,\eta) = -\frac{1}{\sqrt{\eta}} \frac{1}{r_w} \frac{\partial \psi}{\partial x} \tag{4.2.9}
\end{align*}

where $\eta = \left( \frac{r}{r_w} \right)^2$

which satisfies the continuity equation (4.2.3). For constant wall velocity $v_w$ and the given boundary conditions, a suitable choice of stream function is

\begin{equation}
\psi(x,\eta) = \frac{r_w}{2} \left[ r_w \bar{u}_0 - 2v_w x \right] F(\eta) \tag{4.2.10}
\end{equation}

solving (4.2.8)- (4.2.10), we obtain the expressions to velocity components.
Substituting equations (4.2.11) and (4.2.12) into the equations of motion (4.2.1) - (4.2.2) and eliminating pressure term $P$, we obtain the following non-linear ordinary differential equations

$$\eta F'' + 2F'' + \frac{Re}{4} (F' F'' - F F''') = 0 \quad (4.2.13)$$

the boundary conditions on the function $F(\eta)$ and its derivations are readily obtained from equations (4.2.4) through (4.2.7) and equations (4.2.11) and (4.2.12).

Thus, we have

$$F(0) = 0 ; \quad \sqrt{\eta} F'(\eta) \to 0 \quad \text{as} \quad \eta \to 0 \quad (4.2.14)$$

$$F(1) = 1 ; \quad F'(1) + 2\phi F''(1) = 0 \quad (4.2.15)$$

where $\phi = \frac{\sqrt{k}}{\alpha r_w}$ represents the slip coefficient.

### 4.2 Method of Solution

The solution of equation (4.2.13) for small values of $Re_w$ may be expressed in the form of a power series as

$$F(\eta) = F_0(\eta) + \sum_{n=1}^{\infty} Re_w^n F_n(\eta) \quad (4.3.1)$$
substituting (4.3.1) into (4.2.13) and comparing like powers of \( Re_a \) on both sides, we get

\[
\eta F_n^p + 2F_n^p = \frac{1}{4} \left( \sum_{i=0}^{n-l-r} F_i^p F_{n-l-r}^p - F_i^p F_{n-l-r}^p \right) \quad n = 1, 2, 3, \ldots
\]  

(4.3.2)

The boundary conditions are

\[
\begin{align*}
F_a(0) &= 0, \quad \sqrt{\eta} F_a^p(\eta) \to 0 \quad \text{as} \quad \eta \to 0 \quad \forall n \geq 0 \\
F_a(1) &= 1, \quad F_a(1) = 0, \quad \forall n \geq 1 \\
F_a'(1) + 2 \varphi F_a^p(1) &= 0 \quad \forall n \geq 1
\end{align*}
\]  

(4.3.3)

The solutions of the above equations, up to the term in \( Re_a^2 \) are (Singh and Laurence[52])

\[
\begin{align*}
F_0 &= -\frac{\eta^2}{1 + 4\varphi} + \frac{2\eta(1 + 2\varphi)}{1 + 4\varphi} \\
F_1 &= -\frac{\eta^4}{72(1 + 4\varphi)^2} + \frac{\eta^3(1 + 2\varphi)}{12(1 + 4\varphi)^2} - \frac{\eta^2}{8(1 + 4\varphi)} + \frac{\eta(1 + 6\varphi)}{18(1 + 4\varphi)^2} \\
F_2 &= -\frac{\eta^6}{21600(1 + 4\varphi)^3} + \frac{\eta^5(1 + 2\varphi)}{1440(1 + 4\varphi)^3} - \frac{\eta^4}{288(1 + 4\varphi)^2} + \frac{\eta^3(11 + 66\varphi + 72\varphi^2)}{864(1 + 4\varphi)^3} \\
&\quad - \frac{\eta^2(19 + 228\varphi + 870\varphi^2 + 1080\varphi^3)}{1080(1 + 4\varphi)^4} + \frac{\eta(83 + 1162\varphi + 5040\varphi^2 + 7200\varphi^3)}{10800(1 + 4\varphi)^4}
\end{align*}
\]  

(4.3.4)

4.4. Computer Extended Perturbation Series

For the analysis of the perturbation series we need sufficiently large number of terms. Manually it is difficult to calculate beyond \( F_2 \) as it involves
heavy algebraic labour. We solved the equations (4.3.2) together with boundary conditions. Systematically by MATHEMATICA and able to generate universal polynomial functions \( F_n(\eta),\ n = 1,2,3,...,23 \)

The expressions for velocity profile in the axial and transverse directions are given by

\[
U = \frac{u}{u_0} = \left[ 1 - \frac{2Re_w}{Re} \left( \sum_{n=1}^\infty Re_w F_n^\prime \right) \right] \tag{4.4.1}
\]

and (in the transverse direction is)

\[
V = \frac{v}{v_0} = \left[ \frac{F_0(\eta)}{\sqrt{\eta}} + \sum_{n=1}^\infty \left( Re_w F_n(\eta) \right) \right] \tag{4.4.2}
\]

Normalized axial velocity component is given by

\[
u_s = \frac{u}{u} = F'(\eta) = F_0'(\eta) + \sum_{n=1}^\infty Re_w F_n'(\eta) \tag{4.4.3}
\]

For normalized slip velocity, we have

\[
u_s = \left( \frac{u}{u} \right)_{\eta=0} \tag{4.4.4}
\]

The expression for shear stress is of the form

\[
F'(1) = F_s'(1) + \sum_{n=1}^\infty Re_w F_s'(1) \tag{4.4.5}
\]

In this case the coefficients of the series \( F'(1) \) have same sign and are decreasing in magnitude. Using these coefficients of the above series \(4.4.5\) we draw Domb-Sykes plots (Figure 4.2) to find the nature of the nearest singularities which restrict
the convergence of the series and is found to be. The validity of the series (4.4.5) is increased by reverting the corresponding series. This type of reversion was successfully employed earlier by Bujurke et al [53] in different studies. The reversion of the series (4.4.5) representing $F''(l)$ is performed as follows.

4.5. Reversion of series

Consider

$$f''(l) = \sum_{n=1}^{\infty} a_n \Re e^w$$

(4.5.1)

Let

$$Y = f''(l) = \sum_{n=1} a_n \Re e^w$$

Reverting the above series, we have

$$R(Y) = \sum_{n=1} B_n Y^n$$

where $B_1 = \frac{1}{e(l, 1)}$

$$B_n = \frac{1}{e(l, m)} \sum_{i=0}^{n-2} B_{(i+1)} e(m-i, i+1)$$

$m = 2, 3, \ldots, n$

$e(l, \alpha) = (a_1)^\alpha$

$e(k+1, \alpha) = \frac{1}{k \ a_1} \sum_{i=0}^{k-1} [(k-i)\alpha - i] e(i+1, \alpha) a_{k+i}$

$k = 1, 2, \ldots, n; \alpha = 1, 2, \ldots, n$
(we repeat the above procedure for slip coefficients $\phi = 0.0, 0.1, 0.5$). Besides reversion we use Pade’ approximants for summing the reverted series (4.5.2) which yields analytic continuation. The results are given in Table 4.1.

4.7 Results and Discussion

The present analysis deals with the study of tubular membrane system by series method. We have proposed series method for the solution of Navier-Stoke’s equations describing the above problem. The universal polynomial functions $F_n(\lambda)(n = 1,2,3,\cdots,23$ are obtained by solving equations (4.2.13)-(4.2.15) systematically with the help of MATHEMATICA for different slip coefficient $\phi$. Unfortunately power series solution is found to be valid only up to $\text{Re}_w = 1.5$. The series (4.4.2) and (4.4.3) representing velocity profiles in the axial and transverse direction are analysed using Pade’ approximants. The region of validity of the series(4.4.4) representing normalised velocity profiles for tube flow with suction and injection is enhanced by the use of Pade’ approximants from $\text{Re}_w = 5$ (earlier findings) to $\text{Re}_w = 15$ and are shown, respectively in figures (4.3a) and (4.3b). The velocity field is symmetric about the tube central line. Figure shows only half profiles covering the range from $r = 0$ to $r = r_w$. It is noticed from the figure that as the slip velocity increases, the wall shear decreases and the profiles become flatter, approaching these for a plug flow. From
figure (4.4), it is seen that normalized slip velocity (series (4.4.5)) \( u_s = \begin{bmatrix} u \\ \bar{u} \end{bmatrix} \), increased with \( \phi \). As \( u_s \) increases with \( \phi \), the profiles become flatter. As \( \text{Re}_w \) increases \( u_s \) decreases. Where as in channel flow we have reverse/opposite trend.

The coefficients \( C_n \) of the series (4.4.5) representing shear stress \( f'(1) \) for various slip coefficients are decreasing in magnitude, but have fixed sign pattern. Domb-Sykes plot after extrapolation, confirms the radius of convergence of the series (4.4.5) to be 4.59982, 4.70108 and 5.25624 for \( \phi = 0.0, 0.1, 0.5 \) (\( \phi = 0.0 \), corresponds to the result obtained by Terrill). The direct sum of the series for \( f'(1) \) is valid only up to the radius of convergence. The region of validity of the series for shear stress is increased by reverting the series (by changing the role of dependent and independent variables). Later we use Padé approximants for summing the reverted series (4.5.2) which accelerates the convergence of the series. The result of the shear stress \( f'(1) \) (Table(4.1)) agree favorably with numerical (the slip coefficient \( \phi = 0.0 \)). Terrill was able to calculate the shear stress up to for large injection (\( R = -241.9763 \)). We are able to obtain the solution beyond \( \text{Re}_w = -241.9763 \) without any difficulty. We could get accurate results valid up to \( \text{Re}_w = -350 \). But in the case of suction we are able to calculate the results only up to \( R = 5 \). The above procedure is repeated for the slip coefficients \( \phi = 0.1 \) and \( \phi = 0.5 \). The results are given in tables 4.1.
Table 4.1: Values of $f''(l)$ against Reynolds number ($Re_w$) for large injection for various slip coefficients $\varphi = 0.0$, 0.1 and 0.5

<table>
<thead>
<tr>
<th>$Re_w$</th>
<th>$F^*(l)$ ($\varphi = 0.0$) (Terrill[48])</th>
<th>$F^*(l)$ ($\varphi = 0.1$)</th>
<th>$F^*(l)$ ($\varphi = 0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-1.902546</td>
<td>-1.3792622</td>
<td>-0.8892513498</td>
</tr>
<tr>
<td>3.0</td>
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<td>-1.2057595</td>
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<td>-0.75164108</td>
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<td>-1.722170357</td>
<td>-1.0400046</td>
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Figure 4.2a: Velocity profiles for tube flow with suction, Re* = 2.0
Figure 4.2b: Velocity profiles for the tube flow with suction: $Re_w = 5.0$
Figure 4.3a: Velocity profiles for tube flow with injection, $Re_w = -5.0$
Figure 4.3b: Velocity profiles for tube flow with injection, $Re = -10.0$.
Figure 4.4: The effect of slip coefficient on the normalized slip velocity for various Re.

- Re = 6.0
- Re = 3.0
- Re = 1.0
- Re = 0.1
Figure 4.5: Domb-Sykes plot for the coefficients of $F''(1)$ for various $\phi$. 

<table>
<thead>
<tr>
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<tbody>
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