Chapter 2

MULTIGRID SOLUTION OF MODIFIED REYNOLDS EQUATION INCORPORATING POROELASTICITY AND COUPLE STRESSES
2.1. Introduction

The problem considered in this Chapter is the squeeze film lubrication between a rigid spherical indenter and poroelastic matrix in modeling synovial knee joint. Solution of this modified Reynolds equation incorporating poroelasticity of articular cartilage and couple-stress fluid as lubricant is important since many applications in computational science and engineering depend on it. Even though years of research have been spent on finding fast and reliable methods for modified Reynolds equation on very fine meshes for wide range of parameters involved, the recent efficient solution technique—multigrid method which is more efficient and accurate is used for solving large systems of equations.

Synovial joints act as mechanical bearings which facilitate the work of musculoskeletal machines. These diarthrodial joints provide frictionless motion under a wide range of loading conditions with no appreciable wear. The ability of joints to provide ideal performance inspite of the severe operating conditions is the result of interaction between the articular cartilage and synovial fluid. Articular cartilage is a soft tissue, which covers the articulating bone ends of synovial joints. The cartilage is a two-phase deformable porous material that can imbibe or exude the fluid owing to the pressure gradients generated as a result of either squeeze film action of the synovial fluid or consolidation of the solid matrix due to tissue deformation. Poroelasticity is a continuum theory for the analysis of porous media with elastic matrix consisting of an interconnected fluid filled pores. When fluid permeated into a poroelastic material, the drag
force between the fluid and the porous medium may cause deformation in porous matrix. This leads to volumetric changes in the pores. Since the pores are filled with fluid, the presence of the fluid not only acts as a stiffener of the material, but also results in the flow of pore fluid between regions of higher and lower pore pressure. A suitable poroelastic model of cartilage is presented in detail by Torzilli and Mow (1976). Collins (1982) considered appropriate poroelastic models of cartilage which is assumed to satisfy generalized form of Darcy’s law for unsteady flow in an elastic porous medium. Later, Sachs et al. (1994) and Jensen et al. (1994) have studied the linear and non-linear deformation in a poroelastic disk with a free surface having applications in the study of biological tissues.

A synovial joint is completely enclosed in a fibrous capsule that acts with the ligaments and intra-articular structures to provide stability and guidance during motion. The inner surface of the capsule is the synovial membrane which secretes a highly viscous non-Newtonian synovial fluid. This synovial fluid bathes and lubricates both the articular surfaces of the joint. Hou et al. (1990) have presented a squeeze film lubrication of synovial joints by modeling a rigid impermeable spherical indenter approaching a thin permeable cartilage layer. Bujurke and Patil (1991) have modeled articular cartilage as poroelastic material and synovial fluid as Stokes (1966) couple-stress fluid and shown the significant increase in load carrying capacity that can be sustained by synovial joint. Recently, Lin (1996) and Walicki and Walicka (2000) also have studied squeeze film characteristics of hemispherical bearings
lubricated with couple-stress fluid model having application to lubrication mechanism of synovial joints. The presence of small amount of additives in a lubricant can enhance the lubricant viscosity which increases the load capacity and also reduces the coefficient of friction. The long chain Hyaluronic Acid (HA) molecules found as polar additives in synovial fluid are characterized by two material constants $\mu$ and $\eta$. These two material constants $\mu$ and $\eta$ are also found in couple-stress fluid. According to Stokes theory, couple stresses are found to be in noticeable magnitudes in fluids with large molecules. Couple stresses stabilize the flow properties and minimize the sensitivity of the lubricant to changes in the shear rate.

This Chapter comprises five sections. The squeeze film lubrication between a rigid spherical indenter and flat poroelastic matrix in modeling of synovial knee joint is presented. A simplified mathematical formulation of the problem is given in Section 2. Section 3 is devoted to derive the modified Reynolds equation, and in Section 4 numerical procedure with multigrid ideas and principles is given for its solution. In the subsequent sections, the analyses of bearing characteristics are presented for various parameters involved.

2.2. Formulation of the Problem

The governing equations for the motion of an incompressible couple-stress fluid are (Stokes, (1966))

$$\text{div} \vec{V} = 0,$$

$$\rho \frac{d\vec{V}}{dt} = -\text{grade} \ p + \mu \nabla^2 \vec{V} - \eta \nabla \cdot \vec{V},$$
where $\vec{V}$ is the velocity vector, $\rho$ is the density, $p$ is the pressure, $\mu$ is the shear viscosity and $\eta$ is a new material constant responsible for the couple-stress property. The introduction of $\eta$ is due to polar additives in the non-polar lubricant, the ratio $\eta/\mu$ has the dimension of length squared and hence characterizes the material length of the fluid. The physical flow parameters are the velocity components $u, v, w$ and the pressure $p$.

The geometry and co-ordinates of the problem are shown in Fig. 2.1 which is a simplified model of synovial knee joint. This chapter deals with the compression of an impermeable rigid surface, a thin layer of fluid over a thin layer of poroelastic material. The upper rigid impervious spherical indenter approaching the lower flat poroelastic matrix normally with a constant velocity $dH/dt$. The upper rigid indenter has radii $R_x$ and $R_z$ in the $x$ and $z$ directions and $h_0$ is the minimum film thickness. The film thickness is characterized by

$$H = h_0 + \frac{x^2}{2R_x} + \frac{z^2}{2R_z}. \quad (2.2.3)$$

For a sphere $R_x = R_z = R$, and the equation (2.2.3) becomes

$$H = h_0 + \frac{x^2}{2R} + \frac{z^2}{2R}. \quad (2.2.4)$$

The problem considered here would be that of three-dimensional squeeze film lubrication between upper spherical indenter and lower poroelastic material. Lubricant in the joint cavity is assumed as Stokes couple-stress fluid. With usual assumptions of fluid film lubrication, the governing field equations in Cartesian co-ordinates reduce to
Fig. 2.1. Schematic diagram and co-ordinates of simplified synovial knee joint.
where \( u, v \) and \( w \) are velocity components in \( x, y \) and \( z \) directions respectively.

**Poroelastic Region**

Following Torzilli and Mow (1976), we write coupled equations of motion for the deformable cartilage matrix and the flow of fluid contained in its pores in slightly modified form

Matrix: \[ \rho_m \frac{\partial^2 U}{\partial t^2} = \text{div} \sigma + \frac{\mu}{k} \left( \frac{\partial U}{\partial t} - V \right), \] (2.2.9)

Fluid: \[ \rho_f \frac{DV}{Dt} = \text{div} \sigma_f + \frac{\mu}{k} \left( \frac{\partial U}{\partial t} - V \right), \] (2.2.10)

where \( \rho_m \) and \( \rho_f \) denote the densities of solid matrix and fluid respectively, \( U \) is the corresponding displacement vector, \( k \) is the permeability of the cartilaginous matrix to fluid. The left hand terms denote the local forces (mass times acceleration), which are counterbalanced by the right hand terms namely the surface forces, \( \text{div} \sigma \), and the porous medium driving forces (Darcy’s law) respectively. These two component equations may be simply viewed as generalized forms of Darcy’s law for unsteady flow in a deformable porous matrix.
medium in terms of relative velocity $\left(\frac{\partial U}{\partial t} - V\right)$ between the moving cartilage and the fluid contained in its pores. Also, equations (2.2.9) and (2.2.10) denote force balances for the linear elastic solid and viscous fluid components of the cartilage respectively. The classical stress tensor $\sigma$ for a continuous homogeneous medium may be expressed for the matrix (cartilage) and fluid (synovial) respectively, in the forms

$$\sigma_\text{m} = PI + 2Ne + Ael, \quad (2.2.11)$$

$$\sigma_\text{f} = -PI + Eel, \quad (2.2.12)$$

in terms of the elastic parameters $N, E$ and $A$ of the cartilage and the hydrostatic pressure $P$ and $I$ the identity tensor, $e$ the cartilage dilation. The inertial terms in (2.2.9) and (2.2.10) are neglected because in the balance of momentum equation the fluid-fluid viscous stress is negligible compared with the drag between the fluid and solid matrix (Barry and Holmes, 2001). After neglecting inertia terms, addition of equations (2.2.9) and (2.2.10) eliminates the pressure and normal relative velocity of fluid and taking divergence of the result, gives

$$\nabla^2 e = 0, \quad (2.2.13)$$

The cartilage dilatation is characterized by a simple linear equation in terms of the corresponding average bulk modulus $K$ and pressure $P$ (Hori and Mockers, 1976)

$$e = e_0 + P \frac{1}{K}, \quad (2.2.14)$$
The equation describing pressure in the poroelastic region obtained using (2.2.14) in (2.2.13), is
\[ V^2 P = 0. \] (2.2.15)

**Boundary Conditions**

The relevant boundary conditions for the velocity field \((0 < y < H)\) are
\[ u(x,0,z) = u(x,H,z) = w(x,0,z) = w(x,H,z) = 0, \] (2.2.16)
\[ v(x,0,z) = -v_y, \quad v(x,H,z) = -dH \frac{dx}{dt}, \] (2.2.17)
\[ \frac{\partial^2 u}{\partial y^2} \bigg|_{y=0} = \frac{\partial^2 u}{\partial y^2} \bigg|_{y=H} = \frac{\partial^2 w}{\partial y^2} \bigg|_{y=0} = \frac{\partial^2 w}{\partial y^2} \bigg|_{y=H} = 0, \] (2.2.18)

where \(v_y\) represents the normal component of the relative velocity of the fluid at the cartilage surface. Conditions (2.2.16) are no-slip conditions and (2.2.18) are due to vanishing of couple stresses.

### 2.3. Solution Procedure

Due to equation (2.2.7), \(p\) is independent of \(y\), the solution of equations (2.2.6) and (2.2.8) using (2.2.16) and (2.2.18) are
\[ u = \frac{1}{2\mu} \frac{\partial p}{\partial x} \left[ y^3 - yH + \frac{2}{\tau} \left( 1 - \frac{\cosh(\tau y - \tau H/2)}{\cosh(\tau H/2)} \right) \right], \] (2.3.1)
\[ w = \frac{1}{2\mu} \frac{\partial p}{\partial z} \left[ y^3 - yH + \frac{2}{\tau} \left( 1 - \frac{\cosh(\tau y - \tau H/2)}{\cosh(\tau H/2)} \right) \right], \] (2.3.2)

where \[ \tau = \sqrt[3]{\eta/\mu}, \] the couple-stress parameter.

Integrating equation (2.2.15) with respect to \(y\) over the porous layer thickness...
\(-\delta < y < 0\) and using the solid backing boundary condition \(\frac{\partial P}{\partial y} = 0\) at \(y = -\delta\),

we get,

\[
\left. \frac{\partial P}{\partial y} \right|_{y=0} = -A \int_{-\delta}^{0} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} \right) dy,
\]

(2.3.3)

where \(\delta\) is the thickness of the poroelastic layer. Using the Cameron-Morgan approximation (Morgan and Cameron, (1957)) valid for the poroelastic layer thickness \(\delta \ll 0.1\) and incorporating pressure continuity condition \((p = P)\) at the porous interface \((y = 0)\), we get

\[
\left. \frac{\partial P}{\partial y} \right|_{y=0} = -\delta \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} \right).
\]

(2.3.4)

After neglecting inertia terms, equation (2.2.10) may be arranged in terms of relative velocity in the form

\[
\left( \nu - \frac{dU}{dt} \right) = -\frac{k}{\mu} (\nabla P - E \nabla e).
\]

(2.3.5)

Elimination of \(e\) through (2.2.14) and (2.3.5) gives

\[
\left( \nu - \frac{dU}{dt} \right) = -\nabla P \left( \frac{k}{\mu} \left( 1 - \frac{E}{K} \right) \right).
\]

(2.3.6)

The normal component of the relative fluid velocity at the cartilage surface is

\[
v_n = \left( \nu - \frac{dU}{dt} \right)_n = -\frac{k}{\mu} \left( \frac{E}{K} - 1 \right) \left. \frac{\partial P}{\partial y} \right|_{y=0}.
\]

(2.3.7)

Using equation (2.3.4) in equation (2.3.7), we get

\[
v_n = \frac{k\delta}{\mu} \left( \frac{E}{K} - 1 \right) \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} \right).
\]

(2.3.8)
Integrating continuity equation (2.2.5) with respect to \(y\) using boundary condition (2.2.17) and the expressions (2.3.1), (2.3.2) and (2.3.8) for \(u, w\) and \(v_n\), the modified form of Reynolds equation is obtained in the form

\[
\frac{\partial}{\partial x} \left[ \left( F(H, r) + 12k\delta \left( \frac{E}{K} - 1 \right) \right) \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \left( F(H, r) + 12k\delta \left( \frac{E}{K} - 1 \right) \right) \frac{\partial p}{\partial z} \right] = 12\mu \frac{dH}{dt}, \tag{2.3.9}
\]

where

\[
F(H, r) = -H^3 + \frac{12}{r^2} \left( H - \frac{2}{r} \tanh(Hr/2) \right).
\]

For large indenter radius \(R \to \infty, r \to \infty\) and \(k = 0\), the analyses correspond to classical case (Pinkus and Sternlicht, (1961)).

In order to solve the modified Reynolds equation for fluid pressure, following relevant boundary conditions are to be used

\[
p = 0 \text{ at } x = \pm a, \tag{2.3.10}
\]

\[
p = 0 \text{ at } z = \pm a,
\]

where \(2a\) is the dimension of the plate.

Introducing the non-dimensional parameters and variables

\[
\tilde{x} = \frac{x}{a}, \tilde{z} = \frac{z}{a}, \tilde{R} = \frac{R}{a}, \tilde{h}_1 = \frac{h_1}{a}, \tilde{H} = \frac{H}{h_0}, \tilde{h}_0 = 1 + \frac{\tilde{x}^2 + \tilde{z}^2}{2}, \tilde{k} = \frac{k\delta}{h_1}, \tilde{r} = \frac{\rho}{\mu a^2 \frac{dH}{dt}},
\]

into (2.3.9) and (2.3.10), we get
\[
\frac{\partial}{\partial \bar{x}} \left[ F(\bar{H}, \bar{r}) + 12 \bar{k} \left( \frac{E'}{K} - 1 \right) \frac{\partial \bar{p}}{\partial \bar{x}} \right] = 12,
\]

(2.3.11)

where

\[
F(\bar{H}, \bar{r}) = -\bar{H}^3 + \frac{12}{\bar{r}^3} \left( \bar{H} + \frac{2}{\bar{r}} \tanh(\bar{H} \bar{r}/2) \right),
\]

and boundary conditions are

\[
\bar{p} = 0 \text{ at } \bar{x} = \pm 1,
\]

\[
\bar{p} = 0 \text{ at } \bar{r} = \pm 1.
\]

(2.3.12)

**Numerical Solution**

The Modified Reynolds Equation (2.3.11) is of elliptic type equation with variable coefficients, which do not allow analytical solution. Finite difference based multigrid method is found to be most suitable scheme for the solution of this equation. Using second order finite difference scheme, the derivative terms in equation (2.3.11) are approximated by

\[
\frac{\partial}{\partial \bar{x}} \left[ F(\bar{H}, \bar{r}) + 12 \bar{k} \left( \frac{E'}{K} - 1 \right) \frac{\partial \bar{p}}{\partial \bar{x}} \right] = \frac{\partial}{\partial \bar{x}} \left( \alpha \frac{\partial \bar{p}}{\partial \bar{x}} \right) =
\]

\[
\frac{1}{\Delta \bar{x}} \left( \alpha_{\bar{x}+1/2,j} \bar{p}_{\bar{x}+1/2,j} - \alpha_{\bar{x}-1/2,j} \bar{p}_{\bar{x}-1/2,j} \right)
\]

and

\[
\frac{\partial}{\partial \bar{r}} \left[ F(\bar{H}, \bar{r}) + 12 \bar{k} \left( \frac{E'}{K} - 1 \right) \frac{\partial \bar{p}}{\partial \bar{r}} \right] = \frac{\partial}{\partial \bar{r}} \left( \alpha \frac{\partial \bar{p}}{\partial \bar{r}} \right) =
\]

\[
\frac{1}{\Delta \bar{r}} \left( \alpha_{\bar{r}+1/2,j} \bar{p}_{\bar{r}+1/2,j} - \alpha_{\bar{r}-1/2,j} \bar{p}_{\bar{r}-1/2,j} \right),
\]

where
\[ \alpha = F(\bar{H}, \bar{r}) + 12k \left( \frac{E}{K} - 1 \right) \]

Then, equation (2.3.11) becomes
\[ A_0 \tilde{p}_{i+1,j} + A_1 \tilde{p}_{i,j+1} + A_2 \tilde{p}_{i-1,j} + A_3 \tilde{p}_{i,j-1} + A_4 \tilde{p}_{i,j} = A_s, \]  

(2.3.13)

where
\[ A_0 = \frac{1}{(\Delta x)^2} \alpha_{i+1/2,j}, \quad A_1 = \frac{1}{(\Delta x)^2} \alpha_{i-1/2,j}, \]
\[ A_2 = \frac{1}{(\Delta x)^2} \alpha_{i,j+1/2}, \quad A_3 = \frac{1}{(\Delta x)^2} \alpha_{i,j-1/2}, \]
\[ A_4 = -(A_0 + A_1 + A_2 + A_3), \quad A_s = 12. \]

**Multigrid Method**

In this section, the basic ideas of multigrid algorithms are discussed. The main drawback of smoothers (such as Gauss-Seidel or Gauss-Jacobi) is characterized by their global poor convergence rate. For errors whose length scales are comparable to the grid size, the smoothers provide rapid damping, leaving smooth and longer wavelength errors. These smooth wavelength errors are responsible for the slow global convergence. The multigrid algorithm uses different grid sizes and allows us to damp out all wavelength components and provide rapid convergence rates. The high frequency components of the error are effectively damped out on finer grid by applying the smoothers and low frequency errors are effectively reduced using a coarse grid correction by representing them on coarse grid level. In multigrid strategy one has to select suitable smoother and inter-grid transfer operators (linear Interpolation and full weighting operators).
Let $G^1, G^2, \ldots, G^{N-1}$ be the sequence of computational grid levels with $G^N$ the finer grid level. The easiest and most natural one is full-weighting restriction operator which is linear operator from $G^{N-1}$ to $G^{(N/2)-1}$. The full-weighting restriction operator takes fine grid vectors $r^h (r$ is the residual on finer grid) and produces coarse grid vectors and it is given by

$$A^h = I^h r^h,$$

and is defined for $1 \leq i, j \leq (N/2) - 1$ as

$$A^h = \frac{1}{16} \left[ 4r_{2i,2j}^h + 2(r_{2i,2j-1}^h + r_{2i,2j+1}^h + r_{2i-1,2j}^h + r_{2i+1,2j}^h) + r_{2i-1,2j-1}^h + r_{2i-1,2j+1}^h + r_{2i+1,2j-1}^h + r_{2i+1,2j+1}^h \right].$$

(2.3.14)

The standard interpolation operator is a linear operator from $G^{(N/2)-1}$ to $G^{N-1}$. This takes coarse grid vectors $p^h$ and produces finer grid vectors. This is defined by

$$p^h = I^h p^h,$$

and the linear interpolation is given by

$$p_{2i,2j}^h = \frac{1}{2} (p_{i,j}^h + p_{i+1,j}^h),$$

$$p_{2i+1,2j}^h = \frac{1}{2} (p_{i,j}^h + p_{i+1,j}^h),$$

$$p_{2i,2j+1}^h = \frac{1}{2} (p_{i,j}^h + p_{i,j+1}^h),$$

$$p_{2i+1,2j+1}^h = \frac{1}{4} (p_{i,j}^h + p_{i+1,j}^h + p_{i,j+1}^h + p_{i+1,j+1}^h).$$

(2.3.15)

for $0 \leq i, j \leq (N/2) - 1$.

It is important to note that linear interpolation and full-weighting operators are transpose of each other up to some constant.
The equation (2.3.13) is solved numerically for fluid film pressure using multigrid method. The initial solution is taken as $\bar{p}_{i,j} = 0$ on the finest grid level. To get the solution for next parameter, the solution obtained for previous parameters are used as starting solution. The multigrid method with full weighting restriction and linear interpolation operators is used to solve the resulting algebraic equations. The method enhances the convergence rate. Few Gauss-Seidel iterations are applied for smoothing the high frequency errors on the finest grid level. Since the multigrid method uses the different grid sizes, the low frequency errors can be transferred to next coarse grid level using full weighting operator. Repeat this till the coarsest level is reached. By doing this we have damped successfully all types of errors. Now, prolongation by linear interpolation can be introduced for transferring the solution obtained at the coarsest level to next finer level. On the finest level, the convergence criteria used is

$$|\bar{p}_{i,j}^{\text{ref}} - \bar{p}_{i,j}^{r}| < 10^{-4},$$

where $r$ corresponds to number of V-cycles required to achieve this accuracy.
2.4. Results and Discussion

An analysis of lubrication aspects of squeeze film characteristics of poroelastic bearings in general and that of synovial joint in particular is presented. In order to facilitate numerical computation of the proposed problem, physical constants for the poroelastic and couple-stress fluid model must be selected. At present only limited data exists and these are insufficient to fully describe the model under consideration. The governing non-dimensional physical parameters are \( k (= k_0 h^2) \), \( \bar{r}(= \tau h) \), \( R(= R/a) \), and \( E'/K \). The values for parameters \( E' \), \( K \) and for \( k \) are taken from Torzilli (1978) that are associated with healthy human articular cartilage during normal functioning. The couple-stress parameter \( \bar{r} \) has dimension of length squared and this may be regarded as the chain length of the polar additives in the non-polar lubricants. It is expected that the polar effects should be more prominent either when the minimum film thickness is small or molecular size of the additives is large i.e. when \( \bar{r} \) is small. On the other hand for large value of \( \bar{r} \), the couple-stress effects are not significant. In the graphs, the dot-dashed lines correspond to either Newtonian case (\( \bar{r} \rightarrow \infty \)) or non-elastic case (\( E'/K =0 \)).

**Pressure Distribution**

The non-dimensional fluid film pressure distribution \( \bar{p} \) as function of \( \bar{k}, \bar{r}, E'/K \) is obtained by solving equation (2.3.13) and results are shown in Fig. 2.2. The distribution of fluid film pressure that is generated due to squeeze film action for various values of \( \bar{r} \) is shown in Fig. 2.2(a-c). The couple-stress effect is to increase the pressure distribution compared to Newtonian case.
The two material constants \( \mu \) and \( \eta \) which are present in couple-stress fluid are responsible for long chain hyaluronic acid (HA) molecules. The water and low molecular weight substances present in the lubricant are forced into poroelastic cartilage by the squeeze film action which causes concentration of the polymer additives on surfaces of articular cartilage. This enhances the pressure built up due to increased viscosity of lubricant. As parameter \( \overline{r} \) increases, the fluid becomes Newtonian and the pressure distribution decreases. In certain pathological changes which occur in synovial joints due to process of aging or abnormal joint mechanics the fluid becomes Newtonian which is partly responsible for degenerative joint disease (Sokoloff, (1969)).

### Load Carrying Capacity

Once fluid film pressure is obtained after solving equation (2.3.13), the load carrying capacity that can be sustained by knee joint per unit area of the joint surface in non-dimensional form is expressed as

\[
\overline{W} = \int_{-1}^{1} \int_{-1}^{1} \overline{p}(\overline{x}, \overline{z}) d\overline{x} d\overline{z}.
\]

The variations of load carrying capacity \( \overline{W} \) with cartilage permeability \( \overline{k} \) for different \( \overline{r} \) are shown in Fig. 2.3. As the cartilage permeability \( \overline{k} \) increases, the load capacity \( \overline{W} \) decreases for all values of \( \overline{r} \). Large permeability means there are more voids available to fluid discharge on the poroelastic surfaces, which results in the decrease in pressure and leads to decrease in load carrying capacity. Also, since the macromolecules in the lubricant fluid (such as hyaluronate etc.) are too large to pass into the pores of normal articular cartilage, the fluid passes laterally through these voids.
To observe the relative significance of poroelasticity and couple-stress effects as compared to the classical case, the relative load differences
\[ R = \left( \frac{W_{\text{couple-stress}} - W_{\text{Newtonian}}}{W_{\text{Newtonian}}} \right) \times 100 \]
are listed in Table 2.1. Thus, Table 2.1 shows the values of percentage increase in the load capacity of cartilage compared to Newtonian and non-poroelastic case. It is observed that for \( \bar{r} = 10, \bar{k} = 7.5 \times 10^{-3} \) and \( \bar{E}/\bar{K} = 2.0 \), there is an increase of nearly 10% in \( \bar{W} \) compared to classical viscous fluid. This is quite an useful prediction as the joints are capable of supporting the loads 3-4 times the body weight during their normal walking and also, at the energetic activities the load carried through cartilage surfaces is still higher (Paul, 1975).

Fig. 2.4 shows load capacity profiles \( \bar{W} \) as function of radius \( R \), for \( E/\bar{K} = 2.0, \bar{k} = 7.65 \times 10^{-3} \) and \( \bar{r} = 5 \). In the fluid film region, for large radius, the upper rigid indenter becomes relatively flat and uniform and this large area of a film region broadens the pressure distribution and hence the load carrying capacity. This wide thin film area acts to arrests exit of the lateral fluid from the gap. For large radius \( R \) of spherical indenter, the modified Reynolds equation (2.3.11) reduces to simple form giving the squeeze film characteristics between two parallel surfaces. Variation of non-dimensional load capacity \( \bar{W} \) with minimum film thickness \( \bar{h}_0 \) for different values of elastic parameters \( E/\bar{K} \) is shown in Fig. 2.5. For large radius of rigid indenter, the effect of elasticity is to enhance the load carrying capacity compared to non-elastic case \( (E'/\bar{K} = 0) \). Also, \( \bar{W} \) decreases as the intra-articular gap between rigid indenter and poroelastic matrix decreases for all values of elastic parameter \( E'/\bar{K} \).
2.5. Conclusions

The lubrication aspects of diarthrodial joints are analyzed for the canonical squeeze film problem. The cartilage is modeled as poroelastic matrix and synovial fluid as couple-stress fluid. The spherical indenter modeled as rigid and impermeable which is approaching a flat poroelastic matrix. The modified Reynolds equation with variable coefficients, describing the pressure build up, is solved using finite difference based multigrid method. The fifth place decimal convergent solution is obtained. Multigrid method results in significant accuracy and also it saves computational time and storage capacity. Proposed poroelastic model of the joint predicts the maximal supporting load that can be sustained by a joint increases for smaller values of $\bar{r}$, $\bar{k}$ and larger values of $E'/K$. The increase of indenter radius $R_i$ enlarges the load supporting area. Since, the governing equations describing complex structure of cartilage and synovial fluid are complicated because of non-linearity and also joints have wide range of articulating features, the present model do predict some of the salient features which would enable in selecting suitable design parameters and serves as bench mark for the investigation of more involved problems.
Fig. 2. (a). Pressure distribution for $r = 5$, $\sigma = 7.65 \times 10^{-5}$ and $K/E = 2.0$. 
Figure 2.2(b). Pressure distribution for $r = 10$, $k = 7.65 \times 10^{-5}$ and $E/K = 2.0$. 
Fig. 2. 2(c). Pressure distribution for \( r = \infty \), \( k = 7.65 \times 10^{-5} \) and \( E/K = 2.0 \).
with $R = 1$, $E/K = 2.0$.
Fig. 2.4 Variation of load carrying capacity $W$ and radius $R_x$ for $z = 10$, $k = 7.65 \times 10^{-5}$ and $E/K = 2.0$. 
Fig. 2.5 Variation of load carrying capacity with film thickness $h_0$ for different $E'/K$ with $R_x = \infty$ and $R = 10$ and $R_x = \infty$, $f_\infty = 10$ and $k = 7.65 \times 10^{-6}$.
Table 2.1: Relative load difference $R_w$ for $E/K = 2.0$.

<table>
<thead>
<tr>
<th>$\bar{t}$</th>
<th>$k = 7.65 \times 10^{-3}$</th>
<th>$\bar{k} = 7.65 \times 10^{-3}$</th>
<th>$\bar{k} = 7.65 \times 10^{-1}$</th>
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<td>10</td>
<td>9.85183</td>
<td>9.04536</td>
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