CHAPTER 7

GPS OF HYPERCHAOTIC LORENZ AND HYPERCHAOTIC QI SYSTEMS VIA ACTIVE CONTROL

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CHAPTER 7
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7.0 INTRODUCTION

This chapter has been organized as follows. Section 7.1 presents a description of the
hyperchaotic systems considered in this chapter. Section 7.2 describes new results for the GPS
of two identical hyperchaotic Lorenz systems via active control. Section 7.3 describes new
results for the GPS of two identical hyperchaotic Qi systems via active control. Section 7.4
describes new results the GPS of non-identical hyperchaotic Lorenz and hyperchaotic Qi systems
via active control. Section 7.5 contains a summary of the main results derived in this chapter.

7.1 SYSTEMS DESCRIPTION

The hyperchaotic Lorenz system (Jia, 2007) is described by the 4D dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + cx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{align*}
\]

(7.1)

where \(x_1, x_2, x_3, x_4\) are the state variables and \(a, b, c, d\) are constant, positive parameters of the
system.

The 4D dynamics (7.1) exhibits a hyperchaotic attractor, when the system parameter values
are chosen as

\[a = 10, \quad b = 8/3, \quad c = 28 \quad \text{and} \quad d = 1.3.\]

Figure 7.1 depicts the phase portrait of the hyperchaotic Lorenz system.

The hyperchaotic Qi system (Chen, Yang, Qi and Yuan, 2007) is described by the 4D
dynamics
\[ \begin{align*}
\dot{x}_1 &= p(x_2 - x_1) + \varepsilon x_2 x_3 \\
\dot{x}_2 &= r x_1 - s x_1 x_3 + x_2 + x_4 \\
\dot{x}_3 &= x_1 x_2 - q x_3 \\
\dot{x}_4 &= -\lambda x_2
\end{align*} \] (7.2)

where \( x_1, x_2, x_3, x_4 \) are the state variables and \( p, q, r, s, \varepsilon, \lambda \) are constant, positive parameters of the system.

The 4D dynamics (7.2) exhibits a hyperchaotic attractor, when the parameter values are taken as \( p = 35, \ q = 4.9, \ r = 25, \ s = 5, \ \varepsilon = 35 \) and \( \lambda = 22 \).

Figure 7.2 describes the phase portrait of the hyperchaotic Qi system (7.2).

![Figure 7.1 Phase Portrait of the Hyperchaotic Lorenz System](image)
7.2 GPS OF IDENTICAL HYPERCHAOTIC LORENZ SYSTEMS VIA ACTIVE CONTROL

7.2.1 Theoretical Results

In this section, the active control method has been applied for the generalized projective synchronization (GPS) of two identical hyperchaotic Lorenz systems (2007).

Thus, the master system is described by the hyperchaotic Lorenz dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + cx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{align*}
\]

(7.3)

where \(x_1, x_2, x_3, x_4\) are the states and \(a, b, c, d\) are positive, constant parameters of the system.
The slave system is described by the controlled hyperchaotic Lorenz dynamics

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= -y_1y_3 + cy_1 - y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - by_1 + u_3 \\
\dot{y}_4 &= -y_1y_3 + dy_4 + u_4 
\end{align*}
\]

(7.4)

where \( y_1, y_2, y_3, y_4 \) are the states and \( u_1, u_2, u_3, u_4 \) are the active nonlinear controls to be designed.

For the GPS of the systems (7.3) and (7.4), the synchronization error \( e \) is defined by

\[
e_i = y_i - \alpha_i x_i, \quad (i = 1, 2, 3, 4)
\]

(7.5)

where the scales \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are real numbers.

The error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= a(y_2 - y_1) + y_4 - \alpha_1 \left[a(x_2 - x_1) + x_4\right] + u_1 \\
\dot{e}_2 &= -y_1y_3 + cy_1 - y_2 - \alpha_2 \left[-x_1x_3 + cx_1 - x_2\right] + u_2 \\
\dot{e}_3 &= y_1y_2 - by_1 - \alpha_3 \left[x_1x_2 - bx_1\right] + u_3 \\
\dot{e}_4 &= -y_1y_3 + dy_4 - \alpha_4 \left[-x_1x_3 + dx_4\right] + u_4 
\end{align*}
\]

(7.6)

The active nonlinear controller has been chosen as

\[
\begin{align*}
u_1 &= -a(y_2 - y_1) - y_4 + \alpha_1 \left[a(x_2 - x_1) + x_4\right] - k_1e_1 \\
u_2 &= y_1y_3 - cy_1 + y_2 + \alpha_2 \left[-x_1x_3 + cx_1 - x_2\right] - k_2e_2 \\
u_3 &= -y_1y_2 + by_1 + \alpha_3 \left[x_1x_2 - bx_1\right] - k_3e_3 \\
u_4 &= y_1y_3 - dy_4 + \alpha_4 \left[-x_1x_3 + dx_4\right] - k_4e_4
\end{align*}
\]

(7.7)

where the gains \( k_1, k_2, k_3, k_4 \) are positive constants.

Substituting (7.7) into (7.6), the error dynamics simplifies to

\[
\begin{align*}
\dot{e}_1 &= -k_1e_1 \\
\dot{e}_2 &= -k_2e_2 \\
\dot{e}_3 &= -k_3e_3 \\
\dot{e}_4 &= -k_4e_4
\end{align*}
\]

(7.8)
Theorem 7.1 The active feedback controller (7.7) achieves global chaos generalized projective synchronization (GPS) between the identical hyperchaotic Lorenz systems (7.3) and (7.4).

Proof. We consider the quadratic Lyapunov function defined by

\[ V(e) = \frac{1}{2} e^T e = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 \right) \]  

(7.9)

which is a positive definite function on \( \mathbb{R}^4 \).

Differentiating (7.9) along the trajectories of (7.8), we get

\[ \dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \]  

(7.10)

which is a negative definite function on \( \mathbb{R}^4 \).

Thus, by Lyapunov stability theory, the error dynamics (7.8) is globally exponentially stable.

This completes the proof.

7.2.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-scale \( h = 10^{-8} \) is used to solve the two systems of differential equations (7.3) and (7.4) with the active controller (7.7).

The parameters of the identical hyperchaotic Lorenz systems are chosen as \( a = 10, \ b = 8/3, \ c = 28, \ d = 1.3 \).

The initial values for the master system (7.3) are taken as \( x_1(0) = 5, \ x_2(0) = 11, \ x_3(0) = 28, \ x_4(0) = 20 \).

The initial values for the slave system (7.4) are taken as \( y_1(0) = 18, \ y_2(0) = 22, \ y_3(0) = 7, \ y_4(0) = 30 \).

The GPS scales are taken as \( \alpha_1 = 6.2, \ \alpha_2 = -2.3, \ \alpha_3 = 3.7 \) and \( \alpha_4 = -5.6 \).

The state feedback gains are taken as \( k_i = 4 \) for \( i = 1, 2, 3, 4 \).
Figure 7.3 shows the generalized projective synchronization (GPS) of the identical hyperchaotic Lorenz systems. Figure 7.4 shows the time-history of the GPS errors $e_1, e_2, e_3, e_4$ for the identical hyperchaotic Lorenz systems.
7.3. GPS OF IDENTICAL HYPERCHAOTIC QI SYSTEMS VIA ACTIVE CONTROL

7.3.1 Theoretical Results

In this section, the active control method has been applied for the generalized projective synchronization (GPS) of two identical hyperchaotic Qi systems (2007).

Thus, the master system is described by the hyperchaotic Qi dynamics

\[
\begin{align*}
\dot{x}_1 &= p(x_2 - x_3) + \varepsilon x_2 x_3 \\
\dot{x}_2 &= r x_1 - s x_1 x_3 + x_2 + x_4 \\
\dot{x}_3 &= x_1 x_2 - q x_3 \\
\dot{x}_4 &= -\lambda x_2
\end{align*}
\]  

(7.11)

where \( x_1, x_2, x_3, x_4 \) are the states and \( p, q, r, s, \varepsilon, \lambda \) are positive, constant parameters of the system.

The slave system is described by the controlled hyperchaotic Qi dynamics

\[
\begin{align*}
\dot{y}_1 &= p(y_2 - y_1) + \varepsilon y_2 y_3 + u_1 \\
\dot{y}_2 &= r y_1 - s y_1 y_3 + y_2 + y_4 + u_2 \\
\dot{y}_3 &= y_1 y_2 - q y_3 + u_3 \\
\dot{y}_4 &= -\lambda y_2 + u_4
\end{align*}
\]  

(7.12)

where \( y_1, y_2, y_3, y_4 \) are the states and \( u_1, u_2, u_3, u_4 \) are the active nonlinear controls to be designed.

For the GPS of the hyperchaotic Qi systems (7.11) and (7.12), the synchronization error \( e \) is defined by

\[
e_i = y_i - \alpha_i x_i, \quad (i = 1, 2, 3, 4)
\]  

(7.13)

where the scales \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are real numbers.

The error dynamics is obtained as
\[
\begin{align*}
\dot{e}_1 &= p(y_2 - y_1) + \varepsilon y_2 v_3 - \alpha_1 [p(x_2 - x_1) + \varepsilon x_2 v_3] + u_1 \\
\dot{e}_2 &= r y_1 - s y_1 y_3 + y_2 + y_4 - \alpha_2 [r x_1 - s x_1 v_3 + x_2 + x_4] + u_2 \\
\dot{e}_3 &= y_1 y_2 - q y_3 - \alpha_3 [x_1 x_2 - q x_3] + u_3 \\
\dot{e}_4 &= -\lambda y_2 - \alpha_4 [-\lambda x_2] + u_4
\end{align*}
\] (7.14)

The active nonlinear controller has been chosen as

\[
\begin{align*}
u_1 &= -p(y_2 - y_1) - \varepsilon y_2 v_3 + \alpha_1 [p(x_2 - x_1) + \varepsilon x_2 v_3] - k_1 e_1 \\
u_2 &= -r y_1 + s y_1 y_3 - y_2 - y_4 + \alpha_2 [r x_1 - s x_1 v_3 + x_2 + x_4] - k_2 e_2 \\
u_3 &= -y_1 y_2 + q y_3 + \alpha_3 [x_1 x_2 - q x_3] - k_3 e_3 \\
u_4 &= \lambda y_2 + \alpha_4 [-\lambda x_2] - k_4 e_4
\end{align*}
\] (7.15)

where the gains \( k_1, k_2, k_3, k_4 \) are positive constants.

Substituting (7.15) into (7.14), the error dynamics simplifies to

\[
\begin{align*}
\dot{e}_1 &= -k_1 e_1 \\
\dot{e}_2 &= -k_2 e_2 \\
\dot{e}_3 &= -k_3 e_3 \\
\dot{e}_4 &= -k_4 e_4
\end{align*}
\] (7.16)

**Theorem 7.2** The active feedback controller (7.15) achieves global chaos generalized projective synchronization (GPS) between the identical hyperchaotic Qi systems (7.11) and (7.12).

**Proof.** We consider the quadratic Lyapunov function defined by

\[
V(e) = \frac{1}{2} e^T e = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 \right)
\] (7.17)

which is a positive definite function on \( R^4 \).

Differentiating (7.17) along the trajectories of (7.16), we get

\[
\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2
\] (7.18)

which is a negative definite function on \( R^4 \).
Thus, by Lyapunov stability theory, the error dynamics (7.16) is globally exponentially stable. This completes the proof. ■

7.3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-scale $h = 10^{-6}$ is used to solve the two systems of differential equations (7.11) and (7.12) with the active controller (7.15).

The parameters of the identical hyperchaotic Qi systems are chosen as

$$p = 35, q = 4.9, r = 25, s = 5, \varepsilon = 35, \lambda = 22.$$  

The initial values for the master system (7.11) are taken as

$$x_1(0) = 12, x_2(0) = 7, x_3(0) = 28, x_4(0) = 6$$

The initial values for the slave system (7.12) are taken as

$$y_1(0) = 9, y_2(0) = 17, y_3(0) = 22, y_4(0) = 18$$

The GPS scales are taken as

$$\alpha_1 = -6.8, \alpha_2 = 5.6, \alpha_3 = 4.2 \text{ and } \alpha_4 = -3.7.$$  

The state feedback gains are taken as

$$k_i = 4 \text{ for } i = 1, 2, 3, 4.$$  

Figure 7.5 shows the GPS synchronization of the identical hyperchaotic Qi systems.

Figure 7.6 shows the time-history of the GPS synchronization errors $e_1, e_2, e_3, e_4$ for the identical hyperchaotic Qi systems.
Figure 7.5 GPS of the Identical Hyperchaotic Qi Systems

Figure 7.6 Time History of the GPS Errors
7.4. GPS OF HYPERCHAOTIC LORENZ AND HYPERCHAOTIC QI SYSTEMS VIA ACTIVE CONTROL

7.4.1 Theoretical Results

In this section, the active control method has been applied for the generalized projective synchronization (GPS) of hyperchaotic Lorenz and hyperchaotic Qi systems.

Thus, the master system is described by the hyperchaotic Lorenz dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + cx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{align*}
\] (7.19)

where \(x_1, x_2, x_3, x_4\) are the states and \(a, b, c, d\) are constant, positive parameters of the system.

The slave system is described by the controlled hyperchaotic Qi dynamics

\[
\begin{align*}
\dot{y}_1 &= p(y_2 - y_1) + \varepsilon y_2y_3 + u_1 \\
\dot{y}_2 &= ry_1 - sy_1y_3 + y_2 + y_4 + u_2 \\
\dot{y}_3 &= y_1y_2 - qy_3 + u_3 \\
\dot{y}_4 &= -\lambda y_2 + u_4
\end{align*}
\] (7.20)

where \(y_1, y_2, y_3, y_4\) are the states, \(p, q, r, s, \varepsilon, \lambda\) are positive, constant parameters of the system and \(u_1, u_2, u_3, u_4\) are the active nonlinear controls to be designed.

For the GPS of the hyperchaotic systems (7.19) and (7.20), the synchronization error \(e\) is defined by

\[
e_i = y_i - \alpha_i x_i, \quad (i = 1, 2, 3, 4)
\] (7.21)

where the scales \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) are real numbers.

The error dynamics is obtained as
\[ \dot{e}_1 = p(y_2 - y_1) + \varepsilon y_2 y_3 - \alpha_1 \left[ a(x_2 - x_1) + x_4 \right] + u_1 \]
\[ \dot{e}_2 = ry_1 - sy_1 y_3 + y_2 + y_4 - \alpha_2 \left[ -y_2 x_1 + c x_1 - x_2 \right] + u_2 \]
\[ \dot{e}_3 = y_1 y_2 - q y_3 - \alpha_3 \left[ x_1 x_2 - b x_3 \right] + u_3 \]
\[ \dot{e}_4 = -\lambda y_2 - \alpha_4 \left[ -y_2 x_1 + d x_4 \right] + u_4 \]

(7.22)

The active nonlinear controller has been chosen as

\[ u_1 = -p(y_2 - y_1) - \varepsilon y_2 y_3 + \alpha_1 \left[ a(x_2 - x_1) + x_4 \right] - k_1 e_1 \]
\[ u_2 = -ry_1 + sy_1 y_3 - y_2 + y_4 + \alpha_2 \left[ -y_2 x_1 + c x_1 - x_2 \right] - k_2 e_2 \]
\[ u_3 = -y_1 y_2 + q y_3 + \alpha_3 \left[ x_1 x_2 - b x_3 \right] - k_3 e_3 \]
\[ u_4 = -\lambda y_2 + \alpha_4 \left[ -y_2 x_1 + d x_4 \right] - k_4 e_4 \]

(7.23)

where the gains \( k_1, k_2, k_3, k_4 \) are positive constants.

Substituting (7.23) into (7.22), the error dynamics simplifies to

\[ \dot{e}_1 = -k_1 e_1 \]
\[ \dot{e}_2 = -k_2 e_2 \]
\[ \dot{e}_3 = -k_3 e_3 \]
\[ \dot{e}_4 = -k_4 e_4 \]

(7.24)

**Theorem 7.3** The active feedback controller (7.24) achieves global chaos generalized projective synchronization (GPS) between the hyperchaotic Lorenz system (7.19) and hyperchaotic Qi system (7.20).

**Proof.** We consider the quadratic Lyapunov function defined by

\[ V(e) = \frac{1}{2} e^T e = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 \right) \]

(7.25)

which is a positive definite function on \( R^4 \).

Differentiating (7.25) along the trajectories of (7.24), we get

\[ \dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \]

(7.26)

which is a negative definite function on \( R^4 \).
Thus, by Lyapunov stability theory, the error dynamics (7.24) is globally exponentially stable. This completes the proof. ■

7.4.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time scale \( h = 10^{-8} \) is used to solve the two systems of differential equations (7.19) and (7.20) with the active controller (7.23).

The parameters of the hyperchaotic Lorenz system (7.19) are chosen as
\[
a = 10, \quad b = 8 / 3, \quad c = 28, \quad d = 1.3
\]
The parameters of the hyperchaotic Qi system (7.20) are chosen as
\[
p = 35, q = 4.9, r = 25, \quad s = 5, \quad e = 35, \quad \lambda = 22.
\]
The initial values for the master system (7.19) are taken as
\[
x_1(0) = 15, \quad x_2(0) = 4, \quad x_3(0) = 18, \quad x_4(0) = 20
\]
The initial values for the slave system (7.20) are taken as
\[
y_1(0) = 4, \quad y_2(0) = 20, \quad y_3(0) = 6, \quad y_4(0) = 12
\]
The GPS scales are taken as
\[
\alpha_1 = 2.3, \quad \alpha_2 = 1.8, \quad \alpha_3 = -3.9, \quad \alpha_4 = -1.7
\]
The state feedback gains are chosen as
\[
k_1 = 4, \quad k_2 = 4, \quad k_3 = 4, \quad k_4 = 4
\]
Figure 7.7 shows the generalized projective synchronization (GPS) of the non-identical hyperchaotic Lorenz and hyperchaotic Qi systems.

Figure 7.8 shows the time-history of the GPS errors \( e_1, e_2, e_3, e_4 \) for the non-identical hyperchaotic Lorenz and hyperchaotic Qi systems.
Figure 7.7 GPS of the Hyperchaotic Lorenz and Hyperchaotic Qi Systems

Figure 7.8 Time History of the GPS Errors
7.5 SUMMARY

In this chapter, active control laws have been derived for achieving generalized projective synchronization (GPS) of the following pairs of hyperchaotic systems:

(A) Identical hyperchaotic Lorenz systems (2007)

(B) Identical hyperchaotic Qi systems (2007)

(C) Non-identical hyperchaotic Lorenz and hyperchaotic Qi systems

The synchronization results (GPS) derived in this chapter for the hyperchaotic Lorenz and hyperchaotic Qi systems have been proved using Lyapunov stability theory. Since Lyapunov exponents are not required for these calculations, the proposed active control method is very effective and suitable for achieving GPS of the hyperchaotic systems addressed in this chapter. Numerical simulations are shown to demonstrate the effectiveness of the GPS synchronization results derived in this chapter for the hyperchaotic Lorenz and hyperchaotic Qi systems.