THE FORCED KORTEWEG DE VRIES EQUATION

4.1 INTRODUCTION

When fluid flow is disturbed by a small bump it can generate surface wave. The flow of a fluid over an obstacle is a classical and fundamental problem in fluid mechanics. It is well known that transcritical flow over a localized obstacle generates upstream and downstream nonlinear wavetrains. The flow has been successfully modeled in the framework of the forced Korteweg-de Vries equation (fKdV), where numerical and asymptotic analytical methods are the only methods for solving it. The fKdV equation has been derived for water waves by Akylas [2], Cole [17], Mei [69], Wu [94], Lee et al. [66], and for internal waves by Grimshaw and Smyth [46]. Note that for water waves, the forcing provided by the bottom obstacle, or by an applied surface pressure field, or by a slender ship, are equivalent in the weakly nonlinear approximation. Grimshaw and his co-workers [47] showed that the wave-trains having the structure of unsteady undular bore analytically and also confirmed by numerical solutions using explicit finite difference method.

We consider one-dimensional fKdV equation of water surface elevation \( u(x,t) \) satisfying

\[
\frac{\partial u}{\partial t} + Ku_x - 6uu_x - u_{xxx} = f(x) \tag{4.1.1}
\]

as the flow is disturbed by a bump \( f(x) \) for a small interval derived by Cole [17]. Here \( K \) is the correction factor of small order \( \varepsilon^{\frac{1}{2}} \) to the Froude number \( F \), as defined to the undisturbed water depth, that is \( F^2 = 1 + \varepsilon^{\frac{1}{2}} K \). The equation is solved numerically by
quintic B-spline collocation method as it is applied to the KdVB equation by Zaki [100] and then used to observe wave generation for various disturbances \( f(x) \). The proposed method demonstrates the nonlinear wave trains of the water surface generated by the forcing term.

### 4.2 QUINTIC B-SPLINE COLLOCATION METHOD

Consider the fKdV equation (1) where \( K \) is the parameter and the subscripts \( x \) and \( t \) indicate to the differentiation with respect to \( x \) and \( t \).

On applying a Galerkin approach, we obtained a weak form of Eq. (4.1.1) as

\[
\int_{a}^{b} W_m \left( 2u_t + Ku_x - 3uu_x - \frac{1}{3} u_{xxx} - f_x(x) \right) dx = 0
\]

where \( W_m \) is a weight function. If now the weight functions are identified with Dirac delta functions \( W_m = \delta(x_m - x) \) then, the resulting set of equations takes the form

\[
2u_t + Ku_x - 3uu_x - \frac{1}{3} u_{xxx} - f_x(x) \bigg|_{x=x_m} = 0,
\]

where, \( m = 0, 1, 2, \ldots, N \). These equations may also be identified as point collocation conditions.

The boundary conditions will be chosen from the physical boundary condition associated with the fKdV equation

\[
u(a,t) = u(b,t) = 0
\]

and the collocation boundary conditions required for insuring a unique quintic B-spline solution

\[
u_x(a,t) = u_x(b,t) = 0
\]

\[
u_{xx}(a,t) = u_{xx}(b,t) = 0
\]
Consider \( x_m = a + mh, \ h = (b - a)/N, \ m = -3,-2, \ldots, N + 3, \) then \( \Pi := a = x_0 < x_1 < x_2 < \cdots < x_N = b \) is an equal distance partition of the interval \([a,b]\) by the knots \( x_m. \) The set of functions \( \{\phi_{-2}, \phi_{-1}, \ldots, \phi_{N+2}\} \) forms a basis for functions defined over the finite region \([a,b]\). A global approximation \( u(x,t) \) is given by

\[
u_N(x,t) = \sum_{m=-2}^{N+2} \delta_m(t) \phi_m(x), \tag{4.2.6}
\]

where the \( \delta_m \) are time-dependent parameters to be determined. The B-spline \( \phi_m(x) \) and its principle derivatives vanish outside the region \([x_{m-3}, x_{m+3}]\). The intervals \([x_m, x_{m+1}]\) are identified with finite elements, which are each covered by six B-splines. Over the element \([x_m, x_{m+1}]\) the variation of the function \( u(x,t) \) is taken in the from

\[
u(x,t) = \sum_{j=m-2}^{m+3} \delta_j(t) \phi_j(x) \tag{4.2.7}
\]

In terms of a local coordinate system \( \xi \) given by \( h\xi = x - x_m, \) where \( h = (x_{m+1} - x_m) \) and \( 0 \leq \xi \leq 1, \) expressions for the element splines are [73]

\[
\begin{align*}
\phi_{m-2} &= 1 - 5\xi + 10\xi^2 - 10\xi^3 + 5\xi^4 - \xi^5 \\
\phi_{m-1} &= 26 - 50\xi + 20\xi^2 + 20\xi^3 - 20\xi^4 + 5\xi^5 \\
\phi_m &= 66 - 60\xi^2 + 30\xi^4 - 10\xi^5 \\
\phi_{m+1} &= 26 + 50\xi - 20\xi^2 - 20\xi^3 - 20\xi^4 + 10\xi^5 \\
\phi_{m+2} &= 1 + 5\xi + 10\xi^2 + 10\xi^3 + 5\xi^4 - 5\xi^5 \\
\phi_{m+3} &= \xi^5 
\end{align*}
\]

The nodal values \( u_m, u'_m, u''_m \) and \( u'''_m \) at the knot \( x_m \) are given in terms of \( \delta_m \) by

\[
\begin{align*}
u_m &= \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2} \tag{4.2.9} \\
u'_m &= 5(\delta_{m+2} + 10\delta_{m+1} - 10\delta_{m-1} - \delta_{m-2}) \tag{4.2.10} \\
u''_m &= 20(\delta_{m+2} + 2\delta_{m+1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}) \tag{4.2.11}
\end{align*}
\]
\[ h^3 u'''_m = 60(\delta^{m+2} - 2\delta^{m+1} + 2\delta^{m-1} - \delta^{m-2}) \]  
(4.2.12)

where the dashes denote differentiation with respect to \( x \).

To obtain a recurrence relationship for the numerical solution, time center on \((n+1/2)\Delta t\), where \( \Delta t \) is time step, and use a Crank-Nicholson approach with

\[ (u_t)_m^{n+1/2} = \frac{1}{\Delta t} \left( u_{m+1}^{n+1} - u_m^n \right) \]  
(4.2.13)

\[ u_m^{n+1/2} = \frac{1}{2} \left( u_{m+1}^{n+1} + u_m^n \right) \]  
(4.2.14)

\[ (u_x u)_m^{n+1/2} = \frac{1}{2} \left( (u_x u)_m^{n+1/2} + (u_x u)_m^n \right) \]  
(4.2.15)

where the subscript \( m \) is the node label while the superscripts \( n \) and \( n+1 \) are time labels. Eq. (4.2.15) is a quasi-linearization obtained through an arithmetic mean in time. These results are all second order accurate approximations to the values at time \((n+1/2)\Delta t\). Use equations (4.2.9)-(4.2.15) to evaluate \( u_m \) and its space derivatives and we have for each knot an equation relating parameters at adjacent time levels, \( \delta_m^{n+1} \) and \( \delta_m^n \) in the form:

\[
\begin{align*}
\left\{ \begin{array}{l}
a_m \delta_{m-2}^{n+1} + b_m \delta_m^{n+1} + c_m \delta_m^{n+1} + d_m \delta_{m+1}^{n+1} + e_m \delta_{m+2}^{n+1} = \\
a'_m \delta_{m-2}^n + b'_m \delta_m^n + c'_m \delta_m^n + d'_m \delta_{m+1}^n + e'_m \delta_{m+2}^n + f_x(x_m)
\end{array} \right.
\end{align*}
\]  
(4.2.16)

where

\[
\begin{align*}
a_m &= \frac{2}{\Delta t} - \frac{5(K-3z_m)}{2h} + \frac{10}{h^3}; b_m &= \frac{52}{\Delta t} - \frac{25(K-3z_m)}{h} - \frac{20}{h^3}; c_m &= \frac{132}{\Delta t}; \\
d_m &= \frac{52}{\Delta t} + \frac{25(K-3z_m)}{h} + \frac{20}{h^3}; e_m &= \frac{5(K-3z_m)}{2h} - \frac{10}{h^3}; \\
a'_m &= \frac{2}{\Delta t} + \frac{5(K-3z_m)}{2h} + \frac{10}{h^3}; b'_m &= \frac{52}{\Delta t} + \frac{25(K-3z_m)}{h} + \frac{20}{h^3}; c'_m &= \frac{132}{\Delta t};
\end{align*}
\]
\[ d'_m = \frac{52}{\Delta t} \frac{25(K - 3z_m)}{h} - 20 \frac{h^2}{h^2}; e'_m = \frac{2}{\Delta t} \frac{5(K - 3z_m)}{2h} + \frac{10}{h^3}; \]
\[ z_m = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2} \]

The system of equations (4.2.16) consists of \( N + 1 \) quasi-linear equations in \( N + 5 \) unknowns \( (\delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \delta_{N+1}, \delta_{N+2}) \). To obtain a unique solution to this system four additional constrains on the derivatives at the end points are obtained from the collocation boundary conditions (4.2.4) and (4.2.5). Therefore the set of equations (4.2.16) together with the boundary conditions (4.2.4) and (4.2.5) can be written in the matrix equation of the form
\[
Ad^{n+1} = Bd^n + F \tag{4.2.17}
\]
where \( A \) and \( B \) are defective penta-diagonal \( (N+1) \times (N+1) \) matrices and \( F \) is a column vector whose entries are \( f_x(x_m) \).

The time evolution of the approximate solution \( u_N(x,t) \) is determined from that of vector \( d^n \) which is found by repeatedly solving the matrix equation (4.2.17) once the initial vector \( d^0 \) has been computed from the initial condition. As the initial condition, various functions can be used such as \( u(x,0) = 0 \) representing no wave or a solitary wave in form of secant hyperbolic function. In our case, the initial vector \( d^0 \) is set to zero.

**4.3 Numerical Results**

The numerical procedure describe in the previous section is used to observed waves generated by flow disturbed by a bump. Most of our calculations used \( h = 0.5, \Delta t = 0.1 \) and number of time step \( tN = 1000 \) for observation domain \([0,200]\). The bump is placed in the middle of the domain.
The scheme is first tested for KdV equation by giving the force \( f(x) = 0 \) in the right hand side of Eq. (4.1.1) and initial condition is

\[
u(x,0) = a \sec h^2 b(x - x_0) \tag{4.3.1}
\]

with the crest located at \( x = x_0 \). Analytically the initial condition (4.3.1) produces a solitary wave traveling with constant speed and without changing the shape when

\[
b = \sqrt{\frac{3a}{4}}
\]

for the KdV Eq. (4.1.1) with \( f(x) = 0 \). These characters are demonstrated in Fig. 4.1 as a solution of

![Plot of u(x,t) for some values of t calculated for fKdV equation (1) with f(x) = 0, using K = 1.5.](image)

For smaller values of \( K \) the wave travels to the right slower and it is continued decreasing \( \kappa \) the wave travels in different direction. At \( K = a \) the wave reaches steady. This agrees with the analytical solution, i.e.

\[
u(x,t) = a \sec h^2 \sqrt{\frac{3a}{4}} \left[ (x - x_0) - \frac{K - a}{2} t \right] \tag{4.3.2}
\]
Now the scheme is used to solve the equation (4.1.1) involving the forcing term. The parameter $K$ and the function $f$ are input considered effecting to the solution. First we performed the solution for various values of $K$ combined with a bump

$$f(x) = \begin{cases} 0.02 \sec h^2[0.12(x-90)], & \text{for } 84 \leq x \leq 96 \\ 0, & \text{otherwise} \end{cases} \quad (4.3.3)$$

We show plot of $u(x,t)$ in Fig. 4.2 corresponding to $K = 1.0, 0.5, 0.02$ and $-1.0$. Two points are indicated as the place where the wave appears i.e. at the left and right ends of the bump, $x = 84$ and $x = 96$. The surface above the left end is pushed up and the other end is opposite direction, as flow is disturbed and then runs down freely after passing the bump. This continuously grows up the elevation and followed by appearing other waves. Different values of $K$ the elevation of both waves can be seen in Fig. 2. It is indicated that the elevation above $x = 84$ and $x = 96$ grows up without much propagating other waves when $K$ tends to the amplitude of the bump. We can see this in Fig. 4.2(c) for the same value of $K$ and the amplitude.

We then observed the effect of the forcing term by replacing the secant-hyperbolic function with other function such as sinusoidal. We obtain similar profile of the elevation. But when the forcing term is negative of Eq. (20) we obtain opposite composition of the surface elevation describe above.
Figure 4.2: Plot of $u(x,t)$ at a particular time $t$ for various values of $K$
(a) $K = 1.0$ (b) $K = 0.5$ (c) $K = 0.02$ and (d) $K = -1.0$

4. CONCLUSION

We have developed a quintic B-spline collocation method to solve the forced KdV equation representing wave generation as a uniform flow disturbed by a bump on the bottom of a channel. The method has been tested by comparing the numerical solution of the KdV equation to the analytical solution in the form of solitary wave and we obtain agreement between both solutions. When a forcing term is included we obtain that the solution describes generating a train of waves such as obtained by Cole [17]. The evolution of the waves is observed for various values of the parameter corresponding to Froude number.