GENERAL INTRODUCTION

1.1 INTRODUCTION TO SOLITARY WAVE AND SOLITON

Solitary waves were first observed by J. Scott Russel [76] in 1834 and it was not recognized widely at that time. After 150 years, the significance of this discovery as an important stable state of nonlinear system was realized and since then it has become an open area of research especially in the field of numerical analysis. Actually, such nonlinear system represent a hump-shaped wave of permanent form known as solitary wave and it appears in many areas such as physical phenomena, plasma and laser physics [19]. The type of equation we are dealing with were first derived theoretically (1895) by D. Korteweg and G. de Vries [62] leading to an equation governing the propagation of shallow water waves known as “Korteweg de Vries (KdV)” equation and it possessed a solution that matched the early Russel’s observations. It was deduced that the steady velocity of the wave, \( c \), satisfies the relation

\[
c^2 = g(h + a)
\]  

(1.1.1)

where \( a \) is the amplitude of the wave, \( h \) is the height of the undisturbed water and \( g \) is the acceleration due to gravity.

The original KdV equation derived by Korteweg and de Vries is

\[
\frac{\partial \eta}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{l}} \frac{\partial}{\partial x} \left[ \frac{2}{3} \alpha \eta + \frac{1}{2} \eta^2 + \frac{1}{3} \sigma \frac{\partial^2 \eta}{\partial x^2} \right].
\]

(1.1.2)

where \( l \) and \( l + \eta \) represent the depth and the elevation of the water surface above bottom, \( \sigma = l^3 / 3 - Tl / \rho g \), is an arbitrary constant, \( T \) is the surface tension and \( \rho \) is the density of the fluid. The derivation of equation (1.1.2) from the first principle is not easy. Scaling of the independent and dependent variables the KdV equation has a form,
\[
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1.3}
\]

The second and third terms represent the nonlinear and dispersion effects, respectively. The nonlinear effect causes the steepening of waveform, while the dispersion effect makes the waveform spread. Due to the competition of these two effects, a stationary waveform (solitary wave) exists. The reason why each stationary wave is stable in spite of mutual interactions is that the KdV equation has large number of conserved quantities. Dynamical properties of the system are severely restricted by the existence of a large number of conservation laws. The conserved quantities guarantee the time independence of the parameters which characterize the solitons and therefore the solitons are stable. The detailed study of the conservation laws of the KdV equation is presented in Chapter II.

Boussinesq [12] and Rayleigh [75] independently assumed that a solitary wave has length much greater than depth of the water and thereby Russell’s empirical formula for \(c\) from the equation of motion of an inviscid incompressible fluid. They further showed essentially that the wave height above the mean level \(h\) is given by

\[
u(x,t) = a \sec h^2 \left[ \frac{1}{b} (x - ct) \right] \tag{1.1.4}
\]

where \(b^2 = \frac{4h^2(h + a)}{3a}\), for any positive amplitude \(a\). This also showed that tall waves are narrow, or short, and small waves are wide, or long.

In 1955, E. Fermi, J. Pasta and S. Ulam (FPU) intended to calculate the flow of energy in one-dimensional lattice consisting of equal masses connected by nonlinear springs [36]. They conjectured that energy initially put into a long wavelength mode of the system would eventually be “thermalized”, that is, be shared among all modes of the
system. This conjecture was based on the expectation that the nonlinearities in the system would transfer energy into higher harmonic modes. Much to their surprise, the system did not ‘thermalized’ but rather exhibited energy sharing among the few lowest modes and long time near recurrences of the initial state. This remarkable near recurrence phenomena, known nowadays as the FPU problem was confirmed by some other scientists, that the nonlinear terms did not guarantee the approach of the system to thermal equilibrium.

This work inspired Norman Zabusky at Bell Telephone Laboratories and Martin Kruskal at Princeton University in 1965 to study the continuum limit of the FPU problem solving the corresponding nonlinear KdV equation numerically [97]. In doing so, Zabusky and Kruskal made a surprising discovery when they examined the solitary wave solution. When two solitary waves meet each other, they move through each other so that they emerge with their original shapes, sizes and speeds. It is almost as if the waves were governed by the principle of superposition associated with linear waves. Zabusky and Kruskal coined the word “soliton” for the solitary-wave solution of the KdV equation. Solitary waves keep their character after interacting with one another. In fact they seem to be behaving like interacting elementary particles such as ‘electrons or protons’.

The recent history of the mathematics of solitons begins in 1967 with the remarkable discovery by Gardner et al. [40] of an exact method for solving the initial problem of the Korteweg de Vries equation. In essence, they reduced the nonlinear problem to a linear one, which was well-known as the Sturm-Liouville eigenvalue problem for the Schrödinger operator. They also discussed the properties of the exact solution describing the interaction of solitons. This method, which now bears the name Inverse Scattering Transform, was later extended to a more general form to be applicable
to a wide class of nonlinear evolution and wave equations such as the modified KdV equation, the nonlinear Schrödinger equation, the sine-Gordon equation, and many more. In 1971, Hirota [48] developed an ingenious method for obtaining the exact multi-soliton solutions of the KdV equation and derived an explicit expression for its N-soliton solution. An elegant formulation of this method requires the use of bilinear operator, therefore it is called Hirota’s bilinear method. Over the last decade this method has been shown to be applicable to a large class of nonlinear evolution equations, including difference-differential and integro-differential equations.

1.2 EQUATIONS GIVING SOLITARY WAVE SOLUTION

The Korteweg de Vries (KdV) Equation (1.1.3) is the most important equation giving raise to solitary wave solution as we have described in the above section. This equation has a large number of conservation laws. Indeed the KdV equation (1.1.3) is a particular case of the generalized Kortweg de Vries (gKdV) equation of the form

\[ u_t + \varepsilon u u_x + \mu u_{xxx} = 0, \]  

(1.2.1)

where \( \varepsilon \) and \( \mu \) are constants and \( p \) is a positive integer. When \( p = 2 \), the gKdV equation (1.2.1) is known as the modified Korteweg de Vries (mKdV) equation and this equation has also a large number of conservation laws. But the gKdV equation (1.2.1) has only three conservation laws [7] for \( p \geq 3 \) and it is found that the equation is not integrable [21]. The gKdV equation has solitary wave solution and it will seen the existence of soliton-type solutions with certain radiation in the case \( p \geq 3 \). In this case the three conservation laws are

\[ I_1 = \int_a^b u \, dx, \quad I_2 = \int_a^b u^2 \, dx \quad \text{and} \quad I_3 = \int_a^b \left( u^3 - \frac{3\mu}{\varepsilon} u_x^2 \right) \, dx \]  

(1.2.2)

corresponding to mass, momentum and energy respectively.
The Regularized Long Wave (denoted by RLW) equation of the form

\[ u_t + uu_x - \varepsilon uu_x - \mu u_{xxt} = 0 \]  \hspace{1cm} (1.2.3)

where \( \varepsilon \) and \( \mu \) are positive constants, is a class of partial differential equation, which has soliton solution. This equation was originally introduced by Peregrine [75] for modeling the propagation of unidirectional weakly nonlinear and weakly dispersive water waves. Later on, Benjamin et al. [9] proposed the used of RLW equation as a preferred alternative to the more classical KdV equation, to model a larger class of physical phenomena. The RLW equation (1.2.2) is a special case of the generalized regularized long wave (GRLW) equation of the form

\[ u_t + uu_x + p(p + 1)u^p u_x - \mu u_{xxt} = 0 \]  \hspace{1cm} (1.2.4)

where \( p \) is a positive integer. When \( p = 2 \), this equation is known as the modified regularized long wave (MRLW) equation. The GRLW equation (1.2.3) has in general three conservation laws viz.

\[ I_1 = \int_a^b u \, dx, \quad I_2 = \int_a^b u^2 + \mu u_x^2 \, dx, \quad I_3 = \int_a^b u^4 - \mu u_x^2 \, dx \]  \hspace{1cm} (1.2.5)

corresponding to mass, momentum and energy respectively.

The Equal Width (EW) equation

\[ u_t + uu_x - \delta u_{xxt} = 0 \]  \hspace{1cm} (1.2.6)

which is less well known and proposed by Morrison et al. [71] is an alternative description to the more usual KdV and RLW equations, which also have soliton solutions. The generalized form of the EW equation is the generalized equal width (GEW) equation given by

\[ u_t + uu_x - \delta u_{xxt} = 0 \]  \hspace{1cm} (1.2.7)

where \( p \) is a positive integer. When \( p = 2 \), this equation is known as the modified equal width (MEW) equation. In general the GEW equation has three conservation laws viz.
\[
I_1 = \int_a^b u \, dx, \quad I_2 = \int \left( u^2 + \partial_x u^2 \right) \, dx, \quad I_3 = \int_a^b u^{p+2} \, dx
\] (1.2.8)
corresponding to mass, momentum and energy respectively.

Recently Rosenau and Hyman introduced a class of solitary waves with compact support which are known as compactons that are solutions of two parameter family of fully nonlinear dispersive partial differential equations (PDEs). The general form of the Rosenau-Hyman PDE giving raise to compacton solution is

\[
u_t + \left( u^n \right)_x + \left( u^n \right)_{xxx} = 0
\] (1.2.9)

This equation is generally denoted by \( K(m,n) \). The Rosenau-Hyman \( K(2,2) \) equation is a particular case of the Rosenau-Hyman \( K(m,n) \) with \( m = n = 2 \) and this equation has four conservation laws viz.

\[
\int_{-\infty}^{\infty} u \, dx, \quad \int_{-\infty}^{\infty} u^3 \, dx, \quad \int_{-\infty}^{\infty} u \cos x \, dx, \quad \text{and} \quad \int_{-\infty}^{\infty} u \sin x \, dx
\] (1.2.10)

Compactons are solitary waves with the remarkable soliton property is that after colliding with other compactons, they re-emerge with the same coherent shape. These particle-like waves exhibit elastic collisions that are similar to the soliton interactions associated with completely integrable PDEs supporting an infinite number of conservation laws. However, unlike the soliton collisions in an integrable system, the point where two compactons collide is marked by the creation of low-amplitude compacton-anticompacton pairs.

The equations which have solitary wave solutions are mostly non-linear partial differential equations such as the KdV equation, the RLW equation, the EW equation, the Rosenau-Hyman \( K(2,2) \) equation etc. In this work, we will study the general forms of the equations viz. gKdV, Rosenau-Hyman \( K(2,2) \), GEW, GRLW etc., giving solitary
wave solutions. In fact, finding the analytical solutions for the nonlinear equations generally is difficult and probably impossible in some cases; as in the case of interaction of two and three solitary waves or in case of development of the Maxwellian initial condition into solitary waves. So, finding the accurate approximate solutions for these equations are the main aims of many researchers in order to study solitary waves on a wide range and to investigate their properties.

1.3 FINITE ELEMENT METHOD

1.3.1 INTRODUCTION

Finite element method (FEM) is an efficient and powerful tool to solve mathematical problems numerically. While, there are various numerical methods, FEM attains the largest popularity in many fields of engineering as well. It is used for approximating the solution of partial differential equation arising in all areas of applied mathematics. One advantage of the FEM over finite difference methods is the relative ease with which boundary conditions of the problem are handled. Many physical problems have boundary conditions involving derivatives and irregularly shaped boundaries. Boundary conditions of this type are difficult to handle using finite difference techniques since each boundary condition involving derivative must be approximated by difference quotient at the grid points and irregular shaping of the boundary makes placing the grid point difficult. The finite element method includes the boundary conditions as integrals in a functional that is being minimized. So the construction procedure is independent of the particular boundary conditions of the problems. There are numerous textbooks on finite elements, emphasizing different aspects of the method. Some texts are written in an engineering style with special focus on structural analysis; where the method can be derived directly from physical considerations. Other texts are written in an abstract mathematical framework and emphasize the method as an optimal approach for
solving certain class of PDEs. The detailed information for finite element method and advances of this technique can be found in [1, 77, 98].

### 1.3.2 BASIC PRINCIPLES OF FEM

While the main idea of the finite difference method is to replace derivatives in a partial differential equation by difference approximations, the main idea of the finite element and related methods is to seek an approximation

\[ u_h = \sum_{j=1}^{M} U_j N_j(x) \]  

(1.3.1)

to the unknown function \( u(x) \). The sum in \( u_h \) involves prescribed functions \( N_j(x) \) and unknown coefficients \( U_j \). The functions \( N_j(x) \) are often referred to as basis functions or trial functions. In the finite element community, the word shape function is frequently used.

The ultimate aim is to construct a method for computing \( U_j \) such that the error \( u - u_h \) is minimized. For some special problems it is possible to minimize a problem-dependent norm of the error, \( \|u - u_h\| \), without knowing the exact solution \( u(x) \), but in general we must rely on seemingly less attractive strategies. Although the true error \( u - u_h \) is unknown, the error of the PDE, arising from inserting \( u_h \) instead of \( u(x) \), is easy to measure and work with.

Let \( L(u(x)) = 0, \ x \in \Omega \), denote the PDE, where \( L \) is differential operator. If we insert the approximation \( u_h \) in the PDE, we generally have that \( L(u_h) \neq 0 \). The error in the equation, \( R = L(u_h) \), is termed the residual.
Let us now formulate some procedures for determining \( U_j \). The \( M \) equations we need to determine the \( M \) parameters \( U_j \), \( j = 1, 2, 3, \ldots M \) can be obtained by forcing the residual \( R(U_1, \ldots, U_M; x) = L(u_h) \) to be small in different senses.

**The Least-Squares Method:** In this method we minimize the average square of the residual \( \int_{\Omega} R^2 \, d\Omega \) with respect to \( U_j \), \( j = 1, 2, 3, \ldots M \). This results in \( M \) algebraic equations,

\[
\frac{\partial}{\partial U_i} \int_{\Omega} R^2 \, d\Omega = \int_{\Omega} 2R \frac{\partial R}{\partial U_i} \, d\Omega = 0, \quad i = 1, \ldots, M .
\] (1.3.2)

**The Weighted Residual Method:** The idea in this approach is to find \( U_j \), \( j = 1, 2, 3, \ldots M \) such that the weighted mean of \( R \) over \( \Omega \) vanishes for \( M \) linearly independent weighting functions \( W_i \):

\[
\int_{\Omega} R W_i \, d\Omega = 0, \quad i = 1, \ldots, M
\] (1.3.3)

**The Collocation Method:** Let \( W_i = \delta(x - x^{[i]}) \), where \( \delta(x - x^{[i]}) \) is the Dirac delta function that vanishes when \( x \neq x^{[i]} \) and has the property

\[
\int_{\Omega} f(x) \delta(x - x^{[i]}) \, d\Omega = f(x^{[i]})
\]

for an arbitrary function \( f \). Application of these weighting functions results in discrete equations of the form

\[
R(U_1, \ldots, U_M; x^{[i]}) = 0, \quad i = 1, \ldots, M .
\] (1.3.4)

That is, the PDE is required to be satisfied at \( M \) collocation points \( x^{[i]} \). Observe that the collocation method is closely related to the finite difference method.

**The Galerkin’s & Petrov-Galerkin Method:** The most common choice of \( W_i \) in the method of weighted residuals is to let \( W_i = N_i \), that is, the weighted functions and the
trial functions are same. This choice is referred to as Galerkin’s method. The weighted residual method with $W_i \neq N_i$, that is the weighted function and the trial function are different is known as Petrov-Galerkin method.

In this thesis, we will develop the Petrov-Galerkin method for solving the gKdV, KdVB, GEW and GRLW equations using linear hat function as the trial function and different B-spline functions as test functions. Product approximation technique [16] has been used. The term “product approximation” is used to refer to a finite element technique for nonlinear problems. The advantage of product approximation technique is that - in the determination of product approximation solution we can first compute the inner products involved and then solve the resulting system of nonlinear equations. On the other hand to obtain the usual Galerkin approximation solution, the computation of the inner products and the solution of the equations are not separate process and each step of iteration, numerical quadrature must be used in order to evaluate the contribution of the non-linear term. Thus, in general, the product approximation requires much less computational effort, without loosing accuracy.

In this thesis, we use the Petrov-Galerkin method with the used of linear hat function as the trial function and different B-spline functions (quadratic, cubic and quintic) as the test function has been implemented to solve the GEW, gKdV, KdVB and GRLW equations. Product approximation technique has been used. It is found that our methods developed are more accurate and efficient than the previous methods available in literature so far. We use quintic B-splines collocation method to solve the fKdV equation. Lastly, we use quintic B-splines Galerkin method to solve the Rosenau-Hyman $K(2,2)$. 
1.4 VON NEUMANN STABILITY ANALYSIS

In numerical analysis, von Neumann stability analysis also known as Fourier stability analysis is a procedure used to check the stability of finite difference schemes as applied to linear partial difference equations. The analysis is based on the Fourier decomposition of numerical error and was developed at Los Almos National Laboratory after having been briefly described in a 1947 article by British researchers Crank and Nicolson [18]. Later, it was also published in an article [15] co-authored by von Neumann.

The stability of numerical scheme is closely associated with numerical error. A finite difference scheme is stable if the errors made at one time step of the calculation do not cause the errors to increase as the computations are continued. A naturally stable scheme is one in which errors remain constant as the computations are carried forward. If the errors decay and eventually damp out, the numerical scheme is said to be stable. If, on the contrary, the errors grow with time the numerical solution diverges from the true, correct answer and thus the numerical scheme is said to be unstable. The stability of numerical schemes can be investigated by performing von Neumann stability analysis. For time dependent problems, stability guarantees that the numerical method produces a bounded solution whenever the solution of the exact differential equation is bounded. Stability, in general, can be difficult to investigate, especially when equation under consideration is nonlinear. In such cases, the equation is first linearized and then stability condition is investigated. Let us define the round-off error \( \varepsilon_j^n \) as \( \varepsilon_j^n = N_j^n - u_j^n \), where \( u_j^n \) is the solution of the discretized equation that would be computed in the absence of round-off error, and \( N_j^n \) is the numerical solution obtained in finite precision arithmetic. Since the exact solution \( u_j^n \) must satisfy the discretized equation exactly, the error \( \varepsilon_j^n \)
must also satisfy the discretized equation. Thus, expressing the error as a finite Fourier series of the form \( \sum \varepsilon^n(t)e^{ikx_j} \), and again considering only the growth of a single form:

\[
\varepsilon^n_j = \xi^n e^{ikx_j}, \text{ where } i = \sqrt{-1}
\]

and let:

\[
\xi^{n+1} = g\xi^n
\]

where \( g \) is called the growth of an amplification factor. For stability von Neumann condition requires: \( |g| \leq 1 + O(\Delta t) \). i.e. the error will not increase as \( t \) increases. It should be noted that this method applies only to linear difference equations with periodic initial data. The criterion \( |g| \leq 1 \) is necessary and sufficient for three or more level equations although it is always necessary. In practice the method often gives useful results even when its application is not fully justified.

### 1.5 SELECTION OF SOFTWARE

In an attempt to present the numerical schemes developed in the most straightforward manner possible we employs the software package Mathematica. There are many other computational environments we could have used instead – for example, software packages such as Matlab or Maple have similar graphical and numerical capabilities to Mathematica. Once the principles of one such package are learned, it is relatively easy to master other packages. The mathematica software package is the most flexible and sophisticated package. Mathematica provides us with a computational environment where it is much easier to quickly learn the ideas behind the various numerical methods without the additional baggage of learning an operating system, mathematical and graphical libraries, or the complexities of the computer language itself.
An important feature of Mathematica is its ability to perform analytical calculations such as the analytical solution of linear and nonlinear equations, integrals and derivatives, and Fourier transforms. It is found that these features can help to free the user from the tedium of performing complicated algebra by hand, just as a calculator has freed the user from having to do long division.

In the finite element technique, the method of solving those partial differential equations (PDEs) giving solitary wave solutions is that – first the PDE is reduce to an ordinary differential equation (ODE) and then the ODE is further reduce to a system of linear or nonlinear equations. If the system is a linear one then it is solve by Mathematica function Solve[]. On the other hand if the system is nonlinear then it is solved by the Mathematica function FindRoot[]. This process is done for every time steps until the desire time level is obtained. If we used other softwares like C++, Pascal or Fortran, then we would faced a lot of difficulties. For example, if we are to solve a system of nonlinear equations by Newtons’ method, finding the Jacobian matrix of the system would be very difficult and it would be very complicated if the system is large. In our case we solve more that 1000 system of nonlinear equations by Mathematica. Moreover, plotting the graph of a function or the discrete data can be done very easily by Mathematica functions. The three dimensional plots can also be done by Mathematica functions.

In view of the above features of Mathematica and the problems encounter in our work, we have a strong force of selecting the Mathematica software.