THE ROSENAU-HYMAN K(2,2) EQUATION

5.1 INTRODUCTION

Most of the weakly non-linear and linear dispersive equations studied so far admit solitary waves, called solitons that are infinite in extent [21]. Well known partial differential equations (PDEs) with soliton solution include sine-Gordon (SG) equation, cubic non-linear Schordinger (NLS) equation and Korteweg-de Vries (KdV) equation. Rosenau and Hyman [80] reported a class of PDEs

\[ U_x + (U^m)_x + (U^n)_{xxx} = 0, \quad m > 1, \quad 1 \leq n \leq 3 \]  

(5.1.1)

which is a generalization of the KdV equation. These equations with the values of \( m \) and \( n \) are denoted by \( K(m,n) \). It has been shown that interaction of non-linear dispersion with non-linear convection generates exactly compact structures, called compactons, free of exponential tails. The compacton solutions so generated have immediate applications in the study of pattern formations as the observed stationary and dynamical patterns in nature are usually finite in extent. The compacton speed depends on its height, but unlike soliton, its width is independent of its speed. Beside the compacton structure and the unusual speed-width relation, the compactons have the remarkable soliton like property that they collide elastically. The PDEs (5.1.1) has in general two conserved quantities given by

\[ P = \int u(x,t)dx; \quad Q = \int u^{n+1}(x,t)dx. \]  

(5.1.2)

Various forms of B-spline basis together with finite element methods have been extensively used in solving some partial differential equations. For instance, quintic B-spline collocation finite element method for numerical solution of the KdV, Burgers’ and KdVB equations [22] has been successfully implemented. Idris Dag and
his co-authors [25, 26, 27] have also solved Burger’s and RLW equations using quintic B-spline basis with finite element methods. The results of the calculations showed that accuracy of solution is improved if the Galerkin formulation together with quintic B-spline functions is used in getting the numerical solution of the partial differential equations but computational cost of the B-spline Galerkin algorithm increases.

The Rosenau Hyman, $K(m,n)$ equation have been solved numerically by various methods. Frutos et al. [38] used finite difference method to solved the $K(2,2)$ equation. Ismail and Taha [56] have studied $K(2,2)$ numerically by various finite difference and finite element techniques. They also proved the conditional and unconditional stability of the proposed schemes. A finite difference method has been constructed by Ismail [54] for the solution of $K(2,3)$ equation. Ismail and Al-Solamy [55] have used finite element method with the used of linear hat function and cubic B-spline function as trial and test functions for the numerical solution of the $K(3,2)$ equation. Solitary-wave solution of $K(m,n)$ having compact support has been studied by Wazwaz [96] using Adomain decomposition method.

In this paper we will describe numerical solution of one of the partial differential equations (5.1.1) viz. $K(2,2)$ by Galerkin’s method using quintic B-spline functions. The conserved quantities are calculated using Simpson’s rule. Finally comparison between analytical and numerical solution of the $K(2,2)$ equation for the purposed algorithm is made.

5.2 QUINTIC B- SPLINE GALERKIN METHOD

The Rosenau-Hyman $K(2,2)$ equation has the form

$$U_x + (U^2)_x + (U^2)_{xxx} = 0,$$

(5.2.1)
where subscribes $x$ and $t$ denote differentiation. Boundary conditions are selected from the homogeneous boundary conditions

$$
U(a, t) = 0, \quad U(b, t) = 0,
$$
$$
U_x(a, t) = 0, \quad U_x(b, t) = 0, \quad t \in [0, T],
$$
$$
U_{xx}(a, t) = 0, \quad U_{xx}(b, t) = 0
$$

and initial condition

$$
U(x, 0) = f(x), \quad x \in [a, b].
$$

Applying the Galerkin technique to Eq. (5.2.1) with weight function $W$ yields the integral equation

$$
\int_a^b W [U_t + (U^2)_x + (U^2)_{xx}] \, dx = 0.
$$

We consider the mesh $a = x_0 < x_1 < \cdots < x_N = b$ as a uniform partition of the solution domain $a \leq x \leq b$ by the knots $x_m$ and $h = x_m - x_{m-1}$, $m = 1, \ldots, N$, throughout the paper.

Let $Q_m(x), m = -2, \ldots, N + 2$,

$$
Q_m(x) = \frac{1}{h^3} \begin{bmatrix}
(x - x_{m-3})^5 & [x_{m-3}, x_{m-2}] \\
(x - x_{m-3})^5 - 6(x - x_{m-2})^5 & [x_{m-2}, x_{m-1}] \\
(x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & [x_{m-1}, x_m] \\
(x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & [x_{m}, x_{m+1}] \\
-20(x - x_m)^5 & [x_{m+1}, x_{m+2}] \\
-20(x - x_m)^5 + 15(x - x_{m+1})^5 & [x_{m+1}, x_{m+2}] \\
-20(x - x_m)^5 + 15(x - x_{m+1})^5 & [x_{m+2}, x_{m+3}] \\
-20(x - x_m)^5 + 15(x - x_{m+1})^5 - 6(x - x_{m+2})^5 & [x_{m+2}, x_{m+3}] \\
0 & otherwise
\end{bmatrix}
$$

(5.2.4)
be quintic B-splines with the knots \( x_m, m = -5, ..., N + 5 \). The set of quintic B-splines \( Q_m(x) \) forms a basis over the region \( a \leq x \leq b \). The global approximation defined using the quintic B-splines

\[
U_N(x, t) = \sum_{m=-2}^{N+2} \delta_m(t) Q_m(x),
\]

(5.2.5)

will be sought to the analytical solution \( U \). In this approximate solution \( \delta_m \) is a time dependent parameter to be determined from the quintic Galerkin form of the Eq. (5.2.3). The nodal values of \( U \) and its derivatives up to fourth order are given in terms of the parameter \( \delta_m \) from the use of splines (5.2.4) and the trial solution (5.2.5)

\[
U_m = U(x_m) = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2},
\]

\[
U'_m = U'(x_m) = \frac{5}{h} (\delta_{m+2} + 10\delta_{m+1} - 10\delta_{m-1} - \delta_{m-2}),
\]

\[
U''_m = U''(x_m) = \frac{20}{h^2} (\delta_{m+2} + 2\delta_{m+1} - 6\delta_m + 2\delta_{m-1} + \delta_{m-2}),
\]

(5.2.6)

\[
U'''_m = U'''(x_m) = \frac{60}{h^3} (\delta_{m+2} - 2\delta_{m+1} + 2\delta_{m-1} - \delta_{m-2}),
\]

\[
U''''_m = U''''(x_m) = \frac{120}{h^4} (\delta_{m+2} - 4\delta_{m+1} + 6\delta_m - 4\delta_{m-1} + \delta_{m-2}).
\]

A local coordinate system can be defined using the mapping relation \( \xi = x - x_m \) to transform the finite element \([x_m, x_{m+1}]\) into the interval \([0, h]\). The expressions of quintic B-spline shape functions that are independent of element position are obtained with the relation of the global and local coordinates relation over \([0, h]\) as
\[ Q_{m-2} = 1 - 5 \left( \frac{\xi}{h} \right) + 10 \left( \frac{\xi}{h} \right)^2 - 10 \left( \frac{\xi}{h} \right)^3 + 5 \left( \frac{\xi}{h} \right)^4 - \left( \frac{\xi}{h} \right)^5, \]
\[ Q_{m-1} = 26 - 50 \left( \frac{\xi}{h} \right) + 20 \left( \frac{\xi}{h} \right)^2 + 20 \left( \frac{\xi}{h} \right)^3 - 20 \left( \frac{\xi}{h} \right)^4 + 10 \left( \frac{\xi}{h} \right)^5, \]
\[ Q_m = 66 - 60 \left( \frac{\xi}{h} \right)^2 + 30 \left( \frac{\xi}{h} \right)^3 - 10 \left( \frac{\xi}{h} \right)^4, \]
\[ Q_{m+1} = 26 + 50 \left( \frac{\xi}{h} \right) - 20 \left( \frac{\xi}{h} \right)^2 - 20 \left( \frac{\xi}{h} \right)^3 + 10 \left( \frac{\xi}{h} \right)^4, \]
\[ Q_{m+2} = 1 + 5 \left( \frac{\xi}{h} \right) + 10 \left( \frac{\xi}{h} \right)^2 + 5 \left( \frac{\xi}{h} \right)^3 - 5 \left( \frac{\xi}{h} \right)^4, \]
\[ Q_{m+3} = \left( \frac{\xi}{h} \right)^5. \]

(5.2.7)

From the quintic B-splines covering six successive finite elements, typical finite elements are covered by six quintic B-spline shape functions. So the approximation reduced over the element \([x_m, x_{m+1}]\) is

\[ U_N^e = U(\xi, t) = \sum_{i=m-2}^{m+3} \delta_i(t)Q_i(\xi) \]  

(5.2.8)

where \( \delta_i, i = m-2,...,m+3, \) act as element parameters. Taking the weight functions \( W \) with quintic B-spline shape functions and substituting element trial function \( U_N^e \) in the integral equation (5.2.3) over the element \([0,h]\) leads to

\[
\sum_{j=m-2}^{m+3} \left[ \int_0^h Q_j Q_j' d\xi \right] \delta_j = \sum_{k=m-2}^{m+3} \left[ \int_0^h Q_k (Q_k')' d\xi \right] \delta_k \]

(5.2.9)

\[
+ \sum_{k=m-2}^{m+3} \left[ \int_0^h Q_k (Q_k')'' d\xi \right] \delta_k \]

where \( j \) and \( k \) take only the values \( m-2, m-1, m, m+1, m+2, m+3 \) and \( m=0,1,2, ..., N-1, N, \) and \( \cdot \) denotes derivative with respect to time, which in the matrix form is

\[
A^e \delta^e + (\delta^e)^T L^e \delta^e + (\dot{\delta}^e)^T M^e \dot{\delta}^e, \]  

(5.2.10)
where the element matrices \( L^e, M^e \) are \( 6 \times 6 \), the matrix \( A^e \) is \( 6 \times 6 \), 
\[
\delta^e = (\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}, \delta_{m+3})
\]
and
\[
A^e_{i,j} = \int_0^b Q_i Q_j d\xi, \quad L^e_{i,j} = \int_0^b (Q_i Q_j)' d\xi, \quad M^e_{i,j} = \int_0^b (Q_i Q_j)^{\prime\prime} d\xi
\]
(5.2.11)
The matrices \( L^e \) and \( M^e \) are organized to be in the dimension \( 6 \times 6 \) as matrices \( B^e \) and \( E^e \)
\[
B^e_{i,j} = \sum_{k=m-2}^{m+3} L_{i,j} \delta_k, \quad E^e_{i,j} = \sum_{k=m-2}^{m+3} M_{i,j} \delta_k,
\]
(5.2.12)
so the matrices \( B^e \) and \( E^e \) are expressed as depending on the element parameter \( \delta^e \).
Combining all element matrices for each element we obtain a system of nonlinear ordinary differential equation
\[
\dot{A} \delta + (B + E) \delta = 0
\]
(5.2.13)
where the global parameter is
\[
\delta = (\delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_{N+1}, \delta_{N+2})
\]
(5.2.14)
If we used the Crank-Nicolson discretization formula for the vector of element parameter \( \delta \) and the usual finite difference equation for the time derivatives parameter \( \dot{\delta} \) in the equation:
\[
\delta = \frac{\delta^n + \delta^{n+1}}{2}; \quad \dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t},
\]
(5.2.15)
we reach a nonlinear recurrence relation for the time parameters \( \delta \):
\[
(2A + \Delta t B + \Delta t E) \delta^{n+1} = (2A - \Delta t B - \Delta t E) \delta^n.
\]
(5.2.16)
This system is made up of \( (N+5) \) equation in \( (N+5) \) unknown parameters. We can obtain a solvable system by imposing the boundary conditions at the left end of the region \( U(a,t) = U_{xx}(a,t) = 0 \) and at the right end of the region \( Ux(b,t) = U_{xx}(b,t) = 0 \) to eliminate the parameters \( \delta_{-2}^n, \delta_{-1}^n, \delta_{N+1}^n, \delta_{N+2}^n \). The above said non-linear system is
solved by Newton’s method. Iteration should be repeated two or three times using the following corrector procedure

\[(\delta^*)^{n+1} = \delta^n + \frac{1}{2}(\delta^{n+1} - \delta^n).\]  

(5.2.17)

To start the iteration of the recurrence relation of the system (5.2.16), the initial parameter vector \(\delta^0\) must be determined using the following initial and boundary conditions at the knots.

\[\begin{align*}
(U_N)_x(a,0) &= 0, (U_N)_x(b,0) = 0, \\
(U_N)_x(a,0) &= 0, (U_N)_{xx}(b,0) = 0, \\
U_N(x,0) &= U(x_m,0), m = 0, ..., N.
\end{align*}\]  

(5.2.18)

Conditions with their corresponding quintic B-spline representation give matrix equation of the system

\[
\begin{bmatrix}
-1 & -10 & 0 & 10 & 1 \\
1 & 2 & -6 & 2 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 26 & 66 & 26 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 26 & 66 & 26 & 1 \\
1 & 2 & -6 & 2 & 1 \\
-1 & -10 & 0 & 10 & 1
\end{bmatrix}
\begin{bmatrix}
\delta_2 \\
\delta_{-1} \\
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_{N-1} \\
\delta_N \\
\delta_{N+1} \\
\delta_{N+2}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
U(x_0) \\
U(x_1) \\
\vdots \\
U(x_{N-1}) \\
U(x_N) \\
0
\end{bmatrix}
\]

This matrix system is solved to get the initial condition parameters. On determining the initial parameters from the system above, calculation of the solutions are iterated using system (5.2.16) at successive times. By using the obtained parameters from system (5.2.16), nodal values and its derivatives of order 4 can be worked out from Eqs. (5.2.6).
5.3 NUMERICAL APPLICATIONS

In this section, we present some numerical experiments to find the numerical solution of single compacton waves and determining the solution of two and three compactons interactions. We also show the compactons splitting from an initial data.

5.3.1 MOTION OF A SINGLE COMPACTON

The $K(2,2)$ equation possesses the following conserved quantities:

$$I_1 = \int_{-\infty}^{\infty} U(x,t) \, dx \quad \text{and} \quad I_2 = \int_{-\infty}^{\infty} U^3(x,t) \, dx$$

(5.3.1)

Since the conserved quantities are expected to remain constant during the run of the algorithm to have the efficient numerical scheme. Simpson’s rule will be used to calculate the integrals (5.3.1) at the discrete points numerically. To measure the accuracy of numerical solutions, difference between analytical and numerical solutions at some specified times is computed by using the discrete root mean square error norm

$$L_2 = \left\| U - U_N \right\|_2 = \left[ \sum_{i=1}^{N} \left| U_i - (U_N)_i \right|^2 \right]^{\frac{1}{2}}$$

and maximum error norm $L_\infty = \max_i \left| U_i - (U_N)_i \right|$, $i = 1, 2, ..., N - 1$.

We adopt the single compacton wave solution of the $K(2,2)$ equation with the initial condition at $t = 0$,

$$u(x,0) = \begin{cases} \frac{4\nu}{3} \cos^2 \left( \frac{x - x_0}{4} \right), & |x - x_0| \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

(5.3.2)

The exact solution of the $K(2,2)$ equation (5.1.1) is given by

$$u(x,t) = \begin{cases} \frac{4\nu}{3} \cos^2 \left( \frac{x - \nu t}{4} \right), & |x - \nu t| \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

(5.3.3)

which describes a compactly supported solitary wave called compactons, traveling
with velocity $v$ and amplitude $\frac{4v}{3}$ in the positive x-direction. The parameters $v = 1$, $\Delta t = 0.05$, $h = 0.2$ and $x_0 = 10$ are used. The run of the algorithm is carried up to time $t = 20$ over the problem domain $0 \leq x \leq 40$. The maximum, root mean square errors and conserved quantities are presented in table 1. Single compacton wave solution is drawn in Fig 5.1.

![Figures](a) (b) (c) (d)

**Fig. 5.1:** Single compacton profiles. Solution of $K(2,2)$ equation with initial condition (3.2) in $[0,40]$. (a) Solution at $t = 0$; (b) Solution at $t = 10$; (c) Solution at $t = 20$; (d) Space time graph of the solution up to $t = 25$.

**Table 5.1:** Invariants and error norms for single compacton wave
5.3.2 INTERACTION OF TWO COMPACTONS

In this example, we study the interaction of two compactons of the $K(2,2)$ equation (5.1.1) having different amplitude and traveling in the same direction. We consider the $K(2,2)$ equation with initial condition given by

$$u(x,0) = \sum_{i=1}^{2} u_i(x,0), \quad 0 \leq x \leq 60 \quad (5.3.4)$$

where

$$u_i(x,0) = \begin{cases} \frac{4\nu_i}{3} \cos^2\left(\frac{x - x_i}{4}\right), & |x - x_i| \leq 2\pi \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, 2.$$  

Here we choose the following parameters $h = 0.5$, $\Delta t = 0.08$, $x_1 = 10$, $x_2 = 25$, $\nu_1 = 1$, $\nu_2 = 0.5$. The interaction scenario is displayed in Fig. 5.2 and shows how the two well separated compactons interact and emerge after interaction is unchanged in shaped and velocity.
Fig. 5.2: Double compacton profiles. Solution of $K(2,2)$ equation with initial condition (3.4) in $[0,60]$. (a) Solution at $t = 0$ (b) Solution at $t = 16$ (c) Solution at $t = 32$ (d) Space time graph of the solution up to $t = 32$. 
Fig. 5.3: Triple compacton profiles. Solution of $K(2,2)$ equation with initial condition (3.2) in $[0,80]$. (a) Solution at $t=0$; (b) Solution at $t=15$; (c) Solution at $t=30$; (d) Space time graph of the solution up to $t=35$.

5.3.3 INTERACTION OF THREE COMPACTONS

The triple compacton interaction of the $K(2,2)$ equation (5.1.1) has the initial condition given by
\[
 u(x,0) = \sum_{i=1}^{3} u_i(x,0), \quad 0 \leq x \leq 80 \tag{5.3.5}
\]

where
\[
 u_i(x,0) = \begin{cases} 
 \frac{4v_i}{3} \cos^2 \left( \frac{x-x_i}{4} \right), & |x-x_i| \leq 2\pi \\
 0, & \text{otherwise} 
\end{cases} \quad \text{and} \quad i = 1, 2, 3.
\]

**Fig. 5.4:** Triple compacton splitting profiles. Solution of \( K(2,2) \) equation with initial condition (3.6) in \([0,60] \). (a) Solution at \( t = 0 \) (b) Solution at \( t = 5 \) (c) Solution at \( t = 25 \) (d) Space time graph of the solution up to \( t = 24 \).

Here we choose the following parameters \( h = 0.4, \Delta t = 0.05, x_1 = 8, x_2 = 22, x_3 = 36, v_1 = 1.2, v_2 = 0.8 \) and \( v_3 = 0.4 \) The interaction of the three compactons is shown in Fig. 5.3.
5.3.4 SPLITTING COMPACTONS

The triple compacton splitting case of the $K(2,2)$ equation has the following initial condition given by [66]

$$u(x,0) = \begin{cases} 
\frac{4}{3} \cos^2\left(\frac{x}{8}\right), & |x| \leq 4\pi \\
0, & \text{otherwise}
\end{cases} \quad (5.3.6)$$

The compacton splitting from a more general initial data are shown in Fig. 5.4. Here we choose the following parameters $h = 0.4$ and $\Delta t = 0.08$.

5.4 CONCLUSION

We have successfully applied the Galerkin’s finite element method using quintic B-spline function for solving the highly non-linear dispersive partial differential equation, $K(2,2)$. Numerical examples are shown to illustrate the accuracy of the method. We also showed the interaction of two and three compactons and the splitting of compacton. Numerical accuracy of the method can be increased by reducing spatial and temporal step sizes $h$ and $\Delta t$ respectively. The proposed method has a major drawback of computational complexity. It takes more computational time then the other methods cited in this chapter. But in terms of accuracy this method is acceptable.