Chapter 2

The Brahmagupta Polynomials Extended By Generalized Golden Ratio

2.1 Introduction

In the previous chapter we have noted well that Brahmagupta Polynomials are a pair of homogeneous polynomials \( \{x_n(x, y; t), y_n(x, y; t)\} \) of degree \( n \) in two real variables \( x, y \) and a real parameter \( t \neq 0 \) such that \( x^2 - ty^2 \neq 0 \) computed using one of the following (1)-(5) [78] :

1. The Brahmagupta matrix identity :

\[
\begin{bmatrix}
x_n & y_n \\
t y_n & x_n
\end{bmatrix} = \begin{bmatrix}
x & y \\
t y & x
\end{bmatrix}^n
\]

with \( x_0 = 1, y_0 = 0, n = 0, 1, 2, \ldots \)  

2. The Binary linear recurrences :

\[
\begin{align*}
x_{n+1} &= x x_n + t y y_n \\
y_{n+1} &= x y_n + y x_n
\end{align*}
\]
with \( x_0 = 1, \ y_0 = 0, \ n = 0, 1, 2, \ldots \)

3. The Brahmagupta-Pell Identity:

\[
x_n \pm y_n \sqrt{t} = (x \pm y\sqrt{t})^n
\]  \hspace{1cm} (2.1.3)

4. The Binet Forms:

\[
x_n = \frac{1}{2} \left[ (x + y\sqrt{t})^n + (x - y\sqrt{t})^n \right]
\]
\[
y_n = \frac{1}{2\sqrt{t}} \left[ (x + y\sqrt{t})^n - (x - y\sqrt{t})^n \right]
\] \hspace{1cm} (2.1.4)

5. Three-term recurrences:

\[
x_{n+1} = 2x x_n - (x^2 - ty^2) x_{n-1}, \ x_0 = 1, \ x_1 = x
\]
\[
y_{n+1} = 2x y_n - (x^2 - ty^2) y_{n-1}, \ y_0 = 0, \ y_1 = y
\] \hspace{1cm} (2.1.5)

In the present chapter, the Brahmagupta polynomials are extended with the help of generalized golden ratio. The corresponding matrix identity is stated and proved. All the properties that hold for the Brahmagupta polynomials with one parameter \( t \), are nontrivially extended to the present case.

2.2 The Extended Brahmagupta Matrix Identity And Extended Class Of Brahmagupta Polynomials

DEFINITION: The Brahmagupta polynomials extended by the generalized golden ratio are given by the following identity:

\[
\left[ x_n + \left( \frac{s}{2} \pm \sqrt{\frac{s^2}{4} + t} \right) y_n \right] = \left[ x + \left( \frac{s}{2} \pm \sqrt{\frac{s^2}{4} + t} \right) y \right]^n
\] \hspace{1cm} (2.2.1)
For \( s = 0 \), the extended class of polynomials become the usual Brahmagupta polynomials. The number \( \left( \frac{s}{2} + \sqrt{\frac{s^2}{4} + t} \right) \) is called generalized golden ratio for the reason, if \( s = t = 1 \), then it is nothing but the golden ratio \( \frac{1 + \sqrt{5}}{2} \).

Let \( B \) denote the set of matrices of the form
\[
B = \begin{bmatrix} x & y \\ ty & x + sy \end{bmatrix}
\] (2.2.2)

where \( t \) and \( s \) are fixed real numbers and \( x \) and \( y \) are real variables. Define \( B \) to be the extended Brahmagupta matrix. It is easy to check that \( B \) is a commutative algebra over the reals. Let \( \beta = \det B = x^2 + sxy - ty^2 \neq 0 \). Now, let us consider the identity defining the extended Brahmagupta polynomials and note that it may be rewritten in the following form:
\[
\left( x + \frac{s}{2}y \right) \pm y \sqrt{\frac{s^2}{4} + t} = \left( x + s\frac{y}{2} \right) \pm y \sqrt{\frac{s^2}{4} + t}
\] (2.2.3)

This is of the form (2.1.3) and hence it is equivalent to the following recurrence relations:
\[
\begin{align*}
(x_{n+1} + \frac{s}{2}y_{n+1}) &= \left( x + \frac{s}{2}y \right) (x_n + \frac{s}{2}y_n) + \left( \frac{s^2}{4} + t \right) yy_n, \\
y_{n+1} &= yx_n + (x + sy)y_n
\end{align*}
\] (2.2.4, 2.2.5)

on simplification, one obtains the following equivalent form:
\[
\begin{align*}
x_{n+1} &= xx_n + tyy_n, \\
y_{n+1} &= yx_n + (x + sy)y_n
\end{align*}
\] (2.2.6)

And this is equivalent to the desired matrix identity stated in the following theorem. Actually, the above steps gives one proof of the following theorem.
But a direct proof is also possible by the principle of mathematical induction. Hence we have the following theorem:

**Theorem 2.1**

\[
\begin{bmatrix}
  x & y \\
  t & x + sy
\end{bmatrix}^n =
\begin{bmatrix}
  x_n & y_n \\
  t & x_n + sy_n
\end{bmatrix}
\tag{2.2.7}
\]

Since $B$ is a commutative algebra, the above theorem makes sense.

It is very interesting to note that, if $s = t = y = 1$ and $x = 0$, then

\[
B^n =
\begin{bmatrix}
  0 & 1 \\
  1 & 1
\end{bmatrix}^n =
\begin{bmatrix}
  F_{n-1} & F_n \\
  F_n & F_{n+1}
\end{bmatrix}
\tag{2.2.8}
\]

where $F_n$ is the $n$th Fibonacci number (Please refer the subsection 1.5.1)

\[
F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)
\]

Using the definition, one obtains directly the following Binet forms for $(x_n + \frac{s}{2}y_n)$ and $y_n$:

\[
(x_n + \frac{s}{2}y_n) = \frac{1}{2} [(x + \lambda_+ y)^n + (x + \lambda_- y)^n]
\tag{2.2.9}
\]

\[
y_n = \frac{1}{2\sqrt{\frac{s^2}{4} + t}} [(x + \lambda_+ y)^n - (x + \lambda_- y)^n]
\tag{2.2.10}
\]

where $\lambda_\pm = \frac{s}{2} \pm \sqrt{\frac{s^2}{4} + t}$

They are indeed solutions of the following difference equations:

\[
(x_{n+1} + \frac{s}{2}y_{n+1}) = 2 \left( x + \frac{s}{2} y \right) \left( x_n + \frac{s}{2} y_n \right)
- \left( x^2 + sx - ty^2 \right) \left( x_{n-1} + \frac{s}{2} y_{n-1} \right)
\tag{2.2.11}
\]
\[ x_0 + \frac{s}{2} y_0 = 1, \quad x_1 + \frac{s}{2} y_1 = x + \frac{s}{2} y. \]

\[ y_{n+1} = 2\left(x + \frac{s}{2} y\right)y_n - \left(x^2 + sxy - ty^2\right)y_{n-1} \quad (2.2.12) \]

\[ y_0 = 0, \quad y_1 = y. \]

As a consequence, \( x_n \) and \( y_n \) are solutions of the following difference equations:

\[ x_{n+1} = (2x + sy)x_n - (x^2 + sxy - ty^2)x_{n-1} \quad (2.2.13) \]

\[ x_0 = 1, \quad x_1 = x. \]

\[ y_{n+1} = (2x + sy)y_n - (x^2 + sxy - ty^2)y_{n-1} \quad (2.2.14) \]

\[ y_0 = 0, \quad y_1 = y. \]

The first few polynomials are:

\[ x_0 = 1, \quad x_1 = x, \quad x_2 = x^2 + ty^2, \quad x_3 = x^3 + 3txy^2 + sty^3, \]

\[ x_4 = x^4 + 4stx^3y + 6tx^2y^2 + stxy^3 + (t + s^2)y^4, \ldots \]

\[ y_0 = 0, \quad y_1 = y, \quad y_2 = 2xy + sy^2, \quad y_3 = 3x^2y + 3sxy^2 + (t + s^2)y^3, \]

\[ y_4 = 4x^3y + 6sx^2y^2 + 4(t + s^2)xy^3 + s(2t + s^2)y^4, \ldots \]

If \( s, t > 0 \) then \((x + \lambda_+ y) > (x + \lambda_- y) > 0\), then we have

\[ \lim_{n \to \infty} \frac{x_n}{y_n} = \lambda_+, \quad \lim_{n \to \infty} \frac{x_n}{x_{n-1}} = x - \lambda_- y = \lim_{n \to \infty} \frac{y_n}{y_{n-1}} \quad (2.2.15) \]

A simple application of binomial expansions of Binet forms \((2.2.9)\) and \((2.2.10)\), yields the following:

**Theorem 2.2**

\[ x_n = x^n + 2t \sum_{k=2}^{n} \binom{n}{k} c_{k-1} x^{n-k} y^k \quad (2.2.16) \]
\[ y_n = 2 \sum_{k=1}^{n} \binom{n}{k} c_k x^{n-k} y^k \] (2.2.17)

where \( c_0 = 0 \), \( c_1 = \frac{1}{2} \), \( c_{n+1} = sc_n + tc_{n-1} \).

\( \frac{c_{k-1}}{c_k} \) represents \( k^{th} \) convergent of the continued fraction

\[
\frac{-s + \sqrt{s^2 + 4t}}{2} = \frac{1}{s + \frac{t}{s + \frac{t}{s + \ldots}}} .
\] (2.2.18)

Thus \( x_n \) and \( y_n \) are generating functions for \( c_k \).

For \( s = t = 1 \), one gets the following result:

**Theorem 2.3**

\[ x_n = x^n + \sum_{k=2}^{n} \binom{n}{k} F_{k-1} x^{n-k} y^k \] (2.2.19)
\[ y_n = \sum_{k=1}^{n} \binom{n}{k} F_k x^{n-k} y^k \] (2.2.20)

In (1.1.13) we have noted that Brahmagupta polynomials satisfy \( t \) – Cauchy’s – Reimann equations:

\[
\begin{aligned}
\frac{\partial x_n}{\partial x} &= \frac{\partial y_n}{\partial y} = n \, x_{n-1}, \\
\frac{\partial x_n}{\partial y} &= t \, \frac{\partial y_n}{\partial y} = n \, t \, y_{n-1}
\end{aligned}
\] (2.2.21)

As a consequence, \( x_n \) and \( y_n \) satisfy the wave equation (1.1.14): \[
\left( \frac{\partial^2}{\partial x^2} - \frac{1}{t} \frac{\partial^2}{\partial y^2} \right) U = 0
\] (2.2.22)

The corresponding extended result is the following theorem:
Theorem 2.4

The polynomials $x_n(x, y, s, t)$ and $y_n(x, y, s, t)$ satisfy the following second order linear partial differential equations:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{s}{t} \frac{\partial^2}{\partial x \partial y} - \frac{1}{t \frac{\partial^2}{\partial y^2}} \right) U = 0. \quad (2.2.23)$$

Proof: Partial differentiation of (2.2.9) and (2.2.10) yields,

$$\frac{\partial}{\partial x} \left( x_n + \frac{s}{2} y_n \right) = \left( -\frac{s}{2} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) y_n = n \left( x_{n-1} + \frac{s}{2} y_{n-1} \right) \quad (2.2.24)$$

$$\frac{\partial}{\partial y} \left( x_n + \frac{s}{2} y_n \right) = n \left[ \frac{s}{2} \left( x_{n-1} + \frac{s}{2} y_{n-1} \right) + \left( \frac{s^2}{4} + t \right) y_{n-1} \right] \quad (2.2.25)$$

$$\frac{\partial y_n}{\partial x} = ny_{n-1} \quad (2.2.26)$$

So we may simplify the above as follows-

$$\frac{\partial x_n}{\partial x} = - \left( \frac{s}{2} \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) y_n \quad (2.2.27)$$

$$\frac{\partial x_n}{\partial y} = - \frac{s}{2} \frac{\partial y_n}{\partial y} + \frac{s}{2} \left( -\frac{s}{2} \frac{\partial y_n}{\partial x} + \frac{\partial y_n}{\partial y} \right) + \left( \frac{s^2}{4} + t \right) \frac{\partial y_n}{\partial y} = t \frac{\partial y_n}{\partial x} \quad (2.2.28)$$

They naturally lead to

$$t \frac{\partial^2 y_n}{\partial x^2} + \frac{\partial}{\partial y} \left( \frac{s}{2} \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) y_n = 0 \quad (2.2.29)$$

which is same as

$$\left( \frac{\partial^2}{\partial x^2} + \frac{s}{t} \frac{\partial^2}{\partial x \partial y} - \frac{1}{t \frac{\partial^2}{\partial y^2}} \right) y_n = 0 \quad (2.2.29)$$

Also, the Partial differential equation for $x_n$ may be derived as follows-

$$\frac{\partial x_n}{\partial x} + \frac{s}{t} \frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial y} \quad (2.2.30)$$
As a direct consequence, \( x_n \) satisfies the following Partial differential equation:

\[
\frac{1}{t} \frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial x} \tag{2.2.31}
\]

As a direct consequence, \( x_n \) satisfies the following Partial differential equation-

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{s}{t} \frac{\partial^2}{\partial x \partial y} - \frac{1}{t} \frac{\partial^2}{\partial y^2} \right) x_n = 0 \tag{2.2.32}
\]

### 2.3 Recurrence Relations

From the Binet forms (2.2.9) and (2.2.10) we can derive the following recurrence relations:

\[
x_{m+n} = x_m x_n + t y_m y_n \tag{2.3.1}
\]

\[
y_{m+n} = x_m y_n + y_m x_n + s y_m y_n \tag{2.3.2}
\]

\[
\beta^n x_{m-n} = x_m x_n - t y_m y_n \tag{2.3.3}
\]

\[
\beta^n y_{m-n} = x_n y_m - x_m y_n \tag{2.3.4}
\]

where \( \beta = x^2 + sxy - ty^2 \neq 0 \).

Put \( m = n \) in (2.3.1) and (2.3.2) above. Then we see that

\( x_{2n} = x_n^2 + ty_n^2 \), \( y_{2n} = y_n(2x_n + sy_n) \) and the second relation implies that

\( y_{2n} \) is divisible by \( y_n \) and \( 2x_n + sy_n \); also if \( r \) divides \( s \), then \( x_{rn} \) and \( y_{rn} \) are divisors of \( y_{sn} \).

Let \( \sum_{k=0}^{n} = \sum_{k=0}^{n} \), then using the Binet forms (2.2.9) and (2.2.10) we can also derive the following relations:

\[
\sum x_k = \frac{\beta x_n - x_{n+1} - x - sy + 1}{\beta - 2x + sy + 1} \tag{2.3.5}
\]

\[
\sum y_k = \frac{\beta y_n - y_{n+1} + y}{\beta - 2x - sy + 1} \tag{2.3.6}
\]
The last three relations are the convolution formulas. Now we show an interesting result which generalizes a property that holds between the $F_n$ and $L_n$, namely $e^{L(x)} = F(x)$ [42], where

$$F(x) = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots, \tag{2.3.10}$$

and

$$L(x) = L_1x + \frac{L_2}{2}x^2 + \frac{L_3}{3}x^3 + \cdots + \frac{L_n}{n}x^n + \cdots. \tag{2.3.11}$$

**Theorem 2.5**

Let $X$ and $Y$ be generating functions of $x_n$ and $y_n$ respectively; that is,

$$X = \sum_{n=1}^{\infty} \frac{2x_n + sy_n}{n}u^n, \quad Y = \sum_{n=1}^{\infty} y_nu^n,$$

then $Y(u) = uye^{X(u)}$.

**Proof**: To prove this result consider

$$Y(u) = y_1u + y_2u^2 + y_3u^3 + \cdots + y_nu^n + \cdots. \tag{2.3.12}$$

Then substituting the power series for $Y(u)$ into the expression

$$Y(u) - (2x + sy)uY(u) + \beta u^2Y(u),$$

we obtain

$$[1 - (2x + sy)u + \beta u^2]Y(u) = yu + \sum_{k=1}^{\infty} [y_{k+1} - (2x + sy)y_k + \beta y_{k-1}]u^{k+1},$$

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where we have put $y_1 = y$. Now, use the recurrence relation (2.2.14)

$$y_{k+1} - (2x + sy)y_k + \beta y_{k-1} = 0,$$

to find that the above expression reduces to

$$[1 - (2x + sy)u + \beta u^2]Y(u) = yu. \quad (2.3.13)$$

Now consider the series

$$X(u) = (2x + sy_1)u + \frac{2x_2 + sy_2}{2}u^2 + \frac{2x_3 + sy_3}{3}u^3 + \cdots + \frac{2x_n + sy_n}{n}u^n + \cdots.$$

Set $\xi = x + \lambda_+ y$ and $\eta = x + \lambda_- y$ where $\lambda_\pm = \frac{s}{2} \pm \sqrt{\frac{s^2}{4} + t}$. Then

$$\xi^n = (x + \lambda_+ y)^n, \eta^n = (x + \lambda_- y)^n, \xi \eta = \beta, \xi + \eta = 2x + sy.$$

Since,

$$2x_n + sy_n = x_n + \alpha_- y_n + x_n - \alpha_+ y_n = \xi^n + \eta^n,$$

we have,

$$X(u) = (\xi + \eta)u + \frac{1}{2}(\xi^2 + \eta^2)u^2 + \frac{1}{3}(\xi^3 + \eta^3)u^3 + \cdots + \frac{1}{n}(\xi^n + \eta^n)u^n + \cdots, \quad (2.3.14)$$

which can be rewritten in the form

$$X(u) = \xi u + \frac{1}{2}\xi^2 u^2 + \frac{1}{3}\xi^3 u^3 + \cdots + \eta u + \frac{1}{2}\eta^2 u^2 + \frac{1}{3}\eta^3 u^3 + \cdots. \quad (2.3.15)$$

Therefore,

$$X(u) = -\ln(1 - \xi u) - \ln(1 - \eta u),$$

$$= -\ln[(1 - \xi u)(1 - \eta u)].$$

Since $(1 - \xi u)(1 - \eta u) = 1 - (2x + sy)u + \beta u^2$, we have

$$X(s) = -\ln[1 - (2x + sy)u + \beta u^2]. \quad (2.3.16)$$

Now compare (2.3.13) and (2.3.16) and obtain the desired result:

$$Y(u) = uye^{X(u)}.$$
2.3.1 THE ZEROS OF $x_n$ AND $y_n$

The $n$-zeros of $y_n$ are given by \((x - \alpha_- y)^n - (x - \alpha_+ y)^n = 0\). Now,

\[ A^n - B^n = (A - B)(A - \omega B)(A - \omega^2 B) \cdots (A - \omega^n B), \tag{2.3.17} \]

where $\omega = \cos(2\pi/n) + i \sin(2\pi/n)$; it is clear that the numbers $\omega^k$ for $k = 1, 2, \cdots, n$, form the $n^{th}$ roots of unity, and their arguments are $\frac{2k\pi}{n}$ lie on a regular polygon of $n$ sides, inscribed in a unit circle about the origin. Therefore, the zeros of $y_n$ are

\[ (x - \alpha_- y) - \omega^k(x - \alpha_+ y) = 0, \quad k = 1, 2, \cdots, n. \]

Since $\alpha_+ + \alpha_- = -s$, $\alpha_+ - \alpha_- = \sqrt{s^2 + 4t}$, the above equation implies

\[ \frac{\alpha_+ - \alpha_-}{2x - (\alpha_+ + \alpha_-)y} = \frac{(1 - \omega^k)}{(1 + \omega^k)} \tag{2.3.18} \]

which when expressed in terms of $s$, $t$ and sine and cosine functions, we get

\[ \frac{y\sqrt{s^2 + 4t}}{2x + sy} = -i \frac{\sin(k\pi/n)}{\cos(k\pi/n)} \tag{2.3.19} \]

The above result can be written in the form

\[ y\sqrt{s^2 + 4t} = i\sqrt{\beta} \sin \left(\frac{k\pi}{n}\right), \quad 2x + sy = \sqrt{\beta} \cos \left(\frac{k\pi}{n}\right) \tag{2.3.20} \]

Therefore the zeros of $y_n$ lie on the central conic $x^2 + sxy - ty^2 = \beta$.

The zeros of $x_n$ are given by the equation

\[ \alpha_+(x - \alpha_-y)^n - \alpha_-(x - \alpha_+y)^n = 0. \tag{2.3.21} \]

By similar arguments we followed in finding the zeros of $y_n$ will show that $x$ and $y$ should satisfy
\[ y\sqrt{s^2 + 4t} = i\beta \sin \left(\frac{(2k-1)\pi}{2n}\right), \quad 2x + sy = \sqrt{\beta} \cos \left(\frac{(2k-1)\pi}{2n}\right) \]  

(2.3.22)

As in the case of zeros of \( y_n \), the zeros of \( x_n \) also lie on the conic \( x^2 + sxy - ty^2 = \beta \).

For the case when \( s = 0 \), the zeros of \( x_n \) and \( y_n \) lie on the conic \( x^2 - ty^2 = \beta \). If \( t = -1 \), the the polynomials \( x_n \) and \( y_n \) become Chebychev polynomials of the first and second kind respectively [66] and their zeros lie on a unit circle centered about the origin. Notice that the zeros \( T_n(x) \) and \( U_n(x) \) are interspersed on the \( x \)-axis.

### 2.4 Series Summation Involving Reciprocals Of Polynomials

Let us look at some infinite series summation involving \( x_n \) and \( y_n \): First we shall show that

\[ \sum_{k=1}^{\infty} \frac{1}{x_{k+1}} \left( \frac{2x + s}{x_{k-1}} - \frac{\beta + 1}{x_k} \right) = \frac{1}{x} \]  

(2.4.1)

To show the above result consider

\[
\frac{1}{x_{k-1}x_k} - \frac{1}{x_kx_{k+1}} = \frac{x_{k+1} - x_{k-1}}{x_{k-1}x_kx_{k+1}} = \frac{(2x + s)x_k - \beta x_{k-1} - x_{k-1}}{x_{k-1}x_kx_{k+1}} = \frac{2x + s}{x_{k-1}x_{k+1}} - \frac{\beta + 1}{x_kx_{k+1}}.
\]

where we have used the property \( x_{k+1} = (2x + s)x_k - \beta x_{k-1} \). Therefore,
Similarly, we show that
\[
\sum_{k=1}^{\infty} \frac{1}{x_{k+1}} \left( \frac{2x + s}{x_{k-1}} - \frac{\beta + 1}{x_k} \right) = \sum_{k=1}^{\infty} \left( \frac{1}{x_{k-1}x_k} - \frac{1}{x_kx_{k+1}} \right) = \frac{1}{x_0x_1} = \frac{1}{x}
\]

Similarly, we obtain
\[
\sum_{k=2}^{\infty} \frac{1}{y_{k+1}} \left( \frac{2x + s}{y_{k-1}} - \frac{\beta + 1}{y_k} \right) = \frac{1}{yy_2} \quad (2.4.2)
\]
and
\[
\begin{align*}
\sum_{k=r+1}^{\infty} \left( \frac{2x + s}{x_{k-1}x_{k+1}} - \frac{\beta + 1}{x_kx_{k+1}} \right) &= \frac{1}{x_rx_{r+1}} \\
\sum_{k=r+1}^{\infty} \left( \frac{2x + s}{y_{k-1}y_{k+1}} - \frac{\beta + 1}{y_ky_{k+1}} \right) &= \frac{1}{y_ry_{r+1}} \quad (2.4.3)
\end{align*}
\]
If \( \beta = -1 \) the above results become,
\[
\sum_{k=r+1}^{\infty} \frac{1}{x_{k-1}x_{k+1}} = \frac{1}{(2x + s)(x_rx_{r+1})} \quad (2.4.4)
\]
and
\[
\sum_{k=r+1}^{\infty} \frac{1}{y_{k-1}y_{k+1}} = \frac{1}{(2x + s)(y_ry_{r+1})} \quad (2.4.5)
\]
Next, using \((2x + s)x_k = x_{k+1} + \beta x_{k-1}\) we can derive
\[
\sum_{k=r+1}^{\infty} \frac{(2x + s)x_k}{x_{k-1}x_{k+1}} = \sum_{k=r+1}^{\infty} \left( \frac{1}{x_{k-1}} + \frac{\beta}{x_{k+1}} \right) \quad (2.4.6)
\]
If \( \beta = -1 \), then we have
\[
\sum_{k=r+1}^{\infty} \frac{x_k}{x_{k-1}x_{k+1}} = \frac{1}{(2x + s)} \left( \frac{1}{x_r} + \frac{1}{x_{r+1}} \right) \quad (2.4.7)
\]
Similarly, we obtain
\[
\sum_{k=r+1}^{\infty} \frac{(2x + s)y_k}{y_{k-1}y_{k+1}} = \sum_{k=r+1}^{\infty} \left( \frac{1}{y_{k-1}} + \frac{\beta}{y_{k+1}} \right) \quad (2.4.8)
\]
If \( \beta = -1 \), then we have,

\[
\sum_{k=r+1}^{\infty} \frac{y_k}{y_{k-1}y_{k+1}} = \frac{1}{(2x + s)} \left( \frac{1}{y_r} + \frac{1}{y_{r+1}} \right) \quad (2.4.9)
\]

Similar other series properties can be obtained.

### 2.5 An Application Of Extended Brahmagupta Matrix Identity

In this section, as an application of extended Brahmagupta matrix identity, we shall derive a matrix identity to original Brahmagupta polynomials which is quite similar to the following identity of Fibonacci Numbers [42]:

\[
\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}^n = \begin{bmatrix}
F_{n-1} & F_n \\
F_n & F_{n+1}
\end{bmatrix} \quad (2.5.1)
\]

We also derive identities similar to those satisfied by Lucas numbers:

\[
\begin{bmatrix}
2 & 1 \\
1 & 3
\end{bmatrix}^{2n} = 5^n \begin{bmatrix}
F_{2n-1} & F_{2n} \\
F_{2n} & F_{2n+1}
\end{bmatrix} \quad (2.5.2)
\]

\[
\begin{bmatrix}
2 & 1 \\
1 & 3
\end{bmatrix}^{2n+1} = 5^n \begin{bmatrix}
L_{2n} & L_{2n+1} \\
L_{2n+1} & L_{2n+2}
\end{bmatrix} \quad (2.5.3)
\]

Let us denote a general Brahmagupta matrix involving two parameters \( s \) and \( t \) by

\[
B_{s,t} = \begin{bmatrix}
u & v \\
tv & u + sv
\end{bmatrix} \quad (2.5.4)
\]

and begin with the following identity :

\[
B_{s,t}^n = \begin{bmatrix}
u_n & v_n \\
tv_n & u_n + sv_n
\end{bmatrix} \quad (2.5.5)
\]
Binet forms for $u_n$ and $v_n$:

$$u_n = \frac{1}{\alpha_+ - \alpha_-} [\alpha_+ (u - \alpha_- v)^n - \alpha_- (u - \alpha_+ v)^n]$$

$$v_n = \frac{1}{\alpha_+ - \alpha_-} [(u - \alpha_- v)^n - (u - \alpha_+ v)^n]$$

where $\alpha_\pm = \frac{-s \pm \sqrt{s^2 + 4t}}{2}$. Note that

$$\alpha_+ + \alpha_- = -s, \quad \alpha_+ - \alpha_- = \sqrt{s^2 + 4t}, \quad \alpha_+ \alpha_- = -t.$$ 

1. Put $u = 0, \; v = 1, \; s = t = 1$ in (2.5.4) to get

$$u_n = \frac{1}{\sqrt{5}} \left[ - \left( \frac{1 - \sqrt{5}}{2} \right)^n + \left( \frac{1 + \sqrt{5}}{2} \right)^n \right]$$

$$= \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n-1} - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] = F_{n-1}$$

and on the same lines $v_n = F_n$. Thus we obtain Fibonacci matrix identity (2.5.1).

2. Put $u = 2, \; v = 1, \; s = t = 1$ in (2.5.4) to get

$$u_n = (\sqrt{5})^{n-1} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n-1} + (-1)^{n-1} \left( \frac{1 - \sqrt{5}}{2} \right)^{n-1} \right]$$

and

$$v_n = (\sqrt{5})^{n-1} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - (-1)^n \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

which can also be written as follows:

$$u_{2n} = 5^n \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n-1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2n-1} \right] = 5^n F_{2n-1},$$

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\[ v_{2n} = 5^n \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2n} \right] = 5^n F_{2n}, \]

\[ u_{2n+1} = 5^n \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n} + \left( \frac{1 - \sqrt{5}}{2} \right)^{2n} \right] = 5^n L_{2n} \]

and

\[ v_{2n+1} = 5^n \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n+1} + \left( \frac{1 - \sqrt{5}}{2} \right)^{2n+1} \right] = 5^n L_{2n+1}. \]

Thus we obtain Lucas Matrix Identities (2.5.2) and (2.5.3).

Now for real numbers \( x, y, t \neq 0 \) and \( \beta = x^2 - ty^2 \neq 0, \) let us consider

\[ B_{-\beta,2x} = \begin{bmatrix} -\beta u & v \\ -\beta v & -\beta u + 2xv \end{bmatrix} \]

and using the corresponding matrix identity, we obtain

\[ B_{-\beta,2x}^n = \begin{bmatrix} -\beta u_n & v_n \\ -\beta v_n & -\beta u_n + 2xv_n \end{bmatrix} \quad (2.5.6) \]

The resulting polynomials \( u_n \) and \( v_n \) take the following form

\[ -\beta u_n = \frac{1}{2y\sqrt{t}} \left[ -(x - y\sqrt{t})(u + (x + y\sqrt{t})v)^n + (x + y\sqrt{t})(u + (x - y\sqrt{t})v)^n \right] \]

and

\[ v_n = \frac{1}{2y\sqrt{t}} \left[ (u + (x + y\sqrt{t})v)^n - (u + (x - y\sqrt{t})v)^n \right] \]
(1) Fibonacci-Brahmagupta Matrix Identity:

\[
\begin{bmatrix}
0 & 1 \\
-\beta & 2x
\end{bmatrix}^n = \frac{1}{y} \begin{bmatrix}
-\beta y_{n-1} & y_n \\
-\beta y_n & y_{n+1}
\end{bmatrix}
\]  

(2.5.7)

To derive this identity, it is enough we put \( u = 0 \), \( v = 1 \) in the expressions connected with (2.5.4). The actual expressions are:

\[
-\beta u_n = \frac{1}{2y\sqrt{t}}[-(x - y\sqrt{t})(x + y\sqrt{t})^n + (x + y\sqrt{t})(x - y\sqrt{t})^n]
\]

\[
= -\frac{\beta}{y} \frac{1}{2\sqrt{t}}[(x + y\sqrt{t})^{n-1} - (x - y\sqrt{t})^{n-1}] = -\frac{\beta}{y} y_{n-1},
\]

\[
v_n = \frac{1}{2y\sqrt{t}}[(x + y\sqrt{t})^n - (x - y\sqrt{t})^n] = \frac{y_n}{y}
\]

and

\[
-\beta u_n + 2xv_n = \frac{1}{y}(-\beta y_{n-1} + 2xy_n) = \frac{y_{n+1}}{y}
\]

(ii) Lucas-Brahmagupta Identities:

\[
\begin{bmatrix}
-\beta & x \\
-\beta x & -\beta + 2x^2
\end{bmatrix}^{2n} = t^n y^{2n-1} \begin{bmatrix}
-\beta y_{2n-1} & y_{2n} \\
-\beta y_{2n} & y_{2n+1}
\end{bmatrix}
\]

(2.5.8)

and

\[
\begin{bmatrix}
-\beta & x \\
-\beta x & -\beta + 2x^2
\end{bmatrix}^{2n+1} = t^n y^{2n-1} \begin{bmatrix}
-\beta x_{2n} & x_{2n+1} \\
-\beta x_{2n+1} & x_{2n+2}
\end{bmatrix}
\]

(2.5.9)

Again to derive these identities, it is enough we put \( u = -\beta \), \( v = x \) in the expressions connected with (2.5.4). The actual expressions are:

\[
-\beta u_n = \frac{1}{2y\sqrt{t}}[-(x - y\sqrt{t})(-\beta + (x + y\sqrt{t})x)^n + (x + y\sqrt{t})(-\beta + (x - y\sqrt{t})x)^n]
\]

\[
= \frac{1}{2y\sqrt{t}}[-(x - y\sqrt{t})(ty^2 + x\sqrt{t})^n + (x + y\sqrt{t})(ty^2 + x\sqrt{t})^n]
\]

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\[
\begin{align*}
\frac{(y\sqrt{t})^n}{2y\sqrt{t}} & \left[ - (x - y\sqrt{t})(x + y\sqrt{t})^n + (y\sqrt{t} - x)^n \right] \\
& = -\frac{\beta(y\sqrt{t})^n}{2y\sqrt{t}} \left[ (x + y\sqrt{t})^{n-1} + (-1)^n(x - y\sqrt{t})^{n-1} \right]
\end{align*}
\]
and on the same lines
\[
\begin{align*}
v_n &= \frac{(y\sqrt{t})^n}{2y\sqrt{t}} \left[ (x + y\sqrt{t})^{n-1} - (-1)^n(x - y\sqrt{t})^{n} \right] \\
\beta u_{2n} &= -\frac{\beta(ty^2)^n}{2y\sqrt{t}} \left[ (x + y\sqrt{t})^{2n-1} - (x - y\sqrt{t})^{2n-1} \right] = -\frac{\beta(ty^2)^n}{y} y_{2n-1} \\
v_{2n} &= \frac{(ty^2)^n}{2y\sqrt{t}} \left[ (x + y\sqrt{t})^{2n} - (x - y\sqrt{t})^{2n} \right] = \frac{(ty^2)^n}{y} y_{2n} \\
\beta u_{2n+1} &= -\frac{\beta(ty^2)^n}{2y} \left[ (x + y\sqrt{t})^{2n} + (x - y\sqrt{t})^{2n} \right] = -\frac{\beta(ty^2)^n}{y} x_{2n} \\
\end{align*}
\]
and
\[
\begin{align*}
v_{2n+1} &= \frac{(ty^2)^n}{2y} \left[ (x + y\sqrt{t})^{2n+1} + (x - y\sqrt{t})^{2n+1} \right] = \frac{(ty^2)^n}{y} x_{2n+1}
\end{align*}
\]
Also one can see that
\[
\begin{align*}
-\beta u_{2n} + 2xv_{2n} &= \frac{(ty^2)^n}{y} y_{2n+1} \\

\end{align*}
\]
and
\[
\begin{align*}
-\beta u_{2n+1} + 2xv_{2n+1} &= \frac{(ty^2)^n}{y} x_{2n+2}
\end{align*}
\]
As a special case, we shall derive corresponding matrix identities for Pell and Pell-Lucas Sequences [43, 44, 51, 52] - Put \( x = y = 1 \) and \( t = 2 \). Then \( 2x = 2 \) and \( \beta = x^2 - ty^2 = -1 \).
The corresponding Fibonacci-Brahmagupta identity (2.5.7) is

\[
\begin{bmatrix}
0 & 1 \\
1 & 2 \\
\end{bmatrix}^n = \begin{bmatrix} y_{n-1} & y_n \\
y_n & y_{n+1} \end{bmatrix}
\] (2.5.10)

where \(y_{n+1} = 2y_n + y_{n-1}\), \(y_0 = 0\), \(y_1 = 1\) i.e. \(y_n : 0, 1, 2, 5, 12, \ldots\) is the Pell sequence (please see subsection 1.5.2).

The other two identities (2.5.8) and (2.5.9) are-

\[
\begin{bmatrix} 1 & 1 \\
1 & 3 \\
\end{bmatrix}^{2n} = 2^n \begin{bmatrix} y_{2n-1} & y_{2n} \\
y_{2n} & y_{2n+1} \end{bmatrix} (2.5.11)
\]

\[
\begin{bmatrix} 1 & 1 \\
1 & 3 \\
\end{bmatrix}^{2n+1} = 2^n \begin{bmatrix} x_{2n} & x_{2n+1} \\
x_{2n+1} & x_{2n+2} \end{bmatrix} (2.5.12)
\]

where \(x_{n+1} = 2x_n + x_{n-1}\), \(x_0 = 1 = x_1\) i.e. \(x_n : 1, 1, 3, 7, 17, \ldots\) is the Pell-Lucas sequence (please see subsection 1.5.2). Similar matrix identities can be worked out for many other special cases of Brahmagupta polynomials (please refer section 3.3).