Chapter 4

Brahmagupta Sequences Arising From
Continued Fractions Of Quadratic Irrationals

4.1 Introduction

In the present chapter, by Brahmagupta sequences we shall mean the following pair of sequences of numbers:

\[ x_n = \frac{1}{2}[(x_1 + y_1 \sqrt{N})^n + (x_1 - y_1 \sqrt{N})^n] \]  (4.1.1)

\[ y_n = \frac{1}{2\sqrt{N}}[(x_1 + y_1 \sqrt{N})^n - (x_1 - y_1 \sqrt{N})^n] \]  (4.1.2)

where \( x_1, y_1 \) are positive rational numbers and \( N \) is a square free positive integer.

In section 4.2, we describe a general periodic continued fraction with period 1 of the form

\[ a + \frac{b}{a + \frac{b}{a + \ldots}} \]  (4.1.3)

and derive naturally Fibonacci type and Lucas type sequences given by

\[ x_n = \frac{1}{2} \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n + \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \]  (4.1.4)
\[ y_n = \frac{1}{2\sqrt{a^2 + 4b}} \left[ \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n - \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \right] \quad (4.1.5) \]

In section 4.3, we derive purely periodic Simple Continued Fraction of period L, written in the standard form

\[ [a_1, a_2, a_3, \ldots, a_L] = \frac{P + \sqrt{N}}{Q} \quad (4.1.6) \]

The main result is that if \([a_1, a_2, a_3, \ldots, a_L] = \frac{p_L}{q_L}\), then \((x_1, y_1) = (Qp_L - Pq_L, q_L)\) satisfies \(x^2 - N y^2 = (-1)^2 Q^2\) and hence naturally defines the following Brahmagupta sequences:

\[ x_n = \frac{1}{2} \left[ \left( \frac{x_1 + y_1\sqrt{N}}{Q} \right)^n + \left( \frac{x_1 - y_1\sqrt{N}}{Q} \right)^n \right] \quad (4.1.7) \]

\[ y_n = \frac{1}{2\sqrt{N}} \left[ \left( \frac{x_1 + y_1\sqrt{N}}{Q} \right)^n - \left( \frac{x_1 - y_1\sqrt{N}}{Q} \right)^n \right] \quad (4.1.8) \]

In section 4.4, we describe the classical Lagrange’s method of finding least positive solution of \(x^2 - Ny^2 = \pm 1\) using Simple Continued Fraction of \(\sqrt{N}\). The standard results of Lagrange are highlighted more effectively with the help of identities involving 2x2 matrices. Two standard examples with historical significance, \(N = 67\) and \(N = 61\) are given.

In section 4.5, a Sulbhasutra of Bhodhayana [28, 72]

\[ \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \quad (4.1.9) \]

is derived starting from least positive solution \((3, 2)\) of \(x^2 - 2y^2 = 1\), as follows:

\((x_0, y_0) = (3, 2)\), \((x_1, y_1) = (17, 12)\), \((x_2, y_2) = (577, 408)\)

\(\sqrt{2} \approx \frac{577}{408}\)
\[
\begin{align*}
&= \frac{17^2 + 2(12^2)}{2 \times 17 \times 12} \\
&= \frac{2(17^2) - 1}{2 \times 17 \times 12} \\
&= \frac{17}{12} - \frac{1}{2 \times 17 \times 12} \\
&= \frac{3^2 + 2 \times 2^2}{2 \times 3 \times 2} - \frac{1}{3 \cdot 4 \cdot 34} \\
&= \frac{(2 \times 2^2 + 1) + 2 \times 2^2}{12} - \frac{1}{3 \cdot 4 \cdot 34} \\
&= \frac{4^2 + 1}{3 \times 4} - \frac{1}{3 \cdot 4 \cdot 34} \\
&= \frac{4}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \\
&= 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}
\end{align*}
\]

we make an attempt to derive similar Sulabhasutra type formula to \(\sqrt{N}\) starting from the least positive solution of \(x^2 - Ny^2 = 1\).
4.2 The generalized golden ratio: Fibonacci type and Lucas type sequences

We recall that a quadratic irrational of the form

$$\frac{p + \sqrt{N}}{Q} = \frac{a + \sqrt{a^2 + 4b}}{2}$$  \hspace{1cm} (4.2.1)

$$a^2 + ab > 0, a, b$$ are positive integers is regarded as generalized golden ratio for the following reasons:

1. For $$a = b = 1$$, it is exactly same as the golden ratio

$$\frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}}$$  \hspace{1cm} (4.2.2)

2. It has continued fraction expansion

$$\frac{a + \sqrt{a^2 + 4b}}{2} = a + \frac{b}{a + \frac{b}{a + \ldots}}$$  \hspace{1cm} (4.2.3)

Actually, the continued fraction converges to $$X > a$$ satisfying

$$X = a + \frac{b}{X}$$

$$X = a + \frac{\sqrt{a^2 + 4b}}{2}$$  \hspace{1cm} (4.2.4)

Also, it is interesting to note that for $$b = 1$$, one obtains simple periodic continued fractions of period 1 [54] for any $$a = 1, 2, 3, \ldots$$

$$\frac{a + \sqrt{a^2 + 4}}{2} = a + \frac{1}{1 + \frac{1}{a + \ldots}}$$  \hspace{1cm} (4.2.5)

The case $$a = 1$$, is also quite interesting

$$\frac{1 + \sqrt{1 + 4b}}{2} = 1 + \frac{b}{1 + b}$$  \hspace{1cm} (4.2.6)
because it belongs to the family of continued fraction containing the following
two continued fractions
\[
\left[ e^{\frac{1}{2}} \int_{1}^{\infty} e^{-\frac{x^2}{2}} dx \right]^{-1} = 1 + \frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}} \quad (4.2.7) \\
\left[ e \int_{1}^{\infty} \frac{e^{-x}}{x} dx \right]^{-1} = 1 + \frac{1}{1+\frac{1}{1+\frac{2}{1+\frac{2}{1+\frac{1}{1+\ldots}}}}} \quad (4.2.8)
\]
(For more details please refer [86])

\[
e^{\frac{1}{2}} \int_{1}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{1}{e^{-\frac{1}{2}}} \int_{0}^{e^{-\frac{1}{2}}} \frac{1}{t\sqrt{2\log_{e}^1}} dt \quad (4.2.9)
\]
(put \( t = e^{-\frac{x^2}{2}} \))

\[
e \int_{1}^{\infty} \frac{e^{-x}}{x} dx = \frac{1}{e-1} \int_{0}^{e^{-1}} \frac{1}{\log_{e}(\frac{1}{i})} dt \quad (4.2.10)
\]
(put \( t = e^{-x} \))

In fact we write
\[
\frac{1 + \sqrt{1 + 4b}}{2} = \left[ \frac{1}{b} \int_{0}^{b} \frac{dt}{\sqrt{1 + 4t}} \right]^{-1} \quad (4.2.11)
\]
because
\[
\frac{1}{b} \int_{0}^{b} \frac{dt}{\sqrt{1 + 4t}} = \frac{1}{2b} \int_{0}^{b} \frac{4}{2\sqrt{1 + 4t}} dt = \frac{1}{2b} \frac{[-1 + \sqrt{1 + 4b}]}{2b} = \frac{2}{1 + \sqrt{1 + 4b}}
\]
(4.2.12)

In this way the continued fractions show nice interconnections between cer­
tain sequences of numbers (which appear as partial numerators or partial
denominators) and certain functions (which appear as integrands) serve as their generating functions. The convergent of continued fraction of generalized golden ratio \((p_n, q_n)\) are given by

\[
p_{n+1} = ap_n + bp_{n-1}, \quad P_0 = 1, \quad p_1 = a
\]

\[
q_{n+1} = p_n, \quad p_{-1} = 0
\]

They also satisfy (following [77])

\[
\begin{bmatrix}
p_n & p_{n+1} \\
p_{n-1} & p_n
\end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}^n \tag{4.2.15}
\]

Let us prove this by induction on \(n\).

For \(n = 0\)

\[
\begin{bmatrix} p_0 & p_1 \\ p_{-1} & p_0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}
\]

\[
(4.2.16)
\]

Suppose

\[
\begin{bmatrix} p_k & p_{k+1} \\ p_{k-1} & p_k \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}^k
\]

\[
(4.2.17)
\]

Consider

\[
\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}^{k+1} = \begin{bmatrix} p_k & p_{k+1} \\ p_{k-1} & p_k \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}
\]

\[
= \begin{bmatrix} p_{k+1} & ap_{k+1} + bp_k \\ p_k & ap_k + bp_{k-1} \end{bmatrix}
\]

\[
= \begin{bmatrix} p_{k+1} & p_{k+2} \\ p_k & p_{k+1} \end{bmatrix}
\]

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On taking determinants both sides, we get

\[ p_n^2 - p_{n+1}p_{n-1} = (-b)^n \]  \hfill (4.2.18)

Motivated by [29, 78], we define a pair of sequences \( x_n \) and \( y_n \) as follows -

\[
\begin{align*}
    x_0 &= 1, \quad y_0 = 0 \\
    y_n &= \frac{1}{2}p_n \\
    x_n &= x_{n+1} + by_{n-1}
\end{align*}
\]

\( n = 1, 2, 3, \ldots \)

The sequences look like

\[ x_n : 1, \frac{a}{2}, \frac{1}{2}(a^2 + 2b), \frac{1}{2}(a^3 + 3ab), \ldots \]

\[ y_n : 0, \frac{1}{2}, \frac{a}{2}, \frac{1}{2}(a^2 + b), \ldots \]

Thus satisfy the following Brahmagupta matrix identity:-

\[
\begin{bmatrix}
    x_n & y_n \\
    (a^2 + 4b)x_n & y_n
\end{bmatrix}
= \begin{bmatrix}
    \frac{a}{2} & \frac{1}{2} \\
    \frac{a^2 + 4b}{2} & \frac{a}{2}
\end{bmatrix}^n
\]

\( n = 1, 2, 3, \ldots \)

Let us prove this by induction on \( n \). For \( n = 1 \), the identity is obvious. Let us assume the identity for \( n = k \). For \( n = k + 1 \), we may write

\[
\begin{bmatrix}
    \frac{a}{2} & \frac{1}{2} \\
    \frac{a^2 + 4b}{2} & \frac{a}{2}
\end{bmatrix}^{k+1}
= \begin{bmatrix}
    x_k & y_k \\
    (a^2 + 4b)y_k & x_k
\end{bmatrix}
\begin{bmatrix}
    \frac{a}{2} & \frac{1}{2} \\
    \frac{a^2 + 4b}{2} & \frac{a}{2}
\end{bmatrix}
\]

\( n = 1, 2, 3, \ldots \)
because

\[
\frac{a}{2} x_k + (a^2 + 4b) \frac{1}{2} y_k = \frac{a}{4} (p_{k+1} + b p_{k-1}) + (a^2 + 4b) \frac{1}{4} p_k
\]

\[
= \frac{a}{4} (a p_k + b p_{k-1}) + \frac{ab}{4} p_{k-1} + \left( \frac{a^2}{4} + b \right) p_k
\]

\[
= \left( \frac{a^2}{2} + \frac{b}{2} \right) p_k + \frac{ab}{2} p_{k-1} + b \frac{1}{2} p_k
\]

\[
= \frac{1}{2} \left[ a(a p_k + b p_{k-1}) + b p_k \right]
\]

\[
= \frac{1}{2} \left[ (a p_{k+1} + b p_k) + b p_k \right]
\]

\[
= \frac{1}{2} [p_{k+2} + b p_k]
\]

\[
= x_{k+1}
\]

\[
\frac{1}{2} x_k + \frac{a}{2} y_k = \frac{1}{4} (p_{k+1} + b p_{k-1}) + \frac{a}{4} p_k
\]

\[
= \frac{1}{4} p_{k+1} + \frac{1}{4} (a p_k + b p_{k-1})
\]

\[
= \frac{1}{4} p_{k+1} + \frac{1}{4} p_{k+1}
\]

\[
= \frac{1}{2} p_{k+1}
\]

\[
= y_{k+1}
\]

Hence \( \{x_n\} \) and \( \{y_n\} \) satisfy the following relation satisfied by Brahmagupta Polynomials [78]

\[
\begin{bmatrix}
  x_{n+1} \\
  y_{n+1}
\end{bmatrix}
= \begin{bmatrix}
  \frac{a}{2} & \frac{1}{2} (a^2 + ab) \\
  \frac{1}{2} & \frac{a}{2}
\end{bmatrix}
\begin{bmatrix}
  x_n \\
  y_n
\end{bmatrix}
\]

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\( (2) \quad x_n \pm y_n \sqrt{a^2 + 4b} = \left( \frac{a \pm \sqrt{a^2 + 4b}}{2} \right)^n \)

\( (3) \quad x_n^2 - (a^2 + 4b)y_n^2 = (-b)^n \)

\( (4a) \quad x_n = \frac{1}{2} \left[ \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n + \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \right] \)

\( (4b) \quad y_n = \frac{1}{2\sqrt{a^2 + 4b}} \left[ \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n - \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n \right] \)

\( (5a) \quad x_{n+1} = ax_n + bx_{n-1} \), \( x_0 = 1 \), \( x_1 = \frac{a}{2} \)

\( (5b) \quad y_{n+1} = ax_n + by_{n-1} \), \( y_0 = 0 \), \( y_1 = \frac{1}{2} \)

Thus \( \{x_n\} \) and \( \{y_n\} \) are indeed Brahmagupta sequences arising from the Continued Fraction of generalized golden ratio.

For \( a = b = 1 \), the sequences \( \{2y_n\} \) and \( \{2x_n\} \) are nothing but Fibonacci and Lucas sequences. In this sense \( y_n \) and \( x_n \) may be regarded as, Fibonacci type and Lucas type sequences described by convergents of the generalized golden ratio (4.2.1).

### 4.3 Purely periodic Simple Continued Fractions

If all the partial numerators are 1’s and if the integer part and the partial denominators are given by a sequence \( a_1, a_2, a_3, \ldots a_n \ldots \) of positive integers, then the infinite Continued Fraction is called Simple Continued
Fractions. The standard notation popular in the literature is [8, 16, 21, 38, 54]

\[ x = [a_1; a_2, a_3, a_4 \cdots a_n \cdots]. \] (4.3.1)

Some of the well known examples are

(a) \[ \frac{1 + \sqrt{5}}{2} = [1; 1, 1, 1, 1, \ldots] \]

(b) \[ 1 + \sqrt{3} = [2; 1, 2, 1, 2, \ldots] \]

(c) \[ e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots] \]

(d) \[ \pi = [3; 7, 15, 1, 292, \ldots] \]

The convergent

\[ c_n = [a_1; a_2, a_3, \ldots a_n] = \frac{p_n}{q_n} \] (4.3.2)

satisfy the infinite chain of inequalities

\[ c_0 < c_2 < c_4 < c_6 < c_8 < \ldots c_{2n} < \ldots c_{2n+1} < \ldots < c_5 < c_3 < c_1 \] (4.3.3)

and \( \{p_n\}, \{q_n\} \) satisfy

\[ p_n \frac{q_{n-1}}{q_n} - p_{n-1} \frac{q_n}{q_{n+1}} = (-1)^n \] (4.3.4)

\[ 0 < |x - \frac{p_n}{q_n}| < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}} \] (4.3.5)

The infinite simple continued fraction converges to a real number \( x \) which cannot be rational.
Because \( x = \frac{r}{s} \) will lead to
\[
0 < |r q_n - s p_n| < 1
\]
which is impossible. Also, it is worth to quote the result that two distinct infinite simple continued fraction cannot represent the same irrational number. Thus every irrational number has a unique infinite SCF. By a purely periodic simple continued fraction, we mean
\[
x = [a_1; a_2, a_3, \ldots a_L, x] = [a_1; a_2, a_3, \ldots a_L]
\]

**Theorem 4.1** [8, 16]

A quadratic irrational \( \frac{P + \sqrt{Q}}{Q} \), \( P, Q \) are positive integers, \( N \) is a square free positive integer has purely periodic SCF if and only if \( \frac{P + \sqrt{Q}}{Q} > 1 \) and 
\(-1 < \frac{P - \sqrt{Q}}{Q} < 0 \). If
\[
x = \frac{P + \sqrt{Q}}{Q} = [a_1; a_2, a_3, \ldots a_L],
\]
then \( x \) satisfies
\[
x = \frac{xp_L + p_{L-1}}{xq_L + q_{L-1}}
\]
or
\[
q_Lx^2 - (p_L - q_{L-1})x - p_{L-1} = 0
\]
Hence, \( x \) is also given by
\[
x = \frac{(p_L - q_{L-1}) + \sqrt{(p_L - q_{L-1})^2 + 4q_{L}p_{L-1}}}{2q_L}
\]
As a result
\[
P = p_L - q_{L-1}, \quad Q = 2q_L, \quad N = (p_L - q_{L-1})^2 + 4q_{L}p_{L-1}
\]
Theorem 4.2

\[(x_1, y_1) = (Q_{PL} - P_{QL}, q_L)\]  \hspace{1cm} (4.3.13)

is a solution of the Brahmagupta equation \(x^2 - Ny^2 = (-1)^LQ^2\) and naturally defines the Brahmagupta sequences given by

\[
x_n = \frac{1}{2} \left[ \left( \frac{x_1 + y_1\sqrt{N}}{Q} \right)^n + \left( \frac{x_1 - y_1\sqrt{N}}{Q} \right)^n \right] \hspace{1cm} (4.3.14)
\]

\[
y_n = \frac{1}{2\sqrt{N}} \left[ \left( \frac{x_1 + y_1\sqrt{N}}{Q} \right)^n - \left( \frac{x_1 - y_1\sqrt{N}}{Q} \right)^n \right] \hspace{1cm} (4.3.15)
\]

**Proof:** Let us simplify

\[
x_1 = Q_{PL} - p_{QL}
\]

\[
= (2q_L)p_L - (p_L - q_{L-1})q_L
\]

\[
= (p_L + q_{L-1})q_L
\]

Then one can directly verify that

\[
x_1^2 - Ny_1^2
\]

\[
= (p_L + q_{L-1})^2q_L^2 - [(p_L - q_{L-1})^2 + 4q_Lp_{L-1}]q_L^2
\]

\[
= 4[p_Lq_{L-1} - q_Lp_{L-1}]q_L^2 = (-1)^LQ^2
\]

The following two examples illustrate the above theorem.

**Example 1**

Let us consider the simple continued fractions of the golden ratio

\[
\frac{1 + \sqrt{5}}{2} = [1]
\]

Let us note that

\[
P = 1, \quad Q = 2, \quad N = 5, \quad L = 1, \quad p_L = 1, \quad q_L = 1
\]

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\[ x_1 = Qp_L - Pq_L = 1, \quad y_1 = 1, \quad x_1^2 - 5y_1^2 = (-1)^1Q^2 \]

From the theorem 4.2, we get

\[ x_n = \frac{1}{2} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] = \frac{L_n}{2} \]  \hfill (4.3.16)

\[ y_n = \frac{1}{2\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] = \frac{F_n}{2} \]  \hfill (4.3.17)

The \((F_n, L_n)\) are the well known Fibonacci and Lucas numbers (please refer subsection 1.5.1).

**Example 2**

Consider the following simple periodic continued fraction of period 2

\[ 1 + \sqrt{3} = [2 : 1] \]

Let us note that

\[ P = 1, \quad Q = 1, \quad N = 3, \quad L = 2, \quad p_L = 3, \quad q_L = 1 \]

\[ x_1 = Qp_L - Pq_L = 2, \quad y_1 = 1, \quad x_1^2 - 3y_1^2 = (-1)^2Q \]

From the theorem 4.2, we get

\[ x_n = \frac{1}{2} [(2 + \sqrt{3})^n + (2 - \sqrt{3})^n] \]

\[ y_n = \frac{1}{2\sqrt{3}} [(2 + \sqrt{3})^n - (2 - \sqrt{3})^n] \]

The sequences \((x_n, y_n)\) are applied in [12] to compute Brahmagupta triangles.
4.4 A Pair Of Brahmagupta Sequences Of Simple Continued Fraction Of $\sqrt{N}$

The Simple Continued Fraction of $\sqrt{N}$ plays a crucial role for finding least positive solution of $x^2 - Ny^2 = \pm 1$. It was the master mind of the great mathematician Lagrange to compute five sequences, namely, $\{(P_n, Q_n)\}, \{a_n\}$, $\{(p_n, q_n)\}$ starting from $\sqrt{N}$. The first pair is called $(P, Q)$-sequence, the second one partial denominators of the simple continued fraction of $\sqrt{N}$ and the last pair is nothing but the sequences of numerator and denominator of convergent. The method of construction is as follows -

**Step-1**
Write $X_1 = \sqrt{N} = \frac{P_1 + \sqrt{N}}{Q_1}$ so that $P_1 = 0$, $Q_1 = 1$ compute $\lfloor \sqrt{N} \rfloor = a_1$ (greatest integer $\leq \sqrt{N}$) and the first convergent $C_1 = \frac{P_1}{Q_1}$, so that $p_1 = a_1$, $q_1 = 1$.

**Step-2**
Write $\sqrt{N} = a_1 + \left( \frac{P_1 + \sqrt{N}}{Q_1} - a_1 \right)$

Set $X_2 = \frac{P_2 + \sqrt{N}}{Q_2}$

$$= \frac{1}{\frac{P_1 + \sqrt{N}}{Q_1} - a_1}$$

$$= \frac{Q_1}{(P_1 - a_1 Q_1) + \sqrt{N}}$$

$$= Q_1 \left\{ \frac{(a_1 Q_1 - P_1) + \sqrt{N}}{N - (a Q_1 - P_1)^2} \right\},$$
\[ P_2 = a_1 Q_1 - p_1 = a_1, \]
\[ Q_2 = \frac{\lfloor N - p_2^2 \rfloor}{Q_1} = N - a^2. \]

Compute \( a_2 = \lfloor X_2 \rfloor \) and the second convergent
\[ C_2 = \frac{p_2}{q_2}, \quad p_2 = a_1 a_2 + 1, \quad q_2 = a_2 \]

**General Step**

Suppose \( X_{n-1} = \frac{P_{n-1} + \sqrt{N}}{Q_{n-1}} \) and \( X_n = \frac{P_n + \sqrt{N}}{Q_n} \) are computed.

Set \( X_{n+1} = \frac{P_{n+1} + \sqrt{N}}{Q_{n+1}} \), \( P_{n+1} = a_n Q_n - P_n \), \( Q_{n+1} = N - P_{n+1}^2 \).

We may rewrite as \( Q_n Q_{n+1} = N - P_{n+1}^2 \).

Using \( Q_{n-1} Q_n = N - P_n^2 \) and subtracting the above two expressions we get
\[ Q_n (Q_{n+1} - Q_{n-1}) = P_n^2 - P_{n+1}^2 \]
\[ = (P_n - P_{n+1})(P_n + P_{n+1}) \]
\[ = (P_n - P_{n+1})a_n Q_n \]

Thus
\[ Q_{n+1} = Q_{n-1} + (P_n - P_{n+1})a_n \] \hspace{1cm} (4.4.1)

is indeed a positive integer. As before compute
\[ a_{n+1} = \lfloor X_{n+1} \rfloor \] and the \((n + 1)^{th}\) convergent
\[ C_{n+1} = \frac{p_{n+1}}{q_{n+1}} \] where \( p_{n+1} = a_n p_n + p_{n-1} \) and \( q_{n+1} = a_n q_n + q_{n-1} \).
Theorem 4.3 [8, 16, 77] (The celebrated result of Lagrange)

\[ \sqrt{\mathcal{N}} + \sqrt{\mathcal{N}} \] has purely periodic continued fraction of period \( L \).

So that \( X_{L+1} = P_2 + \sqrt{\mathcal{N}}, \quad a_{L+1} = 2a_1 \).

The \( n \)th convergent satisfy the Brahmagupta equation \( p_n^2 - \mathcal{N}q_n^2 = (-1)^n Q_{n+1} \).

Since \( Q_{L+1} = 1 \), \((p_L, q_L)\) provides the least positive solution. The sequence of solutions are in fact a Brahmagupta pair given by

\[
x_n = \frac{1}{2} [(p_L + q_L \sqrt{\mathcal{N}})^n + (p_L - q_L \sqrt{\mathcal{N}})^n]
\]

\[
y_n = \frac{1}{2 \sqrt{\mathcal{N}}} [(p_L + q_L \sqrt{\mathcal{N}})^n - (p_L - q_L \sqrt{\mathcal{N}})^n]
\]

\( n = 1, 2, 3, \ldots \)

Stark [77] has described the whole picture in a powerful way using the idea of matrices as follows -

\[
A_n = \begin{bmatrix}
0 & 1 \\
1 & a_n
\end{bmatrix}
\]

\[
M_n = \begin{bmatrix}
q_{n-1} & p_{n-1} \\
q_n & p_n
\end{bmatrix}
\]

By mathematical induction one can show that

\[
M_n = A_n A_{n-1} \cdots A_2 A_1
\]

This identity agrees well with the identity

\[
q_{n-1}p_n - p_{n-1}q_n = (-1)^n
\]
It is an interesting fact that

\[
\begin{bmatrix}
p_{kL+1} \\
q_{kL+1}
\end{bmatrix} =
\begin{bmatrix}
p_1 & q_1 \\
q_1 & p_1
\end{bmatrix}
\begin{bmatrix}
p_{kL} \\
q_{kL}
\end{bmatrix}
\]  \hspace{1cm} (4.4.8)

\[k = 1, 2, \cdots\]

As a consequence

\[
M_{kL+1}^{-1} =
\begin{bmatrix}
-a_1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q_{kL} & p_{kL} \\
q_{kL+1} & p_{kL+1}
\end{bmatrix}
= 
\begin{bmatrix}
p_{kL} & Nq_{kL} \\
q_{kL} & p_{kL}
\end{bmatrix}
\]

On taking determinants both sides we get

\[(-1)^{kL} = p_{kL}^2 - Nq_{kL}^2\]  \hspace{1cm} (4.4.9)

where \(k = 1, 2, \cdots\)

The exact picture of the five sequences are described as follows-

(1) \(L = 2n\)

\(P : 0, P_2, P_3, \cdots, P_{n+1}, P_{n+1}, \cdots, P_4, P_3, P_2, P_3, P_4, \cdots\)

\(Q : 1, Q_2, Q_3, \cdots, Q_n, Q_{n+1}, Q_n, \cdots, Q_3, Q_2, 1, Q_2, Q_3, \cdots\)

\(a : a_1, a_2, a_3, \cdots, a_n, a_{n+1}, a_n, \cdots, a_3, a_2, 2a_1, a_2, a_3, \cdots\)

\(p : p_1, p_2, p_3, \cdots, p_n, p_{n+1}, p_{n+2}, \cdots, p_{2n-1}, p_{2n}, p_{2n+1}, p_{2n+2}, p_{2n+3}, \cdots\)

\(q : q_1, q_2, q_3, \cdots, q_n, q_{n+1}, q_n, q_{n+2}, \cdots, q_{2n-1}, q_{2n}, q_{2n+1}, q_{2n+2}, q_{2n+3}, \cdots\)

\((p_{2n}, q_{2n})\) is the least positive solution of \(x^2 - Ny^2 = 1\).

(2) \(L = 2n + 1\)

\(P := 0, P_2, P_3, \cdots, P_{n+1}, P_{n+2}, P_{n+1}, \cdots, P_4, P_3, P_2, P_3, P_4, \cdots\)

\(Q : 1, Q_2, Q_3, \cdots, Q_n+1, Q_{n+1}, \cdots, Q_3, Q_2, 1, Q_2, Q_3, \cdots\)
\[a : a_1, a_2, a_3, \ldots, a_{n+1}, a_{n+1}, \ldots, a_3, a_2, 2a_1, a_2, a_3, \ldots\]

\[p : p_1, p_2, p_3, \ldots, p_{n+1}, p_{n+1}, \ldots, p_{2n}, p_{2n+1}, p_{2n+2}, p_{2n+3}, \ldots\]

\[q : q_1, q_2, q_3, \ldots, q_{n+1}, q_{n+1}, \ldots, q_{2n}, q_{2n+1}, q_{2n+2}, q_{2n+3}, \ldots\]

\((p_{2n+1}, q_{2n+1})\) is the least positive solution of \(x^2 - N y^2 = -1\) and \((p_{4n+2}, q_{4n+2})\) is the least positive solution of \(x^2 - N y^2 = 1\).

Using Brahmagupta Identity

\[p_{4n+2} = 2p_{2n+1}^2 + 1 \quad (4.4.10)\]

\[q_{4n+2} = 2p_{2n+1}q_{2n+1} \quad (4.4.11)\]

Two Classical Examples [13]

(1) \(N = 67(L = 10)\)

\[P : 0, 8, 7, 5, 2, 7, 7, 2, 5, 7, 8, 7, \ldots\]

\[Q : 1, 3, 6, 7, 9, 2, 9, 7, 6, 3, 1, 3, 6, \ldots\]

\[a : 8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, 5, 2, \ldots\]

\[p : 8, 41, 90, 131, 221, 1678, 1899, 3577, 9053, 48842\]

\[q : 1, 5, 11, 16, 27, 205, 232, 437, 1106, 5967\]

\((48842, 5967)\) is the least positive solution of \(x^2 - 67y^2 = 1\).

(2) \(N = 61(L = 11)\)

\[P : 0, 7, 5, 7, 5, 4, 6, 4, 5, 7, 5, 7, \ldots\]

\[Q : 1, 12, 3, 4, 9, 5, 9, 4, 3, 12, 1, \ldots\]

\[a : 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, \ldots\]

\[p : 7, 8, 39, 125, 164, 453, 1070, 1523, 5639, 24079, 29718\]

\[q : 1, 1, 5, 16, 21, 58, 137, 195, 722, 3083, 3805\]

\((29718, 3805)\) is the least positive solution of \(x^2 - 61y^2 = -1\).
Using Brahmagupta Identity

\[ x = 2(29718)^2 + 1 = 1766319049 \]

\[ y = 2(29718)(3805) = 226153980 \]

gives the least positive solution of \( x^2 - 61y^2 = 1 \).

### 4.5 A Sulbhasutratype formula For \( \sqrt{N} \)

A remarkable approximation to \( \sqrt{2} \) occurs in one of the vedic sutras of Bodhayana, Apasthamba and Katyayana called as sulvas [72]

\[ \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \text{(approx)} \]  

(4.5.1)

The approximation is expressed in terms of simple unit fractions which is easy to remember in the form of a sanskrit verse of two lines (‘shloka’). It is accurate to five decimal places because the sutra gives \( \sqrt{2} = 1.4142156 \cdots \), whereas the true value is \( \sqrt{2} = 1.414213 \cdots \). The sulvas contain no clue at all for the methods in which this approximation is arrived at. Many explanations are available to explain this sutra. But they are not interested to look for such sutras to any \( \sqrt{N} \) in general.

We shall describe in detail the derivation of the formula for \( \sqrt{2} \):

Consider \( x^2 - 2y^2 = 1 \). The positive solution is \((3, 2)\). Using Brahmagupta Identity from \((x_0, y_0) = (3, 2)\)

we obtain

\[ (x_1, y_1) = (x_0^2 + 2y_0^2, 2x_0y_0) = (17, 12) \]

and

\[ (x_2, y_2) = (x_1^2 + 2y_1^2, 2x_1y_1) = (577, 408) \]
Now \(\left[\frac{x_2}{y_2}\right]^2 - \left[\frac{1}{y_2}\right]^2 = \left[\frac{1}{408}\right]^2 = 0.6 \times 10^{-5}\) shows it is a good approximation.

\[
\sqrt{2} \approx \frac{x_2}{y_2} = \frac{x_1}{y_1} - \frac{1}{2x_1y_1} = \frac{x_0^2 + 2y_0^2}{2x_0y_0} - \frac{1}{2x_1y_1} = \frac{4y_0^2 + 1}{2x_0y_0} - \frac{1}{2x_1y_1} = \frac{2y_0}{x_0} + \frac{1}{2x_0y_0} - \frac{1}{2x_1y_1} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}
\]

Next, we shall describe in detail the derivation of the formula for \(\sqrt{N}\):

Consider \(x^2 - Ny^2 = 1\), where \(N\) is a square free positive number. Either by direct inspection or by standard methods like Chakravala or continued fraction for \(\sqrt{N}\), one can obtain the least positive solution of \(x^2 - Ny^2 = 1\) [13]. Now, let \((x_0, y_0)\) be a known solution. Now using the standard Brahmagupta identity, let us compute

\[
(x_1, y_1) = (x_0^2 + Ny_0^2, 2x_0y_0)
\]

and

\[
(x_2, y_2) = (x_1^2 + Ny_1^2, 2x_1y_1)
\]

In fact, starting from any \((x_n, y_n)\) we can compute

\[
(x_{n+1}, y_{n+1}) = (x_n^2 + Ny_n^2, 2x_ny_n) \quad (4.5.2)
\]
and

\[(x_{n+2}, y_{n+2}) = (x_{n+1}^2 + N y_{n+1}^2, 2 x_{n+1} y_{n+1})\] (4.5.3)

Now, we check

\[(\frac{x_{n+2}}{y_{n+2}})^2 - N = \left(\frac{1}{y_{n+2}}\right)^2\] (4.5.4)

The following approximation gives desired accuracy:

\[\sqrt{N} \approx \frac{x_{n+2}}{y_{n+2}}\]

\[= \frac{2 x_{n+1}^2 - 1}{2 x_{n+1} y_{n+1}}\]

\[= \frac{x_{n+1}}{y_{n+1}} - \frac{1}{2 x_{n+1} y_{n+1}}\]

\[= \frac{2 N y_n^2 + 1}{2 x_n y_n} - \frac{1}{2 x_{n+1} y_n} - \frac{1}{2 x_{n+1} y_{n+1}}\]

\[= \frac{N y_n}{x_n} + \frac{1}{2 x_n y_n} - \frac{1}{2 x_{n+1} y_{n+1}}\]

\[\therefore \sqrt{N} = p_n + \frac{q_n}{x_n} + \frac{1}{2 x_n y_n} - \frac{1}{2 x_{n+1} y_{n+1}}\] (4.5.5)

Here \(p_n = \left\lfloor \frac{N y_n}{x_n} \right\rfloor\) = integer part of \(\frac{N y_n}{x_n}\), \(q_n = \frac{N y_n}{x_n} - p_n = \frac{N y_n - p_n x_n}{x_n}\)

and \(q_n\) need not be equal to 1.

We shall illustrate numerically with some simple examples in the next subsection.
4.5.1 Numerically Illustrations

1. Let us derive a sutra for $\sqrt{3}$

consider $x^2 - 3 y^2 = 1$

$(x_0, y_0) = (2, 1)$

$(x_1, y_1) = (7, 4)$

$(x_2, y_2) = (97, 56)$

and $(x_3, y_3) = (18817, 10864)$. The accuracy calculation shows

$$\left(\frac{x_3}{y_3}\right)^2 - 3 = \frac{1}{y_3^2} = \frac{1}{56^2} = \frac{1}{10864} = 0.8 \times 10^{-9}$$

Following the general sutra, we obtain

$$\sqrt{3} \approx 1 + \frac{5}{7} + \frac{1}{7 \cdot 8} - \frac{1}{7 \cdot 8 \cdot 194}$$

2. Let us derive a sutra for $\sqrt{5}$

consider $x^2 - 5 y^2 = 1$

The solutions required are

$(x_0, y_0) = (9, 4)$,

$(x_1, y_1) = (161, 72)$,

and $(x_2, y_2) = (51841, 23184)$.

The accuracy calculation shows

$$\left(\frac{x_2}{y_2}\right)^2 - 5 = \frac{1}{y_2^2} = \frac{1}{23184^2} \approx 0.1 \times 10^{-8}$$

Following the general sutra, we obtain

$$\sqrt{5} \approx 2 + \frac{2}{9} + \frac{1}{2 \cdot 9 \cdot 4} - \frac{1}{2 \cdot 161 \cdot 72}$$
3. Let us derive a sutra for \( \sqrt{67} \)

Consider \( x^2 - 67y^2 = 1 \)

The solutions required are
\[
(x_0, y_0) = (48842, 5967), \\
(x_1, y_1) = (x_0^2 + 67y_0^2, 2x_0y_0) \\
(x_2, y_2) = (x_1^2 + 67y_1^2, 2x_1y_1).
\]

The accuracy calculation shows
\[
\frac{x_2^2}{y_2^2} - 67 = \frac{1}{y_2^2} = \frac{1}{4(x_0^2 + 67y_0^2)^2 (2x_0y_0)^2} = 3.232 \times 10^{-38}.
\]

A formula for \( \sqrt{67} \) looks as follows:

\[
\sqrt{67} = p_0 + \frac{q_0}{x_0} + \frac{1}{x_0 (2y_0)} \quad - \quad \frac{1}{x_0 (2y_0)(2x_1)}
\]

where

\[
p_0 = \left\lfloor \frac{67y_0}{x_0} \right\rfloor = 8, \\
q_0 = 67y_0 - p_0x_0 = 9053 \\
x_1 = 4771081927.
\]

4. A similar formula for \( \sqrt{61} \)

Following a similar step by step calculations, we obtain

\[
\sqrt{61} = p_0 + \frac{q_0}{x_0} + \frac{1}{x_0 (2 y_0)} \quad - \quad \frac{1}{x_0 (2 y_0)(2x_1)}
\]

where

\[
x_0 = 1766319049, \quad y_0 = 226153980
\]
\[ p_0 = \left[ \frac{61 \, y_0}{x_0} \right] = 7 \]

\[ q_0 = 61 \, y_0 - p_0 \, x_0 = 1431159437 \]

\[ x_1 = x_0^2 + 61y_0^2 = 6239765965720528801. \]