Chapter 3

SECOND ORDER SYMMETRIC DUALITY IN VECTOR OPTIMIZATION

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is primal itself. However, a nonlinear programming problem may not always possess this property. The first symmetric dual formulation was proposed by Dorn [34]. Dantzig et al. [30] formulated a pair of symmetric dual problems under the assumption of convexity-concavity of the kernel function involved. Later Mond and Weir [105] presented a different pair of symmetric programs with a view to relax convexity-concavity of kernel function.

Further, Mangasarian [89] considered second order duality under certain inequalities, whereas Mond [100] proved the second order duality theorems by using second order convexity. This chapter is devoted to second order symmetric duality where both Mangasarian [89] and Mond [100] indicated a possible computational advantage of the second order dual over the first order dual.

This chapter is divided in two sections. In the first section, pair of second order symmetric dual, multiobjective nonlinear programming problems of Wolfe and Mond-Weir type have been formulated. Weak, strong, and converse duality theorems are proved for these problems by using second order \((F, \rho)\) convexity and its generalization.

In section 3.2 we study a pair of Mond-Weir type second order symmetric dual programs over arbitrary cones and establish weak and strong
duality theorems by using second order cone pseudoconvexity and second order strongly cone pseudoconvexity assumptions.

3.1 Symmetric Duality for Multiobjective Programming Using Second Order \((F, \rho)\) Convexity

Mangasarain [89] identified a second order dual formulation of the nonlinear program. A practical advantage of second order dual is that it provides tighter bounds for the value of the objective function for the primal problem when approximations are used because there are more parameters involved in second order dual problem. Bector and Chandra [6] studied Mond-Weir type second order primal and dual programs and established symmetric duality results using second order pseudoconvexity and quasiconvexity for these programs. Suneja et al. [136] presented a pair of Mond-Weir type multiobjective second order symmetric dual programs and obtained duality results using second order \(\eta\)-convex functions.

Hanson and Mond [53] introduced F-convex functions. Preda [112] extended the concept of F-convexity to \((F, \rho)\) convexity. This definition was enhanced to second order \((F, \rho)\) convexity and its natural generalizations by Mishra and Rueda [95]. In the current section we discuss second order symmetric duality for a pair of multiobjective programming problems using second order \((F, \rho)\) convex and \((F, \rho)\) pseudoconvex functions defined as follows:

Let \(f\) be a real twice differentiable function defined on \(S_1 \times S_2\), where \(S_1 \subseteq \mathbb{R}^n\) and \(S_2 \subseteq \mathbb{R}^m\) are nonempty open subsets. Then \(\nabla_x f\) and \(\nabla_y f\) denote the gradient vectors with respect to \(x\) and \(y\), respectively. \(\nabla_{xx} f\) and \(\nabla_{yx} f\) are respectively, the \(n \times n\) and \(m \times m\) symmetric Hessian matrices. \(\nabla_{yy} f\) is the \(n \times m\) matrix of second order mixed partial derivatives. \((\partial / \partial y_i)(\nabla_{yy} f)\) is the square matrix of order \(m\) obtained by differentiating...
the elements of $\nabla_{yy}f$ with respect to $y_i$. Let $q : R^n \times R^n \rightarrow R^n$, then $\nabla_y(\nabla_{xx}f(x, y)q)$ denotes the matrix of order $m \times n$ whose $(i, j)$th element is $(\partial / \partial y_i)(\nabla_{xx}f(x, y)q)_{ij}$.

Let $\rho$ and $\sigma$ be in $R$, $d_1$ and $d_2$ be pseudometrics defined on $R^n \times R^n$ and $R^m \times R^n$ respectively, $p : R^m \times R^m \rightarrow R^n$ and $q : R^n \times R^n \rightarrow R^n$.

**Definition 3.1.1.** $f(., y)$ is said to be **second order** $(F, \rho)$ **convex** at $x^*$, for fixed $y \in R^m$ if

$$f(x, y) - f(x^*, y) + \frac{1}{2} q^T \nabla_{xx}f(x^*, y)q \geq F(x, x^*; \nabla_x f(x^*, y) + \nabla_{xx}f(x^*, y) q) + \rho d_1^2 (x, x^*);$$

for all $x \in R^n$ and some arbitrary sublinear functional $F : S \times S \times R^n \rightarrow R$.

**Definition 3.1.2.** $f(x, .)$ is said to be **second order** $(G, \sigma)$ **concave** at $y^*$, for fixed $x \in R^n$ if

$$f(x, y^*) - f(x, y) - \frac{1}{2} p^T \nabla_{yy}f(x, y^*) p \geq G(y, y^*; - \nabla_y f(x, y^*) - \nabla_{yy}f(x, y^*) p) + \sigma d_2^2 (y, y^*);$$

for each $y \in R^m$ and some arbitrary sublinear functional $G : S \times S \times R^m \rightarrow R$.

**Definition 3.1.3.** $f(., y)$ is said to be **second order** $(F, \rho)$ **pseudo-convex** at $x^*$, for fixed $y \in R^m$ if,

$$F(x, x^*; \nabla_y f(x^*, y) + \nabla_{xx}f(x^*, y) q) \geq - \rho d_1^2 (x, x^*)$$

$$\Rightarrow f(x, y) \geq f(x^*, y) - \frac{1}{2} q^T \nabla_{xx}f(x^*, y)q ;$$

for each $x \in R^n$ and an arbitrary sublinear functional $F$. 

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Definition 3.1.4. \( f(x, .) \) is said to be second order \((G, \sigma)\) pseudo-concave at \( y^* \), for fixed \( x \in \mathbb{R}^n \) if

\[
G(y, y^*; -\nabla y f(x, y^*) - \nabla_{yy} f(x, y^*) p) \geq -\sigma d_2^2(y, y^*)
\]

\[
\Rightarrow f(x, y) \leq f(x, y^*) - \frac{1}{2} p^T \nabla_{yy} f(x, y^*) p;
\]

for each \( y \in \mathbb{R}^n \) and arbitrary sublinear functional \( G \).

Remark 3.1.1. Taking the above definitions in one variable as considered by Srivastava [127] we have the following observations

1. When \( q = 0 \) second order \((F, \rho)\) convexity reduces to \((F, \rho)\) convexity as defined by Preda [112].

2. When \( q = 0 \) and sublinear functional \( F \) is defined as

\[
F(x, x^*; a) = (x - x^*)^T a, \ a \in \mathbb{R}^n \quad \text{and} \quad d_1(x, x^*) = \| x - x^* \|, \ x, x^* \in \mathbb{R}^n,
\]

then second order \((F, \rho)\) convexity reduces to \( \rho \)-convexity defined by Vial [140].

3. When \( q = 0 \) and \( F \) is defined as \( F(x, x^*; a) = \eta(x, x^*)^T a, \ a \in \mathbb{R}^n \) where \( \eta(x, x^*) \) is an arbitrary vector function defined from \( S_1 \times S_2 \) to \( \mathbb{R}^n \) and

\[
d(x, x^*) = \| \theta(x, x^*) \|, \ x, x^* \in \mathbb{R}^n,
\]

then second order \((F, \rho)\) convexity reduces to \( \rho \)-invexity defined by Jeyakumar [65].

Some more observations are as follows:

1. When \( \rho = 0 \), the second order \((F, \rho)\) convex function reduces to second order \( F \) convex [93].

2. When \( \rho = 0, q = 0 \), the second order \((F, \rho)\) convex function reduces to \( F \)-convex [53].
3. When $\rho = 0$ and $F(x, x^*, a) = (x - x^*)^T a$, $a \in \mathbb{R}^n$, the second order $(F, \rho)$ convexity reduces to second order convexity [100].

4. When $\rho = 0$, $q = 0$ and $F(x, x^*, a) = (x - x^*)^T a$, $a \in \mathbb{R}^n$, the second order $(F, \rho)$ convex function reduces to convex function.

We now give examples to illustrate some of the above observations.

An $(F, \rho)$ convex function need not be $\rho$-invex, $\rho \neq 0$ as shown by the following example.

**Example 3.1.1.** Consider the function $f : \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \to \mathbb{R}$ defined as $f(x) = -\cos^2 x$.

$F(x, x^*; a) = \eta(x, x^*)^T a$, where $\eta(x, x^*) = \cos^2 x^* - \cos^2 x$

$\rho = -1, \ d(x, x^*) = \|x - x^*\| = \|\theta(x, x^*)\|$.

Then $f$ is $(F, \rho)$ convex at $x^* = -\pi/4$, because

$$f(x) - f(x^*) - F(x, x^*; \nabla f(x^*)) - \rho d^2(x, x^*)$$

$$= \left(x + \frac{\pi}{4}\right)^2 \geq 0, \ \text{for every } x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].$$

The function $f$ fails to be $\rho$-invex at $x^* = -\pi/4$, because at $x = 0$

$$f(x) - f(x^*) - \eta(x, x^*) \nabla f(x^*) - \rho \|\theta(x, x^*)\| = -1 + \frac{\pi^2}{16} < 0.$$  

A second order $(F, \rho)$ convex function need not be second order $F$-convex as shown by the following example.
Example 3.1.2. Consider the function \( f : S_1 \times S_2 \rightarrow \mathbb{R} \) where 
\[ S_1 = ]-\sqrt{3}, \sqrt{3}[ , \ S_2 = ]1, 2[ , \] defined as
\[
f(x,y) = \frac{1}{x^2 + 1} - \frac{1}{y^2}.
\]
Let \( F(x,x^*; a) = |a| (x - x^*)^2 - x^2, \ \rho = -1, \ d(x, x^*) = |x - x^*|, \ q = -x^* \).
Then \( f \) is second order \((F, \rho)\) convex in the first variable at \( x^* = -1 \), because
\[
f(x,y) - f(x^*,y) + \frac{1}{2} q^T \nabla_{xx} f(x^*,y) q - F(x,x^*; \nabla_x f(x^*,y) + \nabla_{xx} f(x^*,y) q) - \rho d^2(x,x^*)
\]
\[
= x^2 + \frac{3-x^2}{4(x^2 + 1)} \geq 0, \ \text{for all} \ x \in S_1
\]
The function \( f \) however fails to be second order \( F \) convex at \( x^* = -1 \), because for \( x = 0 \)
\[
f(x,y) - f(x^*,y) + \frac{1}{2} q^T \nabla_{xx} f(x^*,y) q - F(x,x^*; \nabla_x f(x^*,y) + \nabla_{xx} f(x^*,y) q)
\]
\[
= -\frac{1}{4} < 0
\]
The next example shows that a second order \((F, \rho)\) convex function need not be \( F \) convex

Example 3.1.3. The function \( f \) considered in Example 3.1.2 is second order \((F, \rho)\) convex at \( x^* = -1 \) but \( f \) fails to be \( F \) convex at \( x^* = -1 \) because for \( x = 1 \)
\[
f(x,y) - f(x^*,y) - F(x,x^*, \nabla_x f(x^*,y)) = -\frac{3}{2} < 0
\]
A second order \((F, \rho)\) convex function may fail to be second order convex and convex as illustrated in the following example.
Example 3.1.4. The function $f$ considered in Example 3.1.2 is second order $(F, \rho)$ convex at $x^* = -1$ but it fails to be second order convex at $x^* = -1$ because for $x = 0$

$$f(x, y) - f(x^*, y) - (x - x^*)^T (\nabla_x f(x^*, y) + \nabla_{xx} f(x^*, y)q) + \frac{1}{2} q^T \nabla_{xx} f(x^*, y)q$$

$$= -\frac{1}{4} < 0.$$

The function $f$ fails to be convex at $x^* = -1$, because for $x = 1$

$$f(x, y) - f(x^*, y) - (x - x^*)^T \nabla_x f(x^*, y) = -1 < 0.$$

We now consider the following pair of second order symmetric multiobjective programming problems of Wolfe type:

(WMP) \hspace{1cm} minimize $(M_1(x, y, \lambda, p_1), \ldots, M_b(x, y, \lambda, p_b))$

subject to

$$\sum_{i=1}^{b} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i) \leq 0$$

$$\lambda > 0, \sum_{i=1}^{b} \lambda_i = 1$$

(WMD) \hspace{1cm} maximize $(N_1(u, v, \lambda, q_1), \ldots, N_k(u, v, \lambda, q_k))$

subject to

$$\sum_{i=1}^{b} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) \geq 0$$

$$\lambda > 0, \sum_{i=1}^{b} \lambda_i = 1$$

where
\[ M_i(x, y, \lambda, p_i) = f_i(x, y) - y^T(\sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i)) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y)p_i \]

\[ N_i(u, v, \lambda, q_i) = f_i(u, v) - u^T(\sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i)) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v)q_i, \]

\( f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, (1 \leq i \leq k) \) are thrice differentiable functions; \( q_i (1 \leq i \leq k) \) are vectors in \( \mathbb{R}^n \) and \( p_i (1 \leq i \leq k) \) are vectors in \( \mathbb{R}^m \), \( p = (p_1, p_2, \ldots, p_k) \), \( q = (q_1, q_2, \ldots, q_k) \) and \( \lambda_i \in \mathbb{R}, (1 \leq i \leq k) \), \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T \).

We now prove the duality results for the problems (WMP) and (WMD) under second order \((F, p)\) convexity assumptions.

**Theorem 3.1.1 (Weak Duality).** Let \((x, y, \lambda, p)\) be feasible for (WMP) and \((u, v, \lambda, q)\) be feasible for (WMD). Suppose that the following conditions hold:

(i) for \( i = 1, 2, \ldots, k \), \( f_i(., v) \) is second order \((F, p_i)\) convex in the first variable at \( u \) and \( f_i(x, .) \) is second order \((G, \sigma_i)\) concave in the second variable at \( y \).

(ii) \( \sum_{i=1}^{k} \lambda_i (p_i d_{1i}^2(x, u) + \sigma_i d_{2i}^2(v, y)) \geq 0 \)

(iii) \( F(x, u; a) + u^T a \geq 0 \), for all \( a \in \mathbb{R}^n_+ \)

(iv) \( G(v, y; b) + y^T b \geq 0 \), for all \( b \in \mathbb{R}^m_+ \).

Then

\[ (M_1(x, y, \lambda, p_1), M_2(x, y, \lambda, p_2), \ldots, M_k(x, y, \lambda, p_k)) \]

\[ \preceq (N_1(u, v, \lambda, q_1), N_2(u, v, \lambda, q_2), \ldots, N_k(u, v, \lambda, q_k)) \]
Proof. Assume by contradiction that

\[(M_1(x, y, \lambda, p_1), M_2(x, y, \lambda, p_2), \ldots, M_k(x, y, \lambda, p_k)) \leq (N_1(u, v, \lambda, q_1), N_2(u, v, \lambda, q_2), \ldots, N_k(u, v, \lambda, q_k))\]

Since \(\lambda > 0\) and \(\sum_{i=1}^{k} \lambda_i = 1\), we have

\[\sum_{i=1}^{k} \lambda_i M_i(x, y, \lambda, p_i) < \sum_{i=1}^{k} \lambda_i N_i(x, y, \lambda, q_i)\]

\[\Rightarrow \sum_{i=1}^{k} \lambda_i f_i(x, y) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i - y^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(x, y) + \nabla_{yy} f_i(x, y) p_i)\]

\[< \sum_{i=1}^{k} \lambda_i f_i(u, v) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u, v) q_i - u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v) q_i)\]

(3.1)

Since \(f_i(., v)\) is second order \((F, \rho_i)\) convex in the first variable at \(u\), we have

\[f_i(x, v) - f_i(u, v) + \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i\]

\[\geq F(x, u; \nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v) q_i) + \rho_i d_i^2(x, u)\]

Multiplying by \(\lambda_i > 0\) and summing over \(i\), we have

\[\sum_{i=1}^{k} \lambda_i [f_i(x, v) - f_i(u, v)] + \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u, v) q_i\]

\[\geq F \left(x, u; \sum_{i=1}^{k} \lambda_i \nabla_x f_i(u, v) + \sum_{i=1}^{k} \lambda_i \nabla_{xx} f_i(u, v) q_i\right) + \sum_{i=1}^{k} \lambda_i \rho_i d_i^2(x, u)\]

(3.2)

As \(f_i(x, .)\) is second order \((G, \sigma_i)\) concave in second variable at \(y\) therefore we have
\[ f_i(x, y) - f_i(x, v) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i \]

\[ \geq G(v, y; - \nabla f_i(x, y) - \nabla_{yy} f_i(x, y) p_i) + \sigma_i d_2^2(v, y). \]

Again multiplying by \( \lambda_i > 0 \) and summing over \( i \) we get,

\[ \sum_{i=1}^{k} \lambda_i [f_i(x, y) - f_i(x, v)] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i \]

\[ \geq G \left( v, y; - \sum_{i=1}^{k} \lambda_i \nabla f_i(x, y) - \sum_{i=1}^{k} \lambda_i \nabla_{yy} f_i(x, y) p_i \right) + \sum_{i=1}^{k} \lambda_i \sigma_i d_2^2(v, y) \quad (3.3) \]

Adding (3.2) and (3.3) we get

\[ \sum_{i=1}^{k} \lambda_i [f_i(x, y) - f_i(u, v)] + \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u, v) q_i - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i \]

\[ \geq F \left( x, u; \sum_{i=1}^{k} \lambda_i \nabla f_i(u, v) + \sum_{i=1}^{k} \lambda_i \nabla_{xx} f_i(u, v) q_i \right) \]

\[ + G \left( v, y; - \sum_{i=1}^{k} \lambda_i \nabla f_i(x, y) - \sum_{i=1}^{k} \lambda_i \nabla_{yy} f_i(x, y) p_i \right) \]

\[ + \sum_{i=1}^{k} \lambda_i (p_i d_2^2(x, u) + \sigma_i d_2^2(v, y)) \quad (3.4) \]

Choose

\[ a = \sum_{i=1}^{k} \lambda_i \nabla f_i(u, v) + \sum_{i=1}^{k} \lambda_i \nabla_{xx} f_i(u, v) q_i \in \mathbb{R}^n \]

\[ b = - \sum_{i=1}^{k} \lambda_i \nabla f_i(x, y) - \sum_{i=1}^{k} \lambda_i \nabla_{yy} f_i(x, y) p_i \in \mathbb{R}^n \]

On using hypotheses (ii), (iii) and (iv), (3.4) reduces to

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\[
\sum_{i=1}^{k} \lambda_i \left[ f_i(x, y) - f_i(u, v) \right] + \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u, v) q_i - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i \\
\geq -u^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(u, v) - u^T \sum_{i=1}^{k} \lambda_i \nabla_{xx} f_i(u, v) q_i \\
+ y^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(x, y) + y^T \sum_{i=1}^{k} \lambda_i \nabla_{yy} f_i(x, y) p_i
\]

which gives
\[
\sum_{i=1}^{k} \lambda_i f_i(x, y) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i - y^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) p_i \\
\geq \sum_{i=1}^{k} \lambda_i f_i(u, v) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u, v) q_i - u^T \sum_{i=1}^{k} \lambda_i \nabla_y f_i(u, v) + \nabla_{xx} f_i(u, v) q_i
\]

which contradicts (3.1).

Thus
\[
(M_1(x, y, \lambda, p_1), M_2(x, y, \lambda, p_2), ..., M_k(x, y, \lambda, p_k)) \\
\not\in (N_1(u, v, \lambda, q_1), N_2(u, v, \lambda, q_2), ..., N_k(u, v, \lambda, q_k))
\]

**Theorem 3.1.2 (Strong Duality).** Let \((x, y, \bar{\lambda}, p)\) be an efficient solution of (WMP), fix \(\lambda = \bar{\lambda}\) in (WMD) and suppose that

(i) \(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is positive definite for all \(i = 1, 2, ..., k\) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y})] \geq 0
\]
or,
\(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is negative definite for all \(i = 1, 2, ..., k\) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y})] \leq 0.
\]
(ii) The vectors \( \{\nabla_y f_i(\bar{x}, \bar{y}), \nabla_y f_2(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\} \) are linearly independent.

Then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is feasible for (WMD) and the corresponding values of objective functions are equal. Further if the hypotheses of Weak Duality Theorem 3.1.1 are satisfied for all feasible solutions of (WMP) and (WMD) then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is a properly efficient solution of (WMD).

**Proof.** Since \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) is an efficient solution of (WMP), by Fritz John optimality conditions [24], there exist vectors \(\alpha \in R^k, \beta \in R^m, \delta \in R^k\) such that

\[
\sum_{i=1}^{k} \alpha_i [\nabla_x f_i(\bar{x}, \bar{y})] + \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(\bar{x}, \bar{y})] + \nabla x (\nabla_{xy} f_i(\bar{x}, \bar{y})\bar{p}_i)](\beta - \bar{y} \sum_{i=1}^{k} \alpha_i) = 0
\]

\[
-\sum_{i=1}^{k} \alpha_i \left[2 \nabla x (\nabla_{xy} f_i(\bar{x}, \bar{y})\bar{p}_i)\right] = 0 \tag{3.5}
\]

\[
\sum_{i=1}^{k} \alpha_i \nabla_y f_i(\bar{x}, \bar{y}) - \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(\bar{x}, \bar{y})] + \nabla y (\nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i)](\sum_{i=1}^{k} \alpha_i) = 0
\]

\[
-\sum_{i=1}^{k} \lambda_i [\nabla_{yy} f_i(\bar{x}, \bar{y}) + \nabla y (\nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i))\bar{p} - \bar{y} \sum_{i=1}^{k} \alpha_i = 0
\]

\[
-\frac{1}{2} \sum_{i=1}^{k} \alpha_i (\nabla_y (\nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i)) = 0 \tag{3.6}
\]

\[
(\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i)(\beta - \bar{y} \sum_{i=1}^{k} \alpha_i - \delta_i = 0, \quad i = 1, 2, \ldots, k \tag{3.7}
\]

\[
\nabla_{yy} f_i(\bar{x}, \bar{y})((\beta - \bar{y} \sum_{i=1}^{k} \alpha_i)\bar{\lambda}_i - \alpha_i \bar{p}_i) = 0, \quad i = 1, 2, \ldots, k \tag{3.8}
\]

\[
\beta^T \left[\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i)\right] = 0 \tag{3.9}
\]

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\[ \delta^T \bar{\kappa} = 0 \]  
\[ (\alpha, \beta, \delta) \geq 0, \quad (\alpha, \beta, \delta) \neq 0 \]  
As \( \bar{\kappa} > 0, \delta \geq 0 \) (3.10) gives \( \delta_i = 0, \ i = 1, 2, ..., k. \)

As \( \nabla \gamma f_i(\bar{x}, \bar{y}) \) is positive or negative definite for \( i = 1, 2, ..., k, \) (by hypothesis (i)) it follows from (3.8) that

\[ (\beta - \bar{y} \sum_{i=1}^{k} \alpha_i) \bar{\kappa}_i = \alpha_i \bar{p}_i \]  
(3.12)

We now claim that \( \alpha \neq 0, \) if \( \alpha = 0 \) then (3.12) gives \( \beta = 0. \)

Thus \( (\alpha, \beta, \delta) = 0, \) contradicting (3.11).

Hence \( \alpha_i \neq 0 \) for some \( i. \)  
(3.13)

Multiplying by \( \bar{\kappa}_i \) in (3.7) and using (3.12) and (3.13) we have

\[ \bar{\kappa}_i^T (\nabla f_i(\bar{x}, \bar{y}) + \nabla \gamma f_i(\bar{x}, \bar{y}) \bar{p}_i) = 0. \]

As \( \bar{\kappa} > 0, \) we have

\[ \sum_{i=1}^{k} \bar{\kappa}_i \bar{p}_i^T (\nabla f_i(\bar{x}, \bar{y})) + \sum_{i=1}^{k} \bar{\kappa}_i \bar{p}_i^T (\nabla \gamma f_i(\bar{x}, \bar{y}) \bar{p}_i) = 0. \]  
(3.14)

We now prove that \( \bar{p}_i = 0, \) for \( i = 1, 2, ..., k. \) Otherwise, by hypothesis (i) we have

\[ \sum_{i=1}^{k} \bar{\kappa}_i \bar{p}_i^T (\nabla f_i(\bar{x}, \bar{y})) + \sum_{i=1}^{k} \bar{\kappa}_i \bar{p}_i^T (\nabla \gamma f_i(\bar{x}, \bar{y}) \bar{p}_i) < 0 \quad \text{or} \quad > 0 \]

which contradicts (3.14). Hence \( \bar{p}_i = 0, \) for \( i = 1, 2, ..., k. \)

Now from \( \bar{p}_i = 0, \bar{\kappa} > 0, \ i = 1, 2, ..., k \) and (3.12),
we have $\beta = \overline{y} \sum_{i=1}^{k} \alpha_i$. \hfill (3.15)

Using $\overline{p}_i = 0$ and (3.15); (3.6) reduces to

$$\sum_{i=1}^{k} \left( \alpha_i - \overline{\alpha}_i \right) \nabla f_i(X, Y) = 0$$

By hypothesis (ii), we get

$$\alpha_i = \overline{\alpha}_i \left( \sum_{i=1}^{k} \alpha_i \right)$$

Now using $\overline{p}_i = 0$ and (3.12); (3.5) gives

$$\sum_{i=1}^{k} \alpha_i \nabla f_i(X, Y) = 0$$

$$\Rightarrow \sum_{i=1}^{k} \overline{\alpha}_i \nabla f_i(X, Y) = 0$$

Hence $(X, Y, \overline{\alpha}, \overline{q} = 0)$ is feasible for (WMD) and $X^T \sum_{i=1}^{k} \overline{\alpha}_i \nabla f_i(X, Y) = 0$.

From (3.9), $\overline{p}_i = 0$, (3.15) and $\alpha \neq 0$, we have

$$\overline{y}^T \sum_{i=1}^{k} \overline{\alpha}_i \nabla f_i(X, Y) = 0$$

Hence, $M_i(X, Y, \overline{\alpha}, \overline{q}_i) = N_i(X, Y, \overline{\alpha}, \overline{q}_i = 0)$. \hfill (3.16)

If $(X, Y, \overline{\alpha}, \overline{q})$ is not an efficient solution of (WMD) then there exists a feasible solution $(u, v, \overline{\alpha}, q)$ of (WMD) such that

$$N_i(X, Y, \overline{\alpha}, \overline{q}_i) \leq N_i(u, v, \overline{\alpha}, q_i)$$
But from (3.16) we have
\[ M_i(x, y, \lambda, \bar{\lambda}, \bar{\rho}) \leq N_i(u, v, \bar{\lambda}, q_i) \]
which contradicts Weak Duality Theorem 3.1.1.

If \((x, y, \lambda, \bar{\lambda}, \bar{\rho}) = 0\) is not properly efficient for (WMD) then for some feasible \((u, v, \bar{\lambda}, q)\) of (WMD) and some \(i\), satisfying
\[
f_i(u, v) - u^T \sum_{i=1}^{k} \lambda_i \left( \nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i \right) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v)q_i
\]

\[ > f_i(x, y) - x^T \sum_{i=1}^{k} \lambda_i \left( \nabla_x f_i(x, y) \right) \]
we have
\[
f_i(u, v) - u^T \sum_{i=1}^{k} \lambda_i \left( \nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i \right) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v)q_i
\]
\[
-(f_i(x, y) - x^T \sum_{i=1}^{k} \lambda_i \nabla_x f_i(x, y)) \]
\[ > M[(f_j(x, y) - x^T \sum_{i=1}^{k} \lambda_i \nabla_x f_i(x, y)) - (f_j(u, v) - u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i)) + \frac{1}{2} q_j^T \nabla_{xx} f_j(u, v)q_j] \]

for all \(M > 0\) and all \(j\) satisfying
\[
f_j(x, y) - x^T \sum_{i=1}^{k} \lambda_i \nabla_x f_i(x, y) \]
\[ > (f_j(u, v) - u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) - \frac{1}{2} q_j^T \nabla_{xx} f_j(u, v)q_j) \]
which further implies that
\[
(f_i(u,v) - u^T \sum_{i=1}^k \lambda_i \nabla_{x_i} f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u,v) q_i
\]
\[
- [f_i(\bar{x}, \bar{y}) - \bar{x}^T \sum_{i=1}^k \lambda_i \nabla_{x_i} f_i(\bar{x}, \bar{y})]
\]
can be made arbitrarily large.

Thus for \( \lambda > 0 \), \( \sum_{i=1}^k \lambda_i = 1 \)

\[
\sum_{i=1}^k \lambda_i f_i(u,v) - \frac{1}{2} \sum_{i=1}^k \lambda_i q_i^T \nabla_{xx} f_i(u,v) q_i - u^T \sum_{i=1}^k \lambda_i (\nabla_{x_i} f_i(u,v) + \nabla_{xx} f_i(u,v) q_i)
\]
\[
> \sum_{i=1}^k \lambda_i f_i(\bar{x}, \bar{y}) - \bar{x}^T \sum_{i=1}^k \lambda_i \nabla_{x_i} f_i(\bar{x}, \bar{y})
\]

which contradicts Weak Duality Theorem 3.1.1 for the feasible solutions \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)\) and \((u,v,\lambda, q)\) of (WMP) and (WMD) respectively.

**Theorem 3.1.3 (Converse Duality).** Let \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})\) be an efficient solution of (WMD). Fix \( \lambda = \bar{\lambda} \) in (WMP) and suppose that

(i) \( \nabla_{xx} f_i(\bar{u}, \bar{v}) \) is positive definite for all \( i = 1, 2, ..., k \) and 
\( \sum_{i=1}^k \lambda_i q_i^T [\nabla_{x_i} f_i(\bar{u}, \bar{v})] \geq 0 \), or \( \nabla_{xx} f_i(\bar{u}, \bar{v}) \) is negative definite for all \( i = 1, 2, ..., k \) and 
\( \sum_{i=1}^k \lambda_i q_i^T [\nabla_{x_i} f_i(\bar{u}, \bar{v})] \leq 0 \).

(ii) the vectors \( \{\nabla_{x_1} f_i(\bar{u}, \bar{v}), \nabla_{x_2} f_i(\bar{u}, \bar{v}),...,\nabla_{x_k} f_i(\bar{u}, \bar{v})\} \) are linearly independent

Then \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)\) is a feasible solution of (WMP) and the corresponding values of objective functions are equal.
Moreover, if the hypotheses of Weak Duality Theorem 3.1.1 are satisfied for all feasible solutions of (WMP) and (WMD), then \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)\) is a properly efficient solution of (WMP).

**Proof.** It follows on the lines of Theorem 3.1.2.

We now consider the following symmetric pair of second order multiobjective programming problems of Mond-Weir type.

\[(\text{MSP}) \quad \text{minimize} \quad (f_1(x, y) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y)p_i,\ldots, f_k(x, y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x, y)p_k) \]

subject to

\[\sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i) \leq 0\]

\[y^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)p_i) \geq 0\]

\[\lambda > 0\]

\[(\text{MSD}) \quad \text{maximize} \quad (f_1(u, v) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v)q_i,\ldots, f_k(u, v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u, v)q_k) \]

subject to

\[\sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) \geq 0\]

\[u^T \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q_i) \leq 0\]

\[\lambda > 0\]

**Theorem 3.1.4 (Weak Duality).** Let \((x, y, \lambda, p)\) be feasible for (MSP) and \((u, v, \lambda, q)\) for (MSD). Suppose that the following conditions hold:

(i) \[\sum_{i=1}^k \lambda_i f_i(., y)\] is second order \((F, \rho)\) pseudoconvex at \(u\) and \[\sum_{i=1}^k \lambda_i f_i(x, .)\]

is second order \((G, \sigma)\) pseudoconcave at \(v\).
(ii) \( F(x,u;\alpha) + u^T \alpha + \rho d_1^2(x,u) \geq 0 \), for all \( \alpha \in \mathbb{R}^n \)

(iii) \( G(v,y;\beta) + y^T \beta + \sigma d_2^2(v,y) \geq 0 \), for all \( \beta \in \mathbb{R}^n \)

Then

\[
(f_1(x,y) - \frac{1}{2} p_1^T \nabla_{yy} f_1(x,y)p_1,\ldots,f_k(x,y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x,y)p_k) \\
\leq (f_1(u,v) - \frac{1}{2} q_1^T \nabla_{xx} f_1(u,v)q_1,\ldots,f_k(u,v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u,v)q_k)
\]

**Proof.** Assume by contradiction that

\[
(f_1(x,y) - \frac{1}{2} p_1^T \nabla_{yy} f_1(x,y)p_1,\ldots,f_k(x,y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x,y)p_k) \\
\leq (f_1(u,v) - \frac{1}{2} q_1^T \nabla_{xx} f_1(u,v)q_1,\ldots,f_k(u,v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u,v)q_k)
\]

Since \( \lambda > 0 \) we have

\[
\sum_{i=1}^{k} \lambda_i f_i(x,y) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x,y)p_i \\
< \sum_{i=1}^{k} \lambda_i f_i(u,v) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u,v)q_i \\
\tag{3.17}
\]

Choose \( \alpha = \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \in \mathbb{R}^n \) then by (ii),

\[
F(x,u;\sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)) \\
+ u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) + \rho d_1^2(x,u) \geq 0
\]

which implies
\[ F(x, u; \sum_{i=1}^{k} \lambda_i (\nabla, f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)) + \rho d^2_i(x,u) \geq -u^T \sum_{i=1}^{k} \lambda_i (\nabla, f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \geq 0 \]

which gives,

\[ F(x, u; \sum_{i=1}^{k} \lambda_i (\nabla, f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)) \geq -\rho d^2_i(x,u) \]

Since \( \sum_{i=1}^{k} \lambda_i f_i(., v) \) is second order \((F, \rho)\) pseudoconvex at \(u\), we have

\[ \sum_{i=1}^{k} \lambda_i f_i(x,v) \geq \sum_{i=1}^{k} \lambda_i f_i(u,v) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u,v)q_i \quad (3.18) \]

Again choose \( b = -\sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \in R^m \) then by (iii)

\[ G(v,y; -\sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i)) + \sigma d^2(v,y) \]

\[ \geq y^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \geq 0 \]

which implies

\[ G(v,y; -\sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i)) \geq -\sigma d^2(v,y) \]

Since \( \sum_{i=1}^{k} \lambda_i f_i(.,.) \) is second order \((G, \sigma)\) pseudoconcave at \(y\) we have

\[ -\sum_{i=1}^{k} \lambda_i f_i(x,v) \geq -\sum_{i=1}^{k} \lambda_i f_i(x,y) + \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x,y)p_i \quad (3.19) \]

Adding (3.18) and (3.19) we have
\[ 0 \geq \sum_{i=1}^{k} \lambda_i f_i(u,v) - \sum_{i=1}^{k} \lambda_i f_i(x,y) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u,v) q_i + \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x,y) p_i \]

This implies

\[ \sum_{i=1}^{k} \lambda_i f_i(x,y) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x,y) p_i \]
\[ \geq \sum_{i=1}^{k} \lambda_i f_i(u,v) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q_i^T \nabla_{xx} f_i(u,v) q_i \quad (3.20) \]

which contradicts (3.17).

Hence

\[ (f_i(x,y) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x,y) p_i, ..., f_k(x,y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x,y) p_k) \]
\[ \preceq (f_i(u,v) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u,v) q_i, ..., f_k(u,v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u,v) q_k) \]

**Theorem 3.1.5 (Strong Duality).** Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) be a weak efficient solution of (MSP), fix \(\lambda = \bar{\lambda}\) in (MSD) and suppose that

(i) \(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is positive definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T [\nabla_{yy} f_i(\bar{x}, \bar{y})] \geq 0\); or \(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is negative definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T [\nabla_{yy} f_i(\bar{x}, \bar{y})] \leq 0\).

(ii) The set \(\{\nabla_{yy} f_1(\bar{x}, \bar{y}) + \nabla_{yy} f_1(\bar{x}, \bar{y}) \bar{p}_1, \nabla_{yy} f_2(\bar{x}, \bar{y}) + \nabla_{yy} f_2(\bar{x}, \bar{y}) \bar{p}_2, \ldots, \nabla_{yy} f_k(\bar{x}, \bar{y}) + \nabla_{yy} f_k(\bar{x}, \bar{y}) \bar{p}_k\}\) is linearly independent.
Then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is feasible for (MSD) and the corresponding values of the objective functions are equal. Moreover, if the hypotheses of Weak Duality Theorem 3.1.4 hold for all feasible solution of (MSP) and (MSD) then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is a properly efficient solution of (MSD).

**Proof.** It follows on the lines of the corresponding results by Yang et al. [156].

**Theorem 3.1.6 (Converse Duality).** Let \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})\) be efficient for (MSD), fix \(\lambda = \bar{\lambda}\) in (MSP) and suppose that

(i) \(\nabla_{xx}f_i(\bar{u}, \bar{v})\) is positive definite for all \(i = 1, 2, ..., k\) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T [\nabla_{xx}f_i(\bar{u}, \bar{v})] \geq 0; \text{ or } \nabla_{xx}f_i(\bar{u}, \bar{v}) \text{ is negative definite for all } i = 1, 2, ..., k \text{ and } \sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T [\nabla_{xx}f_i(\bar{u}, \bar{v})] \leq 0.
\]

(ii) The set \(\{\nabla_{xx}f_1(\bar{u}, \bar{v}) + \nabla_{xx}f_1(\bar{u}, \bar{v})\bar{q}_1, ..., \nabla_{xx}f_k(\bar{u}, \bar{v}) + \nabla_{xx}f_k(\bar{u}, \bar{v})\bar{q}_k\}\) is linearly independent.

Then \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)\) is feasible for (MSP) and the corresponding values of the objective functions are equal.

Moreover, if the hypotheses of Theorem 3.1.4 are satisfied for all feasible solutions of (MSP) and (MSD), then \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)\) is a properly efficient solution for (MSP).

**Proof.** The proof follows on the lines of Theorem 3.1.5.
3.2 Second Order Symmetric Duality in Vector Optimization Over Cones

Symmetric duality with cone constraints has been a subject of investigation for several authors. Suneja et al. [130] formulated a pair of symmetric dual multiobjective programs of Wolfe type over arbitrary cones and established duality results using cone-convexity assumptions. Khurana [75] formulated a pair of Mond-Weir type symmetric dual multiobjective problems and established duality results by using cone pseudoinvexity of the functions involved. In this section we formulate a pair of Mond-Weir type second order symmetric dual programs over arbitrary cones and establish weak and strong duality results by using second order cone pseudoconvexity assumptions. The following notations will be used throughout the section.

If \( f_i : S_1 \times S_2 \to \mathbb{R} \), \( i = 1, 2, \ldots, p \), \( S_1 \subseteq \mathbb{R}^n \), \( S_2 \subseteq \mathbb{R}^m \), are thrice continuously differentiable real valued functions of \( x \) and \( y \) and 
\[
 f(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_p(x, y)),
\]
where \( q = (q_1, q_2, \ldots, q_p) \), where \( q_i \in \mathbb{R}^n \) and 
\[
 r = (r_1, r_2, \ldots, r_p),
\]
where \( r \in \mathbb{R}^m \), then we have the following notations:

\[
 \nabla_x f(x, y) = (\nabla_x f_1(x, y), \ldots, \nabla_x f_p(x, y)),
\]

\[
 \nabla_y f(x, y) = (\nabla_y f_1(x, y), \ldots, \nabla_y f_p(x, y)),
\]

\[
 \nabla_{xx} f(x, y) = (\nabla_{xx} f_1(x, y), \ldots, \nabla_{xx} f_p(x, y)),
\]

\[
 \nabla_{yy} f(x, y) = (\nabla_{yy} f_1(x, y), \ldots, \nabla_{yy} f_p(x, y)),
\]

\[
 \nabla_{xy} f(x, y) = (\nabla_{xy} f_1(x, y), \ldots, \nabla_{xy} f_p(x, y)),
\]

\[
 \nabla_{xx} f(x, y) q = (\nabla_{xx} f_1(x, y) q_1, \ldots, \nabla_{xx} f_p(x, y) q_p),
\]

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\[ \nabla_{yy} f(x, y) r = (\nabla_{yy} f_1(x, y) r_1, \ldots, \nabla_{yy} f_p(x, y) r_p), \]
\[ q^T \nabla_{xx} f(x, y) q = (q_1^T \nabla_{xx} f_1(x, y) q_1, \ldots, q_p^T \nabla_{xx} f_p(x, y) q_p) \]
and \[ r^T \nabla_{yy} f(x, y) r = (r_1^T \nabla_{yy} f_1(x, y) r_1, \ldots, r_p^T \nabla_{yy} f_p(x, y) r_p) \]

Suneja et al. [131] studied second order cone convex functions in vector optimization. Based on their work, we give the following definitions:

Let \( K \subseteq R^n \) be a closed convex pointed cone with \( \text{int } K \neq \phi \).

**Definition 3.2.1.** The function \( f \) is said to be **second order K-convex** in the first variable at \( u \in S_1 \) for fixed \( v \in S_2 \) with respect to:

\[ q: R^n \times R^n \rightarrow R^n \] if for every \( x \in S_1 \)

\[ [f(x, v) - f(u, v) - (x - u)^T \nabla_x f(u, v) - (x - u)^T \nabla_{xx} f(u, v) q]^+
+ \frac{1}{2} q^T \nabla_{xx} f(u, v) q \in K \]

and \( f \) is said to be **second order K-convex** in the second variable at \( v \in S_2 \) for fixed \( u \in S_1 \) with respect to \( r: R^n \times R^n \rightarrow R^n \) if for every \( y \in S_2 \)

\[ [f(u, y) - f(u, v) - (y - v)^T \nabla_x f(u, v) - (y - v)^T \nabla_{yy} f(u, v) r]^+
+ \frac{1}{2} r^T \nabla_{yy} f(u, v) r \in K \]

**Remark 3.2.1.** If \( K = R^n_+ \) then second order K-convex functions reduce to second order convex functions.

We now give an example of a function which is second order K-convex but fails to be second order convex.
**Example 3.2.1.** Consider the function $f : S_1 \times S_2 \rightarrow R^2$, where 
$S_1 = [-\sqrt{2}, \sqrt{2}]$, $S_2 = R$, defined as $f(x, y) = (f_1(x, y), f_2(x, y))$,
where $f_1(x, y) = -x^2 + y^2$, $f_2(x, y) = x^3 - x + \frac{1}{y^2 + 1}$.

Then $f$ is second order K-convex in the first variable at $u = 1$, with respect to 
$q = \left(-u, -\frac{u}{2}\right)$, where $K = \{(x, y) : y \leq -\sqrt{2}x, x \leq 0\}$, as

$$f(x, v) - f(u, v) - (x - u)^T [\nabla_x f(u, v) + \nabla_{xx} f(u, v)q] + \frac{1}{2} q^T \nabla_{xx} f(u, v)q = \left(-x^2, x^3 - \frac{1}{4}\right) \in K \text{ for all } x \in S_1$$

The function $f$ fails to be second order convex at $u = 1$ because for $x = -1$,

$$f(x, v) - f(u, v) - (x - u)^T (\nabla_x f(u, v) + \nabla_{xx} f(u, v)q) + \frac{1}{2} q^T \nabla_{xx} f(u, v)q = \left(-1, -\frac{5}{4}\right) \not\geq 0$$

**Remark 3.2.2.** (i) If $q = 0$ then second order K-convex function reduces to K-convex function.

(ii) For $q = 0$, $K = R^p$, second order K-convex function reduces to convex function.

We now give an example of second order K-convex function that fails to be K convex and convex.
Example 3.2.2. The function $f$ considered in Example 3.2.1 is second order $K$-convex at $u = 1$, but $f$ fails to be $K$-convex at $u = 1$, because for $x = 0$

$$f(x, v) - f(u, v) - (x - u)^T \nabla_x f(u, v) = (-1, 2) \notin K.$$ 

The function $f$ also fails to be convex at $u = 1$, because for $x = -1$

$$f(x, v) - f(u, v) - (x - u)^T \nabla_x f(u, v) = (-4, 4) \geq 0.$$ 

Further we consider second order cone pseudoconvex and second order strongly cone pseudoconvex functions.

Definition 3.2.2. The function $f$ is said to be second order $K$-pseudoconvex in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to $q$ if for every $x \in S_1$, 

$$- (x-u)^T(\nabla_x f(u, v) + \nabla_{xx} f(u, v) q) \notin \text{int } K$$ 

$$\Rightarrow - (f(x, v) - f(u, v) + \frac{1}{2} q^T \nabla_{xx} f(u, v) q) \notin \text{int } K$$

and $f$ is said to be second order $K$-pseudoconvex in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to $r$ if for every $y \in S_2$, 

$$-(y-v)^T(\nabla_y f(u, v) + \nabla_{yy} f(u, v) r) \notin \text{int } K$$ 

$$\Rightarrow -(f(u, y) - f(u, v) + \frac{1}{2} r^T \nabla_{yy} f(u, v) r) \notin \text{int } K$$

Remark 3.2.3. Every second order $K$-convex function is second order $K$-pseudoconvex, but the converse is not true as can be seen by the following example.
Example 3.2.3. Consider the function $f : S_1 \times S_2 \to \mathbb{R}^2$ defined by

$$
\begin{align*}
  f(x, y) &= (f_1(x, y), f_2(x, y)) \quad \text{where} \quad S_1 = R, S_2 = [0, 5]. \\
  f_1(x, y) &= -(x^2 + y^2 + 1), \quad f_2(x, y) = -(x^4 + 1/y).
\end{align*}
$$

Then $f$ is second order $K$-pseudoconvex in the second variable at $v = 1$, with respect to $r = (v^2, 2v^2 - 1)$, where $K = \{(x, y) : y \leq x, x \geq 0\}$ because

$$
-(y - v)^T(\nabla_y f(u, v) + \nabla_{yy} f(u, v) r) = (4(y - 1), y - 1) \notin \text{int } K \Rightarrow y \leq 1
$$

$$
\Rightarrow -f(u, y) - f(u, v) + \frac{1}{2} r^T \nabla_{yy} f(u, v) r = (y^2, 1/y) \notin \text{int } K.
$$

But it fails to be second order $K$-convex at $v = 1$ because for $y = 1$

$$
\begin{align*}
  f(u, y) - f(u, v) - (y - v)^T \nabla_y f(u, v) - (y - v)^T \nabla_{yy} f(u, v) r + \frac{1}{2} r^T \nabla_{yy} f(u, v) r
  &= \langle -1, -1 \rangle \notin K
\end{align*}
$$

Definition 3.2.3. The function $f$ is said to be second order strongly $K$-pseudoconvex in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to $q$ if for every $x \in S_1$

$$
-(x-u)^T(\nabla_x f(u, v) + \nabla_{xx} f(u, v) q) \notin \text{int } K
$$

$$
\Rightarrow \quad [f(x, v) - f(u, v) + \frac{1}{2} q^T \nabla_{xx} f(u, v) q] \in K
$$

and $f$ is said to be second order strongly $K$-pseudoconvex in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to $r$ if for every $y \in S_2$

$$
-(y-v)^T(\nabla_y f(u, v) + \nabla_{yy} f(u, v) r) \notin \text{int } K
$$
\[ f(u, y) - f(u, v) + \frac{1}{2} \mathbf{r}^T \nabla_{yy} f(u, v) \mathbf{r} \in K. \]

We now give an example of a function which is second order strongly \( K \)-pseudoconvex in the first variable but fails to be second order \( K \)-convex.

**Example 3.2.4.** Consider the function \( f : S_1 \times S_2 \to \mathbb{R}^2 \) defined by,

\[
\begin{align*}
f(x, y) &= (f_1(x, y), f_2(x, y)) \quad \text{where } S_1 = ]-2, \infty[ \text{, } S_2 = ]0, 5[ \\
f_1(x, y) &= x^2 + y^2 + 1, \quad f_2(x, y) = x^4 + 1/y.
\end{align*}
\]

Then \( f \) is second order strongly \( K \)-pseudoconvex in the first variable at \( u = 2 \) with respect to \( q = (-u - 1, -\frac{u}{4}) \), \( K = \{(x, y) : y \leq x, x \geq 0\} \), because

\[-(x - u)^T (\nabla_x f(u, v) + \nabla_{xx} f(u, v) q) = (2(x - 2), -8(x - 2)) \notin \text{int } K \Rightarrow x \leq 2\]

\[ f(x, v) - f(u, v) + \frac{1}{2} q^T \nabla_{xx} f(u, v) q \]

\[ = (x^2 + 5, x^4 - 10) \in K, \]

\( f \) fails to be second order \( K \)-convex in the first variable at \( u = 2 \), because for \( x = 0 \)

\[
\begin{align*}
f(x, v) - f(u, v) - (x - u)^T \nabla_x f(u, v) - (x - u)^T \nabla_{xx} f(u, v) q + \frac{1}{2} q^T \nabla_{xx} f(u, v) q & \\
& = (1, 6) \notin K.
\end{align*}
\]

We now consider the following pair of second order Mond-Weir type symmetric dual multiobjective programming problems:
(SP) K-minimize \( f(x, y) - \frac{1}{2} r^T \nabla_{xy} f(x, y) r \)

subject to

\[
-\sum_{i=1}^{p} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yx} f_i(x, y) r_i) \in C_2^+ \tag{3.21}
\]

\[
y^T \sum_{i=1}^{p} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yx} f_i(x, y) r_i) \geq 0 \tag{3.22}
\]

\( x \in C_1, \lambda \in \text{int} K^+, \lambda_i \neq 0, r_i \in R^m, i = 1, 2, \ldots, p. \)

(SD) K-maximize \( f(u, v) - \frac{1}{2} q^T \nabla_{uv} f(u, v) q \)

subject to

\[
\sum_{i=1}^{p} \lambda_i (\nabla_x f_i(u, v) + \nabla_{ux} f_i(u, v) q_i) \in C_1^+ \tag{3.23}
\]

\[
u^T \sum_{i=1}^{p} \lambda_i (\nabla_x f_i(u, v) + \nabla_{ux} f_i(u, v) q_i) \leq 0 \tag{3.24}
\]

\( v \in C_2, \lambda \in \text{int} K^+, \lambda_i \neq 0, q_i \in R^n, i = 1, 2, \ldots, p \)

where \( f_i: R^n \times R^n \to R, i = 1, 2, \ldots, p \) are thrice continuously differentiable functions. \( C_1 \) and \( C_2 \) are closed convex cones with nonempty interiors in \( R^n \) and \( R^m \) respectively, \( C_1^+ \) and \( C_2^+ \) are positive polar cones of \( C_1 \) and \( C_2 \) respectively, \( K \) is a closed convex pointed cone in \( R^p \) such that \( \text{int} K \neq \phi \) and \( K^+ \) is its positive polar cone.

We now establish duality theorems for (SP) and (SD)
**Theorem 3.2.1 (Weak Duality).** Let \((x, y, \lambda, r)\) and \((u, v, \lambda, q)\) be feasible solutions of (SP) and (SD) respectively. Assume that \(f(\cdot, y)\) is second order strongly \(K\)-pseudoconvex in the first variable at \(u\) for fixed \(v\) with respect to \(q = (q_1, q_2, \ldots, q_p)\) where \(q_i \in \mathbb{R}^n\) and \(-f(x, \cdot)\) is second order \(K\)-pseudoconvex in the second variable at \(y\) for fixed \(x\) with respect to \(r = (r_1, r_2, \ldots, r_p), r_i \in \mathbb{R}^m\). Then

\[
\left[ f(u,v) - \frac{1}{2} q^T \nabla_{xz} f(u,v)q \right] - \left[ f(x,y) - \frac{1}{2} r^T \nabla_{yy} f(x,y)r \right] \in \text{int } K
\]

**Proof.** Let if possible

\[
\left[ f(u,v) - \frac{1}{2} q^T \nabla_{xz} f(u,v)q \right] - \left[ f(x,y) - \frac{1}{2} r^T \nabla_{yy} f(x,y)r \right] \in \text{int } K \quad (3.25)
\]

Since \((u, v, \lambda, q)\) is feasible for (SD), from (3.23) and (3.24) we get

\[
(u - x)^T \sum_{i=1}^{p} \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)q) \leq 0
\]

Since \(\lambda \in \text{int } K^+\), therefore

\[
[ - (x-u)^T (\nabla_x f(u, v) + \nabla_{xx} f(u, v)q)] \notin \text{int } K.
\]

As \(f(\cdot, v)\) is second order strongly \(K\)-pseudoconvex in the first variable at \(u\) for fixed \(v\) with respect to \(q \in \mathbb{R}^n\), we get

\[
[f(x, v) - f(u, v) + \frac{1}{2} q^T \nabla_{xx} f(u, v)q] \in K. \quad (3.26)
\]

Similarly, since \((x, y, \lambda, r)\) is feasible for (SP), using (3.21) and (3.22) we get
\[ (-y + v)^T \sum_{i=1}^{n} \lambda_i (\nabla_y f(x, y) + \nabla_{xy} f(x, y)r) \leq 0 \]
which implies that
\[ [(-y + v)^T (\nabla_y f(x, y) + \nabla_{yy} f(x, y)r)] \notin \text{int } K. \]

As \(-f(x, \cdot)\) is second order \(K\)-pseudoconvex in the second variable at \(y\) for fixed \(x\) with respect to \(r \in \mathbb{R}^m\), we have
\[ [-f(x, y) + f(x, v) + \frac{1}{2} r^T \nabla_{yy} f(x, y)r] \notin \text{int } K. \quad (3.27) \]

Adding (3.25) and (3.26), we have
\[ -f(x, y) + f(x, v) + \frac{1}{2} r^T \nabla_{yy} f(x, y)r \in \text{int } K, \]
which contradicts (3.27). Hence
\[ \left[ f(u, v) - \frac{1}{2} q^T \nabla_{xx} f(u, v)q \right] - \left[ f(x, y) - \frac{1}{2} r^T \nabla_{yy} f(x, y)r \right] \notin \text{int } K. \]

In order to prove the strong duality theorem, we will make use of the following proposition which gives generalized form of Fritz-John optimality conditions established by Suneja et al. \[130\] for a point to be weak minimum of the vector optimization problem (VP), given as below

\[(VP) \quad \text{K-minimize } f(x) \]
subject to \(-g(x) \in Q\)
\[ x \in S \]
where \(S \subseteq \mathbb{R}^n\), \(f : S \to \mathbb{R}^p\), \(g : S \to \mathbb{R}^m\), \(K\) and \(Q\) are closed convex cones in \(\mathbb{R}^p\) and \(\mathbb{R}^m\) respectively with nonempty interiors. \(X_0 = \{x \in S : -g(x) \in Q\}\) is the feasible set of (VP).
Proposition 3.2.1. If $x^*$ is a weak minimum of $(VP)$, then there exist $\alpha^* \in K^+, \beta^* \in Q^+$ not both zero such that

$$(\alpha^* \nabla f(x^*)^T + \beta^* \nabla g(x^*)^T)(x - x^*) \geq 0, \text{ for all } x \in S$$

and $\beta^* g(x^*) = 0$.

Theorem 3.2.2 (Strong Duality). Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ be a weak minimum of $(SP)$. Fix $\lambda = \bar{\lambda}$ in $(SD)$ and suppose that

(a) either the Hessian matrix $\nabla_{yy}f_i$ is positive definite for all

$$i = 1, 2, ..., p \text{ and } \sum_{i=1}^p \bar{\lambda}_{i} \bar{r}_i^T [\nabla_{yy}f_i] \geq 0; \text{ or the Hessian matrix } \nabla_{yy}f_i$$

is negative definite for all $i = 1, 2, ..., p \text{ and } \sum_{i=1}^p \bar{\lambda}_{i} \bar{r}_i^T [\nabla_{yy}f_i] \leq 0$.

(b) the set $\{\nabla_{yy}f_{i+1}, \nabla_{yy}f_{i+2}, ..., \nabla_{yy}f_{p}, \nabla_{yy}g_{i+1}, \nabla_{yy}g_{i+2}, ..., \nabla_{yy}g_{p}\}$ is linearly independent,

where $f = (f_1, f_2, ..., f_p)$, $f_i = f_i(\bar{x}, \bar{y}), \ i = 1, ..., p$. Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution for $(SD)$ and the corresponding values of the objective functions are equal. Moreover if the hypotheses of Weak Duality Theorem 3.2.1 are satisfied for all feasible solutions of $(SP)$ and $(SD)$, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a weak maximum of $(SD)$.

Proof. Since $(x, y, \bar{\lambda}, \bar{r})$ is a weak minimum of $(SP)$, by Proposition 3.2.1 $\exists \alpha \in K^+, \beta \in (C_2^\ast)^+ = C_2, \gamma \geq 0, (\alpha, \beta, \gamma) \neq 0$ such that for each $x \in C_1, \lambda \in \text{int } K^+$ and $r_i \in R^m (i = 1, 2, ..., p)$, we have
\[
\left\{ \sum_{i=1}^{p} \alpha_i \left[ \nabla_y f_i - \frac{1}{2} \nabla_x (\nabla_y f_i \bar{r}_i) \right] + \sum_{i=1}^{p} \lambda_i \left[ \nabla_{yy} f_i + \nabla_x (\nabla_{yy} f_i \bar{r}_i) \right] (\beta - \gamma \bar{y}) \right\} (x - \bar{x}) \\
+ \left\{ \sum_{i=1}^{p} \alpha_i \left[ \nabla_y f_i - \frac{1}{2} \nabla_y (\nabla_{yy} f_i \bar{r}_i) \right] + \sum_{i=1}^{p} \lambda_i \left[ \nabla_{yy} f_i + \nabla_y (\nabla_{yy} f_i \bar{r}_i) \right] (\beta - \gamma \bar{y}) \right\} (y - \bar{y}) \\
- \gamma \sum_{i=1}^{p} \lambda_i (\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i) \right\} (y - \bar{y}) + \left\{ \sum_{i=1}^{p} (\beta - \gamma \bar{y})^T (\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i) \right\} (\lambda_i - \bar{\lambda}_i) \\
+ \left\{ -\sum_{i=1}^{p} \alpha_i \nabla_{yy} f_i \bar{r}_i + \sum_{i=1}^{p} \lambda_i \nabla_{yy} f_i (\beta - \gamma \bar{y}) \right\} (r_i - \bar{r}_i) \geq 0
\] 

(3.28)

\[
\beta^T \sum_{i=1}^{p} \lambda_i [\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i] = 0
\] 

(3.29)

\[
\gamma \bar{y}^T \sum_{i=1}^{p} \lambda_i [\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i] = 0
\] 

(3.30)

(\alpha, \beta, \gamma) \neq 0

substituting \( x = \bar{x}, \ y = \bar{y}, \ r_i = \bar{r}_i \) in (3.28) we get

\[
(\beta - \gamma \bar{y})^T \sum_{i=1}^{p} (\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i) (\lambda_i - \bar{\lambda}_i) \geq 0, \ \text{for all} \ \lambda \in \text{int} \ K^+ 
\]

which implies

\[
(\beta - \gamma \bar{y})^T (\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i) = 0, \ i = 1,2,...,p
\] 

(3.31)

Now substituting \( x = \bar{x}, \ y = \bar{y}, \ \lambda = \bar{\lambda} \) in (3.28)

\[
\Rightarrow \left\{ \sum_{i=1}^{p} [(\beta - \gamma \bar{y}) \bar{\lambda}_i - \alpha_i \bar{r}_i] \nabla_{yy} f_i \right\} (r_i - \bar{r}_i) \geq 0, \ \forall r_i \in R^n
\]

\[
\Rightarrow [(\beta - \gamma \bar{y}) \bar{\lambda}_i - (\alpha_i \bar{r}_i)]^T (\nabla_{yy} f_i) = 0
\] 

(3.32)
As $\nabla_{yy} f_i$ is positive or negative definite for $i = 1, 2, ..., p$, by hypothesis (a) it follows from (3.32) that

$$\beta - \gamma \bar{y}_i \bar{\lambda}_i = \alpha_i \bar{r}_i, \quad i = 1, ..., p \quad (3.33)$$

We now claim that $\alpha_i \neq 0$ for $i = 1, ..., p$.

If for some index $k_0$, $\alpha_{k_0} = 0$, then since $\bar{\lambda}_i \neq 0$, $i = 1, ..., p$

$$\beta - \gamma \bar{y} = 0 \Rightarrow \beta = \gamma \bar{y} \quad (3.34)$$

Now for $x = \bar{x}$, $\lambda = \bar{\lambda}$, $r_i = \bar{r}_i$ (3.28) gives

$$\sum_{i=1}^{p} (\alpha_i - \gamma \bar{\lambda}_i) \nabla y f_i + \sum_{i=1}^{p} \bar{\lambda}_i \nabla_{yy} f_i (\beta - \gamma \bar{y} - \gamma \bar{r}_i)$$

$$+ \sum_{i=1}^{p} \nabla (\nabla_{yy} f_i \bar{r}_i) \left[ (\beta - \gamma \bar{y}) \bar{\lambda}_i - \frac{1}{2} \alpha_i \bar{r}_i \right] = 0$$

Using (3.33) and (3.34) we get

$$\sum_{i=1}^{p} (\alpha_i - \gamma \bar{\lambda}_i) [\nabla_y f_i + \nabla_{yy} f_i \bar{r}_i] = 0$$

by hypothesis (b) $\alpha_i - \gamma \bar{\lambda}_i = 0$ or $\alpha_i = \gamma \bar{\lambda}_i$, $i = 1, 2, ..., p$ (3.35)

Since $\bar{\lambda}_i \neq 0, i = 1, 2, ..., p$ and $\alpha_{k_0} = 0$ for some $k_0$ we have $\gamma = 0$. Now from (3.34), (3.35) and $\gamma = 0$ we get $\beta = 0, \alpha_i = 0, i = 1, 2, ..., p$ which is a contradiction as $(\alpha, \beta, \gamma) \neq 0$.

Therefore $\alpha_i \neq 0, i = 1, 2, ..., p$.

Premultiplying (3.31) by $\bar{\lambda}_i$ and using (3.33) and $\alpha_i \neq 0$ we get
We now claim that $\bar{r}_i = 0$, else by hypothesis (a)

$$\sum_{i=1}^p \bar{\kappa}_i \bar{r}_i^T [\nabla y f_i + \nabla y y f_i] > 0 \quad \text{or} \quad < 0$$

which contradicts (3.36).

Hence $\bar{r}_i = 0$, $\forall \ i = 1, 2, \ldots, p$.

Further from $\bar{r}_i = 0$, $\bar{\kappa}_i \neq 0$, $i = 1, 2, \ldots, p$, (3.33) gives $\beta = \gamma \bar{y}$. (3.37)

Now $\sum_{i=1}^p (\alpha_i - \gamma \bar{\kappa}_i) [\nabla y f_i + \nabla y y f_i] = 0$

reduces to

$$\sum_{i=1}^p (\alpha_i - \gamma \bar{\kappa}_i) \nabla y f_i = 0.$$ 

By hypothesis (b), using $\bar{r}_i = 0$, we have $\alpha_i = \gamma \bar{\kappa}_i, \ i = 1, 2, \ldots, p$ (3.38)

Again as $\bar{\kappa}_i \neq 0, \alpha_i \neq 0$, we get $\gamma \neq 0$ and hence $\gamma > 0$

Applying (3.33), (3.37), (3.38) and $\bar{r}_i = 0$ in (3.28) we get

$$\sum_{i=1}^p (x - \bar{x})^T (\nabla x f_i) \alpha_i \geq 0, \quad \forall \ x \in C_1$$

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By virtue of (3.38) we have

\[ \sum_{i=1}^{p} (x - \bar{x})^T (\nabla_{x} f_i) \gamma \tilde{\lambda}_i \geq 0 \quad \forall x \in C_1. \]

As \( \gamma > 0 \), \( \sum_{i=1}^{p} (x - \bar{x})^T (\nabla_{x} f_i) \tilde{\lambda}_i \geq 0 \) \hspace{1cm} (3.39)

Since \( C_1 \) is a closed convex cone therefore \( x + \bar{x} \in C_1 \) and (3.39) gives

\[ \sum_{i=1}^{p} x^T \nabla_{x} f_i \tilde{\lambda}_i \geq 0, \quad \forall x \in C_1 \]

\[ \sum_{i=1}^{p} \tilde{\lambda}_i \nabla_{x} f_i \in C_1^* \]

Again for \( x = 0 \), (3.39) reduces to

\[ \sum_{i=1}^{p} \bar{x}^T (\nabla_{x} f_i) \tilde{\lambda}_i \leq 0. \]

Thus, \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is a feasible solution for (SD) and the corresponding values of the objective functions of (SP) and (SD) are equal. We now show that \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is a weak maximum for (SD). Suppose not, then there exists a feasible solution \((u,v,\bar{\lambda},q)\) such that

\[ f(u,v) - \frac{1}{2} q^T \nabla_{xx} f(u,v)q - f(\bar{x},\bar{y}) \in \text{int } K \]

which contradicts the Weak Duality Theorem 3.2.1. \( \square \)