Chapter 2

CONE CONVEX AND RELATED FUNCTIONS IN VECTOR OPTIMIZATION

Convexity and its generalizations can be extended to vector case in two different ways, either by defining the concept component wise or by using the idea of cones. Cone convexity plays a crucial role in optimization problems. Cambini [15] introduced several classes of vector valued functions which are possible extensions of scalar generalized concavity. These classes are defined using three order relations generated by a cone, the interior of the cone and the cone without origin. In this chapter we study cone convex and other related functions in vector optimization problems. This chapter is divided into four sections.

In section 2.1, the concept of cone semistrictly convex functions is introduced as a generalization of semistrictly convex functions. Certain properties of these functions and a relation with cone convex functions have been established. In section 2.2, sufficient optimality conditions are proved for a vector optimization problem over topological vector spaces, involving Gâteaux derivatives by assuming the functions to be cone subconvex. Further a Mond-Weir type dual is associated with the optimization problem and weak and strong duality results are proved.

In section 2.3, we introduce cone semilocally preinvex, cone semilocally quasi preinvex and cone semilocally pseudo preinvex functions and study their properties. These functions are further used to establish necessary and sufficient optimality conditions for a vector optimization problem over cones involving $\eta$-semi differentiable functions. The section ends by
formulating a Mond-Weir type dual and proving various duality theorems.

Section 2.4 introduces generalized type-I, generalized quasi type-I, generalized pseudo type-I, generalized quasi pseudo type-I and generalized pseudo quasi type-I functions over cones, for a nonsmooth vector optimization problem, using Clarke’s generalized gradients of locally Lipschitz functions. Sufficient optimality conditions are established and a Mond-Weir type dual is associated with the optimization problem. Weak and strong duality results are proved for the pair under cone generalized type-I assumptions.

2.1 Cone Semistrictly Convex Functions

Yang [153] defined the class of semistrictly convex functions. These functions were earlier called explicitly convex by Xue and Sheh [152]. This section begins by introducing the notion of cone semistrictly convex functions.

Let $X$ be a topological vector space and $Y$ be a locally convex topological vector space, $C \subset Y$ be a pointed closed convex cone with nonempty interior. The ordering in $Y$ is determined by $C$. Let $Y^*$ be the topological dual space of $Y$. The conjugate cone $C^*$ of $C$ is defined as

$$C^* = \{ y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in C \}.$$

The positive conjugate cone $C^{*+}$ of $C$ is given by

$$C^{*+} = \{ y^* \in Y^* : \langle y^*, y \rangle > 0, \forall y \in C \setminus \{0 \} \}.$$

Let $S \subseteq X$ be a nonempty open convex set and $f : S \to Y$. 


**Definition 2.1.1.** The function \( f \) is said to be \( C \)-semistrictly convex on \( S \), if for every \( x, y \in S \), \( f(x) - f(y) \notin \{0\} \) implies

\[
 tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in \text{int} \, C , \quad \text{for all } t \in (0,1).
\]

**Remark 2.1.1.** For \( X = \mathbb{R}^n \), \( Y = \mathbb{R} \), \( C = \mathbb{R}_+ \), the above definition reduces to that of semistrictly convex function [153] (Definition 1.1.8).

**Remark 2.1.2.** The following examples illustrate that the classes of cone convex and cone semistrictly convex functions are independent.

**Example 2.1.1.** Let \( C = \{(x,y) : -\frac{x}{2} \leq y, \frac{x}{2} \leq y\} \),

\[
 f(x) = \begin{cases} 
 (0,0), & x \neq 0, \\
 (1,1), & x = 0.
\end{cases}
\]

Then \( f \) is a \( C \)-semistrictly convex function. However \( f \) is not \( C \)-convex because for \( x = -1, y = 1, t = \frac{1}{2} \)

\[
 tf(x) + (1-t)f(y) - f(tx + (1-t)y) = (-1,-1) \notin C
\]

**Example 2.1.2.** Let \( C = \{(x,y) : -x \leq y \leq x\} \), \( f : \mathbb{R} \to \mathbb{R}^2 \) be defined as \( f(x) = (x^2, -x^2) \), then \( f \) is \( C \)-convex, but \( f \) fails to be \( C \)-semistrictly convex, because for \( x = 1, y = \frac{3}{2}, t = \frac{1}{2} \),

\[
 f(x) = (1,-1), \quad f(y) = \left( \frac{9}{4}, -\frac{9}{4} \right), \quad f(x) \neq f(y) \quad \text{and}
\]

\[
 tf(x) + (1-t)f(y) - f(tx + (1-t)y) = \left( \frac{1}{16}, -\frac{1}{16} \right) \notin \text{int} \, C.
\]
We now discuss some properties of $C$-convex and $C$-semistrictly convex functions in terms of Gâteaux derivatives, defined below.

**Definition 2.1.2 ([61]).** Let $\bar{x} \in S$, then $f : S \to Y$ is said to be Gâteaux differentiable at $\bar{x}$ if for any $x \in X$, the limit

$$f'_x(x) := \lim_{t \to 0} \frac{f(\bar{x} + tx) - f(\bar{x})}{t}$$

exists and $f'_x(\cdot)$ is a continuous linear map from $X$ to $Y$.

The map $f'_x : x \to f'_x(x)$ is called **Gâteaux derivative** of $f$ at $\bar{x}$ in the direction $x$. We observe that

(i) $f'_x(0) = 0$

(ii) $f'_x(\alpha x + \beta y) = \alpha f'_x(x) + \beta f'_x(y), \ \forall \alpha, \beta \in R$ and $\forall x, y \in S$.

**Theorem 2.1.1.** Let $f : S \to Y$, be Gâteaux differentiable then $f$ is $C$-convex on $S$ if and only if

$$f(x) - f(y) - f'_x(x - y) \in C, \ \forall x, y \in S \tag{2.1}$$

where $f'_x(x - y)$ is the Gâteaux derivative of $f$ at $y$ in the direction $x - y$.

**Proof.** Firstly, let $f$ be $C$-convex on $S$, then for every $x, y \in S, t \in [0,1]$,

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in C,$$

or,

$$f(x) - f(y) - \frac{[f(y + t(x - y)) - f(y)]}{t} \in C.$$

Taking limit as $t \to 0^+$, we get

$$f(x) - f(y) - f'_x(x - y) \in C \text{ as } (f'_x)^+(x - y) = f'_y(x - y).$$
Conversely, let us suppose that (2.1) holds.

Let \( x, y \in S, \) \( 0 \leq t \leq 1, \) then \( \hat{x} = ty + (1-t)x \in S. \)

By using (2.1) we have

\[
f(x) - f(\hat{x}) - f'_\hat{x}(x - \hat{x}) \in C
\]

and

\[
f(y) - f(\hat{x}) - f'_\hat{x}(y - \hat{x}) \in C
\]

\[
\Rightarrow f(x) - f(\hat{x}) - tf'_\hat{x}(x - y) \in C \tag{2.2}
\]

and

\[
f(y) - f(\hat{x}) - (1-t)f'_\hat{x}(y - x) \in C. \tag{2.3}
\]

Multiplying (2.2) by \( (1-t) \) and (2.3) by \( t \) and adding we get,

\[
(1-t)f(x) + tf(y) - f((1-t)x + ty) \in C, \quad \forall x, y \in S, t \in [0,1].
\]

Hence \( f \) is C-convex. \( \square \)

**Corollary 2.1.1.** Let \( f \) be a Gâteaux differentiable C-convex function, then

\[
(f'_y - f'_x)(y - x) \in C, \text{ for all } x, y \in S.
\]

**Proof.** Let \( x, y \in S, \) by Theorem 2.1.1

\[
f(x) - f(y) - f'_x(x - y) \in C, \tag{2.4}
\]

and

\[
f(y) - f(x) - f'_y(y - x) \in C. \tag{2.5}
\]

Also, \( f'_x(\alpha x) = \alpha f'_x(x), \) for any real \( \alpha. \)

Adding (2.4) and (2.5) and using above relation we get
\[ f'_y(y-x) - f'_x(y-x) \in C, \]

or, \[ (f'_y - f'_x)(y-x) \in C. \]

In the following theorem we establish a relation between \( C \)-semistrictly convex and \( C \)-convex functions.

**Theorem 2.1.2.** Let \( f : S \to Y \) be \( C \)-semistrictly convex. If there exists \( \alpha \in (0,1) \) such that for every \( x, y \in S \),

\[ \alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y) \in C, \]

then \( f \) is \( C \)-convex.

**Proof.** Let us suppose to the contrary that \( f \) is not \( C \)-convex. Then, there exist \( x, y \in S \), \( t \in (0,1) \) such that

\[ tf(x) + (1-t)f(y) - f(tx + (1-t)y) \notin C. \]

Thus, there exists \( k^* \in C^* \) such that

\[ \langle k^*, tf(x) + (1-t)f(y) - f(tx + (1-t)y) \rangle < 0, \]

as \( C \) is a closed convex cone.

We have two cases:

**Case 1:** \( f(x) - f(y) \notin \{0, y\} \).

Since \( f \) is \( C \)-semistrictly convex, for \( x, y \in S \), \( 0 < t < 1 \),

\[ tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in \text{int} C \subset C. \]
Case 2: \( f(x) - f(y) \in \{0\} \).

Put \( z = tx + (1-t)y \), then using \( f(x) - f(y) \in \{0\} \),

\[
\langle k^+, tf(x) + (1-t)f(y) - f(tx + (1-t)y) \rangle < 0
\]

reduces to

\[
\begin{align*}
\langle k^+, f(x) - f(z) \rangle < 0 \\
\text{and } \langle k^+, f(y) - f(z) \rangle < 0
\end{align*}
\] (2.7)

(i) Let \( 0 < t < \alpha < 1 \) and \( z_i = \frac{t}{\alpha} x + \left(1 - \frac{t}{\alpha}\right) y \), then,

\[
z = tx + (1-t)y
\]

\[
= \alpha \left[ \frac{t}{\alpha} x + \left(1 - \frac{t}{\alpha}\right) y \right]
\]

\[
= \alpha \left[ \frac{t}{\alpha} x + \left(1 - \frac{t}{\alpha}\right) y \right] + (1-\alpha)y
\]

\[
= \alpha z_i + (1-\alpha)y.
\]

Using (2.6) we get

\[
\alpha f(z_i) + (1-\alpha) f(y) - f(z) \in C.
\]

This gives

\[
\langle k^+, \alpha f(z_i) + (1-\alpha) f(y) - f(z) \rangle \geq 0 .
\] (2.8)

From (2.7) we have

\[
(1-\alpha) \langle k^+, f(z) - f(y) \rangle > 0.
\]
Adding in (2.8) we get,

\[ \langle h^*, f(z_1) - f(z) \rangle > 0. \tag{2.9} \]

Let \( b = \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{t}{1-t} \right) \), \( 0 < b < 1 \)
then,

\[
z_1 = \frac{t}{\alpha} x + \left( 1 - \frac{t}{\alpha} \right) y
= \frac{t}{\alpha} x + \left( 1 - \frac{t}{\alpha} \right) \left[ \frac{z}{1-t} - \frac{tx}{1-t} \right]
= \frac{t}{\alpha} x + \frac{\alpha - t}{\alpha(1-t)} z - \left( \frac{t}{1-t} \right) \left( \frac{\alpha - t}{\alpha} \right) x
= \frac{t(1 - \alpha)}{\alpha(1-t)} x + \left[ 1 - \frac{t(1 - \alpha)}{\alpha(1-t)} \right] z
= bx + (1 - b)z. 
\]

Since \( f(x) - f(z) \notin \{0\} \), using C-semistrict convexity of \( f \) we get,

\[ bf(x) + (1 - b)f(z) - f(z_1) \in \text{int } C, \]

or,

\[ \langle h^*, bf(x) + (1 - b)f(z) - f(z_1) \rangle > 0. \tag{2.10} \]

From (2.7) we have

\[ \langle h^*, bf(x) - bf(z) \rangle < 0. \]

On adding the above inequality in (2.10) we get


\[ \langle k^*, f(z_1) - f(z) \rangle < 0 \] which contradicts (2.9).

(ii) Let \( 0 < \alpha < t < 1 \), that is, \( 0 < \frac{t - \alpha}{1 - \alpha} < 1 \).

Let \( z_2 = \left( \frac{t - \alpha}{1 - \alpha} \right) x + \left( \frac{1 - t}{1 - \alpha} \right) y \)

then \( z_2 = \frac{tx - \alpha x + (1-t)y}{1-\alpha} \)

or, \( (1 - \alpha)z_2 = z - \alpha x \)

or, \( z = \alpha x + (1 - \alpha)z_2 \).

Using (2.6) we get,

\[ \alpha f(x) + (1 - \alpha)f(z_2) - f(z) \in C. \]

That is,

\[ \langle k^*, \alpha f(x) + (1 - \alpha)f(z_2) - f(z) \rangle \geq 0. \] (2.11)

Using (2.7) as earlier we get

\[ \langle k^*, f(z_2) - f(z) \rangle > 0. \] (2.12)

Let \( s = \frac{t - \alpha}{t(1 - \alpha)} \), then \( 0 < s < 1 \), as \( 0 < \alpha < t < 1 \).

Now,

\[ z_2 = \frac{z}{1 - \alpha} - \left( \frac{\alpha}{1 - \alpha} \right) x \]
\[
\begin{align*}
&= \frac{z}{1-\alpha} - \left(\frac{\alpha}{1-\alpha}\right) \left[\frac{z}{t} - \left(\frac{1-t}{t}\right)y\right] \\
&= \left[\frac{1}{1-\alpha} - \frac{\alpha}{t(1-\alpha)}\right]z + \frac{(1-t)\alpha}{t(1-\alpha)}y \\
&= \frac{t-\alpha}{t(1-\alpha)} z + \left[1 - \frac{t-\alpha}{t(1-\alpha)}\right]y \\
&= sz + (1-s)y.
\end{align*}
\]

Again using \(C\)-semistrict convexity of \(f\), we have

\[sf(z) + (1-s)f(y) - f(z_2) \in \text{int} C,\]

or,

\[\langle h^*, sf(z) + (1-s)f(y) - f(z_2) \rangle > 0.\]

Using (2.7), the above inequality reduces to

\[\langle h^*, f(z_2) - f(z) \rangle < 0, \text{ which contradicts (2.12)}.\]

Hence \(f\) is \(C\)-convex. \(\square\)

We now consider the vector minimization problem

\[\text{(UP)} \quad \text{C-min } f(x)\]

\[x \in S,\]

where \(f : S \to Y\), and \(S\) is a nonempty convex subset of \(X\).

**Definition 2.1.3.** \(\bar{x} \in S\) is a **global efficient solution** of (UP) if, there exists no \(x \in S\) such that

\[f(\bar{x}) - f(x) \in C \setminus \{0_Y\}.\]
**Definition 2.1.4.** $\bar{x} \in S$ is a locally efficient solution of (UP) if there exists a neighbourhood $U$ of $\bar{x}$ such that

$$f(\bar{x}) - f(x) \notin C \setminus \{0_y\} \quad \text{for all} \ x \in U \cap S.$$ 

If $f$ is a $C$-convex function then, we know that every locally efficient solution of (UP) is a global efficient solution. This property also holds if $f$ is $C$-semistrictly convex function as shown in the following theorem.

**Theorem 2.1.3.** Let $S$ be a nonempty convex subset of $X$ and let $f : S \to Y$ be $C$-semistrictly convex on $S$. If $\bar{x}$ is a locally efficient solution of (UP) then $\bar{x}$ is a global efficient solution of (UP).

**Proof.** Let $\bar{x}$ be a locally efficient solution of (UP). Then there exists a neighborhood $U$ of $\bar{x}$ such that

$$f(\bar{x}) - f(x) \notin C \setminus \{0_y\} \quad \text{for all} \ x \in U \cap S. \quad (2.13)$$

If $\bar{x}$ is not a global efficient solution for (UP) then, there exists $x^* \in S$ such that

$$f(\bar{x}) - f(x^*) \in C \setminus \{0_y\}. \quad (2.14)$$

$$\Rightarrow \quad f(x^*) - f(\bar{x}) \notin \{0_y\}$$

Then, by $C$-semistrict convexity of $f$, we have, for $0 < t < 1$,

$$tf(x^*) + (1-t)f(\bar{x}) - f(tx^* + (1-t)\bar{x}) \in \text{int } C \subset C. \quad (2.15)$$

From (2.14) we have

$$tf(\bar{x}) - tf(x^*) \in C \setminus \{0_y\}, \quad \text{for} \ 0 < t < 1. \quad (2.16)$$

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Adding (2.15) and (2.16), we get, for $0 < t < 1$,

$$f(\bar{x}) - f(tx^* + (1-t)\bar{x}) \in C \setminus \{0_r\}.$$  

(2.17)

For sufficiently small $t > 0$

$$tx^* + (1-t)\bar{x} = \bar{x} + t(x^* - \bar{x}) \in U \cap S.$$  

Thus (2.17) contradicts (2.13).

2.2 Cone Subconvex Functions: Sufficiency and Duality

Hu and Ling [56] studied the generalized Fritz John and generalized Kuhn-Tucker necessary conditions in terms of Gâteaux derivatives for a vector optimization problem in ordered topological vector spaces. This section is devoted to the study of sufficiency and duality results for this problem by assuming the functions to be cone subconvex defined as follows:

Let $S \subseteq X$ be a nonempty convex set of a topological vector space $X$ and $C \subseteq Y$ be a closed convex pointed cone of a locally convex linear space $Y$ with nonempty interior.

**Definition 2.2.1 ([56]).** The function $f : S \rightarrow Y$ is said to be $C$-subconvex on $S$ if there exists $v \in \text{int} C$ such that for any $t \in (0,1)$ and $\varepsilon > 0$,

$$\varepsilon v + tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in C, \ \forall x, y \in S.$$  

**Remark 2.2.1.** It is evident from the definition that $C$-convexity $\Rightarrow$ $C$-subconvexity. The converse however may not hold true as can be seen from the following example.
Example 2.2.1. Let \( f : S \to \mathbb{R}^2 \) where \( S = [-5, 5] \) be defined as
\[
f(x) = (x^2, -x^2), \quad C = \{(x, y) : -x \leq y, y \geq 0\}.
\]
Then \( f \) is \( C \)-subconvex. However, \( f \) fails to be \( C \)-convex, as for \( x = 1, y = -1, t = \frac{1}{2} \),
\[
 tf(x) + (1-t)f(y) - f(tx + (1-t)y) = (1, -1) \notin C
\]

Remark 2.2.2. Clearly \( C \)-subconvexity \( \Rightarrow \) \( C \)-subconvexlikeness, the converse however fails.

Example 2.2.2. Let \( f : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \) where
\[
 A = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 > 1\} \cup \{(1,0), (0,1)\},
\]
be defined as
\[
 f(x_1, x_2) = (2x_1, x_2 + 1); \quad C = \{(x, y) : -x \leq y \leq x, y \geq 0\}.
\]
Then \( f(A) + \text{int} C \) is convex and hence \( f \) is \( C \)-subconvexlike [62]. But as \( A \) is not a convex set, therefore \( f \) is not \( C \)-subconvex.

We now discuss optimality conditions for the following problem in terms of Gâteaux derivatives.

\[\text{(MP)}\quad \begin{aligned}
\text{C-minimize } & f(x) \\
\text{subject to } & -g(x) \in D,
\end{aligned}\]

where \( f : X \to Y \) and \( g : X \to Z \); \( X \) is a topological vector space and \( Y \) and \( Z \) are locally convex linear spaces. \( C \subseteq Y \) and \( D \subseteq Z \) are closed convex pointed cones with nonempty interiors.
Let $X_2 = \{x \in X : -g(x) \in D\}$ be the feasible set of (MP).

Hu and Ling [56] gave the following Kuhn-Tucker type necessary optimality conditions in terms of Gâteaux derivatives for the problem (MP) by assuming generalized Slater constraint qualification given below.

**Definition 2.2.2.** The constraint map $g$ is said to satisfy generalized Slater constraint qualification, if there exists $\hat{x} \in X_2$ such that $-g(\hat{x}) \in \text{int} D$.

**Theorem 2.2.1.** Let $(f, g)$ be Gâteaux differentiable at $\bar{x} \in X_2$ and $(C \times D)$-subconvex on $X$. If $\bar{x}$ is a weak minimum solution of (MP) and $g$ satisfies generalized Slater constraint qualification, then there exists $\bar{\lambda} \in C^* \setminus \{0\}$, $\bar{\mu} \in D^*$ such that

$$\left\langle \bar{\lambda}, f'_\bar{x}(x) \right\rangle + \left\langle \bar{\mu}, g'_\bar{x}(x) \right\rangle = 0, \quad \forall \ x \in X,$$

$$\left\langle \bar{\mu}, g(\bar{x}) \right\rangle = 0.$$

We now give the sufficient optimality conditions for a feasible solution to be a weak minimum of (MP) in the form of the following theorem.

**Theorem 2.2.2.** Let $(f, g)$ be Gâteaux differentiable at $\bar{x} \in X_2$ and $(C \times D)$-subconvex on $X$. If there exists $\bar{\lambda} \in C^* \setminus \{0\}$, $\bar{\mu} \in D^*$ such that

$$\left\langle \bar{\lambda}, f'_\bar{x}(x) \right\rangle + \left\langle \bar{\mu}, g'_\bar{x}(x) \right\rangle = 0, \quad \forall \ x \in X_2, \quad (2.18)$$

$$\left\langle \bar{\mu}, g(\bar{x}) \right\rangle = 0 \quad (2.19)$$

then $\bar{x}$ is a weak minimum of (MP).
**Proof.** Let if possible, \( \bar{x} \) be not a weak minimum of (MP).

Then, there exists \( x \in X_2 \) such that

\[
f(\bar{x}) - f(x) \in \text{int} C. \tag{2.20}
\]

Since \( f \) is \( C \)-subconvex on \( X \), therefore, there exists \( u \in \text{int} C \), such that for any \( t \in (0,1), \varepsilon > 0 \)

\[
\varepsilon tu + tf(x) + (1-t)f(\bar{x}) - f(tx + (1-t)\bar{x}) \in C,
\]

that is,

\[
\varepsilon u + f(x) - f(\bar{x}) - \left[ \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \right] \in C.
\]

Letting \( t \to 0^+ \), we get

\[
\varepsilon u + f(x) - f(\bar{x}) - f'_x(x - \bar{x}) \in C.
\]

Hence we have

\[
\varepsilon \left\langle \bar{\lambda}, u \right\rangle + \left\langle \bar{\lambda}, f(x) \right\rangle - \left\langle \bar{\lambda}, f(\bar{x}) \right\rangle - \left\langle \bar{\lambda}, f'_x(x - \bar{x}) \right\rangle \geq 0. \tag{2.21}
\]

Again since \( g \) is \( D \)-subconvex on \( X \), therefore there exists \( v \in \text{int} D \), such that for any \( t \in (0,1), \varepsilon' > 0 \)

\[
\varepsilon' \left\langle \bar{\mu}, v \right\rangle + \left\langle \bar{\mu}, g(x) \right\rangle - \left\langle \bar{\mu}, g(\bar{x}) \right\rangle - \left\langle \bar{\mu}, g'_x(x - \bar{x}) \right\rangle \geq 0. \tag{2.22}
\]

Adding (2.21) and (2.22) we get

\[
\varepsilon \left\langle \bar{\lambda}, u \right\rangle + \varepsilon' \left\langle \bar{\mu}, v \right\rangle + \left\langle \bar{\lambda}, f(x) - f(\bar{x}) \right\rangle + \left\langle \bar{\mu}, g(x) \right\rangle - \left\langle \bar{\mu}, g(\bar{x}) \right\rangle

- \left[ \left\langle \bar{\lambda}, f'_x(x - \bar{x}) \right\rangle + \left\langle \bar{\mu}, g'_x(x - \bar{x}) \right\rangle \right] \geq 0. \tag{2.23}
\]
From (2.18), (2.19) and the fact that $f'_x()$ and $g'_x()$ are linear, we have from (2.23)

$$\varepsilon\langle \bar{\lambda}, u \rangle + \varepsilon'\langle \bar{\mu}, v \rangle + \langle \bar{\lambda}, f(x) - f(\bar{x}) \rangle + \langle \bar{\mu}, g(x) \rangle \geq 0.$$  \hspace{1cm} (2.24)

Since, $x \in X_2$ is a feasible solution of (MP), therefore $-g(x) \in D$ which gives that

$$\langle \bar{\mu}, g(x) \rangle \leq 0.$$

Now from (2.24) we have

$$\varepsilon\langle \bar{\lambda}, u \rangle + \varepsilon'\langle \bar{\mu}, v \rangle + \langle \bar{\lambda}, f(x) - f(\bar{x}) \rangle \geq -\langle \bar{\mu}, g(x) \rangle \geq 0.$$

Since $\varepsilon, \varepsilon'$ are arbitrarily chosen, we get $\langle \bar{\lambda}, f(\bar{x}) - f(x) \rangle \leq 0$.

As $\bar{\lambda} \neq 0$, we have $f(\bar{x}) - f(x) \notin \text{int} C$, which is a contradiction to (2.20).

Hence $\bar{x}$ is a weak minimum of (MP). \hfill \Box

The following sufficient optimality conditions can be proved on the similar lines as those of Theorem 2.2.2.

**Theorem 2.2.3.** Let $(f, g)$ be Gâteaux differentiable at $\bar{x} \in X_2$ and $(C \times D)$-subconvex on $X$. If there exist $\bar{\lambda} \in C^* \setminus \{0\}$, $\bar{\mu} \in D^*$ such that

$$\langle \bar{\lambda}, f'_x(x - \bar{x}) \rangle + \langle \bar{\mu}, g'_x(x - \bar{x}) \rangle \geq 0, \quad \forall \ x \in X_2$$

$$\langle \bar{\mu}, g(x) \rangle = 0$$

Then $\bar{x}$ is a weak minimum of (MP).

In the next theorem we give sufficient optimality conditions for a feasible solution to be an efficient solution for (MP).
**Theorem 2.2.4.** Let \((f, g)\) be Gâteaux differentiable at \(\bar{x} \in X_2\) and \((C \times D)\) subconvex on \(X\). If there exist \(\bar{\lambda} \in C^*\), \(\bar{\mu} \in D^*\), \((\bar{\lambda}, \bar{\mu}) \neq 0\) such that (2.18) and (2.19) hold then \(\bar{x}\) is an efficient solution of (MP).

**Proof.** Let \(\bar{x}\) be not an efficient solution of (MP), then there exists \(x \in X_2\) such that \(f(\bar{x}) - f(x) \in C \setminus \{0_y\}\).

Since \(f\) is \(C\)-subconvex and \(g\) is \(D\)-subconvex on \(X\), therefore, proceeding on the lines of Theorem 2.2.2, we get

\[
\langle \lambda, f(\bar{x}) - f(x) \rangle \leq 0,
\]

that is \(f(\bar{x}) - f(x) \in C \setminus \{0_y\}\), which is a contradiction. Hence \(\bar{x}\) is an efficient solution of (MP).

We associate the following Mond-Weir type dual with the problem (MP) and study duality results.

\[
\text{(MD)} \quad \text{C-maximize } f(u)
\]

subject to \(\langle \lambda, f'(x - u) \rangle + \langle \mu, g'(x - u) \rangle \geq 0, \ \forall \ x \in X_2\)

\[
\langle \mu, g(u) \rangle \geq 0,
\]

\[
\mu \in D^*, \ 0 \neq \lambda \in C^*, \ u \in X.
\]

**Theorem 2.2.5 (Weak Duality Theorem).** Let \(\bar{x}\) be a feasible solution for (MP) and \((\bar{u}, \bar{\lambda}, \bar{\mu})\) be feasible for (MD), \((f, g)\) be Gâteaux differentiable at \(\bar{x} \in X_2\) and \((C \times D)\)-subconvex on \(X\) then,

\[
f(\bar{u}) - f(\bar{x}) \notin \text{int } C.
\]
Proof. Since \( g \) is \( D \)-subconvex on \( X \), therefore, there exists \( v \in \text{int} \, D \) such that for \( \bar{x}, \bar{u} \in X \), \( t \in (0,1) \) and \( g' > 0 \),

\[
\epsilon' t v + t g(\bar{x}) + (1-t) g(\bar{u}) - g(t\bar{x} + (1-t)\bar{u}) \in D,
\]
or,

\[
\epsilon' v + g(\bar{x}) - g(\bar{u}) - \left[ \frac{g(\bar{u} + t(\bar{x} - \bar{u})) - g(\bar{u})}{t} \right] \in D.
\]

Let \( t \to 0^+ \), as \( D \) is a closed cone, we get

\[
\epsilon' v + g(\bar{x}) - g(\bar{u}) - g'_\pi(\bar{x} - \bar{u}) \in D,
\]

which implies

\[
\epsilon' \langle \bar{\pi}, v \rangle + \langle \bar{\pi}, g(\bar{x}) \rangle - \langle \bar{\pi}, g(\bar{u}) \rangle - \langle \bar{\pi}, g'_\pi(\bar{x} - \bar{u}) \rangle \geq 0. \tag{2.25}
\]

Similarly as \( f \) is \( C \)-subconvex on \( X \), there exists \( w \in \text{int} \, C \), such that for \( \epsilon > 0 \),

\[
\epsilon \langle \bar{\lambda}, w \rangle + \langle \bar{\lambda}, f(\bar{x}) \rangle - \langle \bar{\lambda}, f(\bar{u}) \rangle - \langle \bar{\lambda}, f'_\pi(\bar{x} - \bar{u}) \rangle \geq 0. \tag{2.26}
\]

Adding (2.25) and (2.26) and using feasibility of \( \bar{x} \) and \( (\bar{u}, \bar{\lambda}, \bar{\pi}) \) for (MP) and (MD) respectively, we get

\[
\epsilon \langle \bar{\lambda}, w \rangle + \epsilon' \langle \bar{\pi}, v \rangle + \langle \bar{\lambda}, f(\bar{x}) - f(\bar{u}) \rangle \geq 0.
\]

Since \( \epsilon, \epsilon' \) are arbitrary, we get \( \langle \bar{\lambda}, f(\bar{u}) - f(\bar{x}) \rangle \leq 0 \).

As \( 0 \neq \bar{\lambda} \in C' \), we have \( f(\bar{u}) - f(\bar{x}) \notin \text{int} \, C \). \( \square \)

**Theorem 2.2.6 (Strong Duality Theorem).** Let \( (f, g) \) be Gâteaux differentiable and \( (C \times D) \)-subconvex on \( X \). Suppose that \( g \) satisfies the generalized Slater constraint qualification. If \( \bar{x} \) is a weak minimum
solution of (MP), then there exist \( 0 \neq \lambda \in C^*, \mu \in D^* \) such that \((\bar{x}, \lambda, \mu)\) is a weak maximum solution of (MD).

**Proof.** Since \( \bar{x} \) is a weak minimum of (MP), therefore by Theorem 2.2.1, there exist \( 0 \neq \lambda \in C^*, \mu \in D^* \) such that

\[
\langle \lambda, f'_x(x) \rangle + \langle \mu, g'_x(x) \rangle = 0 \quad \forall x \in X, \\
\langle \mu, g(\bar{x}) \rangle = 0.
\]

In particular, for \( \bar{x} \in X \)

\[
\langle \lambda, f'_x(\bar{x}) \rangle + \langle \mu, g'_x(\bar{x}) \rangle = 0
\]

\[
\Rightarrow \quad \langle \lambda, f'_x(x) - f'_x(\bar{x}) \rangle + \langle \mu, g'_x(x) - g'_x(\bar{x}) \rangle = 0
\]

\[
\Rightarrow \quad \langle \lambda, f'_x(x - \bar{x}) \rangle + \langle \mu, g'_x(x - \bar{x}) \rangle = 0
\]

Thus, \((\bar{x}, \lambda, \mu)\) is feasible for (MD).

Let if possible \( \bar{x} \) be not a weak maximum of (MD) then, there exists a feasible solution \((u, \lambda, \mu)\) of (MD) such that

\[
f(u) - f(\bar{x}) \in \text{int} C,
\]

which contradicts the Weak Duality Theorem (2.2.5) as \( \bar{x} \) is feasible for (MP) and \((u, \lambda, \mu)\) for (MD).

Thus \( \bar{x} \) must be a weak maximum of (MD). \( \square \)
2.3 Vector Optimization with Cone Semilocally Preinvex Functions

Preda [114] studied optimality and duality for a multiobjective fractional programming problem involving semilocally preinvex functions. Later Suneja et al. [138] introduced $\rho$-semilocally preinvex, $\rho$-semilocally quasi preinvex and $\rho$-semilocally pseudo preinvex functions and proved optimality conditions and duality results for a multiobjective non linear programming problem using the above defined functions. This section begins by introducing cone semilocally preinvex functions as follows:

Let $S \subseteq \mathbb{R}^n$ be an $\eta$-locally star shaped set at $\bar{x} \in S$, $K \subseteq \mathbb{R}^m$ be a closed convex cone with nonempty interior and let $f : S \to \mathbb{R}^m$ be a vector valued function.

**Definition 2.3.1.** The function $f$ is said to be $K$-semilocally preinvex ($K$-slpi) at $x \in S$ with respect to $\eta$ if corresponding to each $x \in S$, there exists a positive number $d_\eta(x,\bar{x}) \leq \alpha_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \text{ for } 0 < t < d_\eta(x,\bar{x}).$$

$f$ is said to be $K$-slpi on $S$ if it is $K$-slpi at each $\bar{x} \in S$.

The following theorem gives a characterization of cone semilocally preinvex functions.

**Theorem 2.3.1.** The function $f$ is $K$-semilocally preinvex with respect to $\eta$ if and only if its epigraph $\text{Epi}(f) = \{(x, y) : x \in S, y \in f(x)+K\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is $\eta$-locally star shaped in the first component and locally star shaped in the second component.
Proof. First suppose that $f$ is $K$-slpi on $S$, with respect to $\eta$.

Let $(x_1,y_1)$ and $(x_2,y_2) \in \text{Epi}(f)$, then

$$x_1, x_2 \in S, \quad y_1 \in f(x_1) + K \text{ and } y_2 \in f(x_2) + K,$$

which implies that $y_1 = f(x_1) + k_1$ and $y_2 = f(x_2) + k_2$ where $k_1, k_2 \in K$.

Since $f$ is $K$-slpi on $S$, there exists a positive number $a_\eta(x_1,x_2) \leq 1$ such that

$x_2 + t\eta(x_1,x_2) \in S$ for $0 < t < a_\eta(x_1,x_2)$ and there exists a positive number $d_\eta(x_1,x_2) \leq a_\eta(x_1,x_2)$ such that

$$tf(x_1) + (1-t)f(x_2) - f(x_2 + t\eta(x_1,x_2)) \in K, \quad \text{for } 0 < t < d_\eta(x_1,x_2)$$

$$\Rightarrow t(y_1 - k_1) + (1-t)(y_2 - k_2) - f(x_2 + t\eta(x_1,x_2)) \in K$$

$$\Rightarrow ty_1 + (1-t)y_2 - f(x_2 + t\eta(x_1,x_2)) \in K + (tk_1 + (1-t)k_2) = K$$

$$\Rightarrow ty_1 + (1-t)y_2 \in f(x_2 + t\eta(x_1,x_2)) + K$$

$$\Rightarrow (x_2 + t\eta(x_1,x_2), ty_1 + (1-t)y_2) \in \text{Epi}(f), \quad \text{for } 0 < t < d_\eta(x_1,x_2)$$

$$\Rightarrow \text{Epi}(f) \text{ is } \eta\text{-locally star shaped in first component and locally star shaped in second component.}$$

Conversely, let $\text{Epi}(f)$ be $\eta$-locally star shaped in the first component and locally star shaped in the second component. Let $x_1, x_2 \in S$, then

$$(x_1, f(x_1)),(x_2,f(x_2)) \in \text{Epi}(f).$$

Thus there exists a positive number $a_\eta(x_1,x_2) \leq 1$ such that
\[(x_2 + t\eta(x_1,x_2), tf(x_1) + (1-t)f(x_2)) \in \text{Epi}(f), \quad \text{for } 0 < t < a_\eta(x_1,x_2)\]

\[\Rightarrow \quad tf(x_1) + (1-t)f(x_2) \in f(x_2 + t\eta(x_1,x_2)) + K\]

\[\Rightarrow \quad tf(x_1) + (1-t)f(x_2) - f(x_2 + t\eta(x_1,x_2)) \in K, \quad \text{for } 0 < t < a_\eta(x_1,x_2).\]

Hence \(f\) is \(K\)-slpi on \(S\).

\[\square\]

**Remark 2.3.1.** If \(m = n\) and \(K = R^n\) then \(K\)-semilocally preinvex functions reduce to semilocally preinvex functions defined by Preda \[114\]. We now give an example of a function which is \(K\)-slpi but fails to be slpi.

**Example 2.3.1.** Consider the set \(S = R \setminus E\) where \(E = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{2\}\).

Thus the set \(S\) is \(\eta\)-locally starshaped, where

\[\eta(x,\bar{x}) = \begin{cases} x - \bar{x}, & x, \bar{x} > \frac{1}{2}, x \neq 2, \bar{x} \neq 2 \quad \text{or} \quad x, \bar{x} < -\frac{1}{2} \\ \bar{x} - x, & x > \frac{1}{2}, x \neq 2, \bar{x} < -\frac{1}{2} \quad \text{or} \quad \bar{x} > \frac{1}{2}, \bar{x} \neq 2, x < -\frac{1}{2} \end{cases}\]

and

\[a_\eta(x,\bar{x}) = \begin{cases} \frac{2-x}{x-\bar{x}}, & \text{if } \frac{1}{2} < \bar{x} < 2, 2 < x \quad \text{or} \quad \frac{1}{2} < x < 2, x < -\frac{1}{2} \\ \frac{\bar{x}-2}{\bar{x}-x}, & \text{if } 2 < \bar{x}, \frac{1}{2} < x < 2 \\ 1, & \text{elsewhere} \end{cases}\]

Consider the function \(f : S \rightarrow R^2\) defined by

\[f(x) = \begin{cases} (x,0), & \frac{1}{2} < x, x \neq 2 \\ (0,-x), & x < -\frac{1}{2} \end{cases}\]
Then, $f$ is $K$-slpi at $\bar{x} = -1$, where $K = \{(x, y) : y \leq 0, \ y \leq -x\}$; because

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad \text{for } 0 < t < d_{\eta}(x,\bar{x}) = a_{\eta}(x,\bar{x}).$$

The function $f$ fails to be slpi at $\bar{x} = -1$ because for $x = 1$, there does not exist any positive number $d_{\eta}(x,\bar{x}) \leq a_{\eta}(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \geq 0 \quad \text{for } 0 < t < d_{\eta}(x,\bar{x}).$$

**Remark 2.3.2.** If $\eta(x, \bar{x}) = x - \bar{x}$ then $K$-semilocally preinvex functions reduce to $K$-semilocally convex functions defined by Weir [145].

We now give an example of a $K$-slpi function which fails to be $K$-semilocally convex.

**Example 2.3.2.** The function $f$ considered in Example 2.3.1 is $K$-slpi at $\bar{x} = -1$ but it fails to be $K$-semilocally convex at $\bar{x} = -1$ because for $x = 1$, there does not exist any positive real number $d < 1$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(tx + (1-t)\bar{x}) \in K \quad \text{for } 0 < t < d.$$

**Theorem 2.3.2.** Let $f : S \to R^m$ be $K$-slpi on an $\eta$-locally star shaped set $S \subseteq R^n$. Then the set $Z = f(S) + K$ is locally star shaped.

**Proof.** Let $z, \bar{x} \in Z$, then there exist $x, \bar{x} \in S, k, \bar{k} \in K$ such that

$$z = f(x) + k, \quad \bar{x} = f(\bar{x}) + \bar{k}.$$  \hspace{1cm} (2.27)

Since $S$ is an $\eta$-locally star shaped set and $x, \bar{x} \in S$, therefore there exists a maximum positive number $a_{\eta}(x, \bar{x}) \leq 1$ such that

$$\bar{x} + t\eta(x, \bar{x}) \in S \quad \text{for } 0 < t < a_{\eta}(x, \bar{x}).$$
As $f$ is $K$-slpi on $S$, there exists a positive number $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x, \bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x, \bar{x})$$

$$\Rightarrow \quad tf(x) + (1-t)f(\bar{x}) \in f(\bar{x} + t\eta(x, \bar{x})) + K.$$

On using (2.27) we have

$$t(z - k) + (1-t)(\bar{z} - \bar{k}) \in f(S) + K$$

$$\Rightarrow \quad tz + (1-t)\bar{z} \in f(S) + tk + (1-t)\bar{k} + K$$

$$\Rightarrow \quad tz + (1-t)\bar{z} \in f(S) + K \quad \text{for } 0 < t < d_\eta(x, \bar{x}),$$

as $K$ is a convex cone.

Hence $Z$ is locally star shaped.

\[\square\]

**Theorem 2.3.3.** Let $f$ be $K$-slpi on $S$ with respect to $\eta$ then for each $y \in \mathbb{R}^m$, the set $S_f(y) = \{x \in S : y \in f(x) + K\}$ is $\eta$-locally star shaped.

**Proof.** Let $x, \bar{x} \in S_f(y), y \in \mathbb{R}^m$, then $x, \bar{x} \in S$ such that

$$y \in f(x) + K \quad \text{and} \quad y \in f(\bar{x}) + K,$$

Thus there exist $k, \bar{k} \in K$ such that

$$y = f(x) + k \quad \text{and} \quad y = f(\bar{x}) + \bar{k}. \quad \text{(2.28)}$$

Since $f$ is $K$-slpi with respect to $\eta$ therefore there exists a positive number $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$ where $\eta : S \times S \rightarrow \mathbb{R}^n$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x, \bar{x})) \in K, \quad 0 < t < d_\eta(x, \bar{x})$$
On using (2.28) we get
\[ t(y - k) + (1 - t)(y - \bar{k}) - f(\bar{x} + t\eta(x, \bar{x})) \in K, \quad 0 < t < d_\eta(x, \bar{x}) \]

\[ \Rightarrow \quad y - (tk + (1 - t)\bar{k}) - f(\bar{x} + t\eta(x, \bar{x})) \in K \]

\[ \Rightarrow \quad y - f(\bar{x} + t\eta(x, \bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x, \bar{x}) \]

\[ \Rightarrow \quad \bar{x} + t\eta(x, \bar{x}) \in S_f(y), \quad \text{for } 0 < t < d_\eta(x, \bar{x}) \]

Hence \( S_f(y) \) is \( \eta \)-locally star shaped.

We now give the definition of \( \eta \)-semi differentiable function.

**Definition 2.3.2.** The function \( f: S \rightarrow \mathbb{R}^m \) is said to be \( \eta \)-semi differentiable at \( x \in S \) if

\[ (df)'(\bar{x}, \eta(x, \bar{x})) = \lim_{t \to 0^+} \frac{1}{t} [f(\bar{x} + t\eta(x, \bar{x})) - f(\bar{x})] \]

exists for each \( x \in S \).

**Remark 2.3.3.**

(1) If \( \eta(x, \bar{x}) = x - \bar{x} \) then \( \eta \)-semi differentiability of \( f \) reduces to (one sided) directional differentiability of \( f \) at \( \bar{x} \) in the direction \( x - \bar{x} \), as considered by Weir [145].

(2) If \( m = 1 \) and \( \eta(x, \bar{x}) = x - \bar{x} \), then \( \eta \)-semi differentiability reduces to semi differentiability [72].

Let \( f: S \rightarrow \mathbb{R}^m \) be \( \eta \)-semi differentiable at \( \bar{x} \in S \).

In the following result, we give another property of \( K \)-slpi functions.

**Theorem 2.3.4.** If \( f \) is \( K \)-slpi at \( \bar{x} \) then \( f(x) - f(\bar{x}) - (df)'(\bar{x}, \eta(x, \bar{x})) \in K \) \( \forall x \in S \).
Proof. Since the function $f$ is $K$-slpi at $\bar{x}$ with respect to $\eta$, therefore corresponding to each $x \in S$ there exists a positive number $d_\eta(x,\bar{x}) \leq a_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \text{ for } 0 < t < d_\eta(x,\bar{x})$$

which implies

$$f(x) - f(\bar{x}) - \frac{1}{t}[f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})] \in K, \text{ for } 0 < t < d_\eta(x,\bar{x})$$

Since $K$ is a closed cone, therefore taking limit as $t \to 0^+$, we get

$$f(x) - f(\bar{x}) - (df)^*(\bar{x},\eta(x,\bar{x})) \in K, \quad \forall x \in S.$$  

We now introduce semilocally naturally quasi preinvex functions over cones.

**Definition 2.3.3.** The function $f$ is said to be $K$-semilocally naturally quasi preinvex ($K$-slnqpi) at $\bar{x}$, with respect to $\eta$ if

$$-(f(x) - f(\bar{x})) \in K \quad \Rightarrow \quad -(df)^*(\bar{x},\eta(x,\bar{x})) \in K.$$  

**Remark 2.3.4.** If $m = n$ and cone $K = R^n_+$, $K$-slnqpi functions reduce to slqpi functions defined by Preda [114].

**Theorem 2.3.5.** If the set $S_f(y) = \{x \in S : y \in f(x) + K\}$ is $\eta$-locally star shaped for each $y \in R^m$, then $f$ is $K$-semilocally naturally quasi preinvex on $S$ with respect to same $\eta$.

**Proof.** Let $S_f(y)$ be $\eta$-locally star shaped for each $y \in R^m$.

Let $x, \bar{x} \in S$ such that $-(f(x) - f(\bar{x})) \in K$.

Denoting $y = f(\bar{x})$, we get $-(f(x) - y) \in K$.  

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\[ y \in f(x) + K, \]
\[ x \in S_f(y) \]

Also \( \bar{x} \in S_f(y) \) as \( 0 \in K \).

Since \( S_f(y) \) is \( \eta \)-locally star shaped, therefore there exists a maximum positive number \( a_\eta(x, \bar{x}) \leq 1 \) such that

\[ \bar{x} + t\eta(x, \bar{x}) \in S_f(y), \quad \text{for } 0 < t < a_\eta(x, \bar{x}) \]

which implies,

\[ y \in f(\bar{x}+t\eta(x, \bar{x}))+K \quad \text{for } 0 < t < a_\eta(x, \bar{x}). \]

Thus,

\[ f(\bar{x}) \in f(\bar{x}+t\eta(x, \bar{x}))+K, \quad \text{for } 0 < t < a_\eta(x, \bar{x}) \]
\[ \Rightarrow -(f(\bar{x}+t\eta(x, \bar{x}))-f(\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x, \bar{x}). \]
\[ \Rightarrow \frac{-1}{t}(f(\bar{x}+t\eta(x, \bar{x}))-f(\bar{x})) \in K, \quad 0 < t < d_\eta(x, \bar{x}) \]

Since \( K \) is a closed cone, therefore taking limit as \( t \to 0^+ \), we get

\[ -(df)^+(\bar{x}, \eta(x, \bar{x})) \in K \]

Thus \( -(f(x)-f(\bar{x})) \in K \quad \Rightarrow \quad -(df)^+(\bar{x}, \eta(x, \bar{x})) \in K \).

Hence \( f \) is \( K \)-slqpi on \( S \).

**Theorem 2.3.6.** If \( f \) is \( K \)-slpi at \( \bar{x} \in S \) with respect to \( \eta \) then \( f \) is \( K \)-slqpi at \( \bar{x} \), with respect to same \( \eta \).
**Proof.** Let $f$ be $K$-slpi at $\bar{x}$, then there exists a positive number $d_\eta(x,\bar{x}) \leq \alpha_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x}). \quad (2.29)$$

Suppose that,

$$-(f(x) - f(\bar{x})) \in K$$

then,

$$-t(f(x) - f(\bar{x})) \in K, \quad t > 0. \quad (2.30)$$

Adding (2.29) and (2.30) we have

$$-[f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})] \in K, \quad 0 < t < d_\eta(x,\bar{x}).$$

$$\Rightarrow \quad \frac{-1}{t}(f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x})$$

Since $K$ is a closed cone, therefore taking limit as $t \to 0^+$, we get

$$-(df)(\bar{x},\eta(x,\bar{x})) \in K$$

Thus

$$-(f(x) - f(\bar{x})) \in K \quad \Rightarrow \quad -(df)(\bar{x},\eta(x,\bar{x})) \in K.$$  

Hence $f$ is $K$-slnqpi at $\bar{x}$ with respect to same $\eta$.

The converse of the above theorem may not hold as can be seen from the following example.

**Example 2.3.3.** Consider the set $S = \mathbb{R} \setminus E$, where $E = \left[ \begin{array}{c} -1/2 \\ -1/2 \end{array} \right] \cup \{2\}$. Then as discussed in Example 2.3.1, $S$ is $\eta$-locally starshaped.
Consider the function \( f : S \to \mathbb{R}^2 \) defined by
\[
f(x) = \begin{cases} 
(-x^2, 0), & x < -\frac{1}{2} \\
(0, -x), & x > \frac{1}{2}, \ x \neq 2.
\end{cases}
\]

The function \( f \) is \( K \)-slqpi at \( \bar{x} = -2 \), where \( K = \{(x, y) : y \leq 0, y \geq x\} \), because
\[
-(f(x) - f(\bar{x})) \in K
\]
\[
\Rightarrow -2 \leq x < -\frac{1}{2}
\]
\[
\Rightarrow -(df)^+(\bar{x}, \eta(x, \bar{x})) = (-4(2) + 2, 0) \in K
\]

The function \( f \) fails to be \( K \)-slpi at \( \bar{x} = -2 \) by Theorem 2.3.4, because for \( x = 1 \)
\[
f(x) - f(\bar{x}) - (df)^+(\bar{x}, \eta(x, \bar{x})) = (16, -1) \notin K
\]

**Definition 2.3.4.** The function \( f : S \to \mathbb{R}^m \) is said to be \( K \)-semilocally quasi preinvex (\( K \)-slqpi) at \( \bar{x} \) with respect to \( \eta \), if
\[
(f(x) - f(\bar{x})) \notin \text{int} K \quad \Rightarrow \quad -(df)^+(\bar{x}, \eta(x, \bar{x})) \in K.
\]

**Theorem 2.3.7.** If \( K \) is a pointed cone and \( f \) is \( K \)-slqpi at \( \bar{x} \) then \( f \) is \( K \)-slqpi at \( \bar{x} \) with respect to same \( \eta \).

**Proof.** Let \( K \) be a pointed cone and \( f \) be \( K \)-slqpi at \( \bar{x} \) with respect to \( \eta \), then,
\[
(f(x) - f(\bar{x})) \notin \text{int} K \quad \Rightarrow \quad -(df)^+(\bar{x}, \eta(x, \bar{x})) \in K . \quad (2.31)
\]
Suppose that

$-(f(x) - f(\bar{x})) \in K$. \hspace{1cm} (2.32)

Since $K$ is pointed, $K \cap (-K) = \{0\}$

$\Rightarrow$ \hspace{1cm} int $K \cap (-K) = \emptyset$

$\Rightarrow$ \hspace{1cm} $-K \subset R^n \setminus \text{int } K$.

In view of (2.32) we get \hspace{1cm} $f(x) - f(\bar{x}) \notin \text{int } K$.

Thus by (2.31) we have \hspace{1cm} $-(df)^+(\bar{x}, \eta(x, \bar{x})) \in K$.

Hence $f$ is $K$-slqpi.

The converse of the above theorem may not hold, as can be seen by the following example.

**Example 2.3.4.** The function $f$ considered in Example 2.3.3 is $K$-slqpi at $\bar{x} = -2$. But $f$ fails to be $K$-slqpi at $\bar{x} = -2$ because for $x = 1$

$$f(x) - f(\bar{x}) = (4, -1) \notin \text{int } K$$

where as \hspace{1cm} $-(df)^+(\bar{x}, \eta(x, \bar{x})) = (12, 0) \notin K$.

The next definition introduces cone semilocal pseudo preinvex functions.

**Definition 2.3.5.** The function $f : S \to R^n$ is said to be $K$-semilocally pseudo preinvex ($K$-slppi) at $\bar{x}$, with respect to $\eta$ if

$-(df)^+(\bar{x}, \eta(x, \bar{x})) \in K \Rightarrow -(f(x) - f(\bar{x})) \in \text{int } K$

We consider the vector optimization problem
(VOP) \( K \)-minimize \( f(x) \)

subject to \( -g(x) \in Q \)

\[ h(x) = \{0_{\ell^k}\} \]

where \( f: S \to R^m \), \( g: S \to R^p \) and \( h: S \to R^k \) are \( \eta \)-semi differentiable functions with respect to same \( \eta \) and \( S \subseteq R^n \) is a nonempty \( \eta \)-locally star shaped set.

Let \( K \subseteq R^m \) and \( Q \subseteq R^p \) be closed convex cones having nonempty interior and let \( X_3 = \{x \in S : -g(x) \in Q, h(x) = \{0_{\ell^k}\}\} \) be the set of all feasible solutions of (VOP).

Let \( F_0 = (f,g,h): S \to S' \) where \( S \subseteq R^n \) is a nonempty set and \( S' = R^n \times R^p \times R^k \) and \( K_0 = (K \times Q \times \{0_{\ell^k}\}) \). If \( F_0 \) is \( K_0 \)-slpi on \( S \), that is, \( f \) is \( K \)-slpi, \( g \) is \( Q \)-slpi and \( h \) is \( \{0_{\ell^k}\} \)-slpi with respect to same \( \eta \), then by Theorem 2.3.2, \( F_0(S) + K_0 \) is locally starshaped. If we assume \( F_0(S) + K_0 \) to be a closed set then \( F_0(S) + K_0 \) becomes convex and the following alternative theorem follows on the lines of Illés and Kassay [60].

**Theorem 2.3.8 (Theorem of Alternative).** Let \( F_0 \) be \( K_0 \)-slpi on \( S \) such that \( F_0(S) + K_0 \) is closed with nonempty interior, then the following assertions hold:

(i) if there is no \( x \in S \) such that \( f(x) \in -\text{int} \ K, \ g(x) \in -Q \) and \( h(x) = \{0_{\ell^k}\} \), then there exist \( \lambda \in K^+, \ \mu \in Q^+ \) and \( v \in R^k \) with \( (\lambda, \mu, v) \neq 0 \) such that

\[ \lambda^T f(x) + \mu^T g(x) + v^T h(x) \geq 0, \ \forall \ x \in S. \]
(ii) if there exist \( \lambda \in K^+ \setminus \{0\} , \mu \in Q^+ \) and \( \nu \in R^k \) such that
\[
\lambda^T f(x) + \mu^T g(x) + \nu^T h(x) \geq 0 , \ \forall \ x \in S ,
\]
then there is no \( x \in S \) such that
\[
f(x) \in -\text{int} \ K , \ g(x) \in -Q \text{ and } h(x) = \{0_{R^k}\} .
\]

We shall be using the following constraint qualification to prove the necessary optimality conditions for (VOP).

**Definition 2.3.6.** The constraint pair \((g, h)\) is said to satisfy **generalized Slater type constraint qualification** at \( \bar{x} \) if \((g, h)\) is \((Q \times \{0_{R^k}\})\)-slpi at \( \bar{x} \) and there exists \( x^* \in S \) such that

\[
-g(x^*) \in \text{int} \ Q \text{ and } h(x^*) = \{0_{R^k}\} .
\]

We now establish the necessary optimality conditions for (VOP).

**Theorem 2.3.9 (Necessary Optimality Conditions).**

Let \( F_1(x) = (f(x) - f(\bar{x}), g(x), h(x)) \ \forall \ x \in S \) and \( F_1(S) + (K \times Q \times \{0_{R^k}\}) \) be closed with nonempty interior. Let \( \bar{x} \in X_3 \) be a weak minimum of (VOP), \( f \) be \( K\)-slpi, \( g \) be \( Q\)-slpi and \( h \) be \( \{0_{R^k}\}\)-slpi with respect to same \( \eta \).

Suppose that the pair \((g, h)\) satisfies generalized Slater type constraint qualification and \( \eta(\bar{x}, \bar{x}) = 0 \), then there exist \( 0 \neq \bar{\lambda} \in K^+ , \bar{\mu} \in Q^+ \) and \( \nu \in R^k \) such that
\[
\bar{\lambda}^T (df)^*(\bar{x}, \eta(x, \bar{x}))) + \bar{\mu}^T (dg)^*(\bar{x}, \eta(x, \bar{x})) + \nu^T (dh)^*(\bar{x}, \eta(x, \bar{x})) \geq 0 ,
\]
for all \( x \in S \). \hspace{1cm} (2.33)

and \( \bar{\mu}^T g(\bar{x}) = 0 \). \hspace{1cm} (2.34)
Proof. Since \( \bar{x} \) is a weak minimum of (VOP), therefore there does not exist any \( x \in S \) such that

\[
f(x) - f(\bar{x}) \in -\text{int} \ K
\]

\[
g(x) \in -Q
\]

and \( h(x) = \{0 \}_{K^*} \).

Then by Theorem 2.3.8, there exist \( \bar{\lambda} \in K^+, \ \bar{\mu} \in Q^+, \ \bar{\nu} \in \mathbb{R}^k, (\bar{\lambda}, \bar{\mu}, \bar{\nu}) \neq 0 \) such that

\[
\bar{\lambda}^T (f(x) - f(\bar{x})) + \bar{\mu}^T g(x) + \bar{\nu}^T h(x) \geq 0, \quad \text{for all } x \in S
\]

\[
\Rightarrow \quad \bar{\lambda}^T f(x) + \bar{\mu}^T g(x) + \bar{\nu}^T h(x) \geq \bar{\lambda}^T f(\bar{x}) \quad \text{for all } x \in S \quad (2.35)
\]

Now, \( \bar{\mu} \in Q^+ \) and \(-g(\bar{x}) \in Q\), therefore \( \bar{\mu}^T g(\bar{x}) \leq 0 \).

By taking \( x = \bar{x} \) in (2.35) and using \( h(\bar{x}) = 0 \), we get \( \bar{\mu}^T g(\bar{x}) \geq 0 \).

Thus

\[
\bar{\mu}^T g(\bar{x}) = 0. \quad (2.36)
\]

From (2.35), (2.36) and \( h(\bar{x}) = 0 \), we have

\[
(\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(x) - (\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(\bar{x}) \geq 0, \quad \text{for all } x \in S
\]

As \( \bar{x} + t\eta(x, \bar{x}) \in S \), for \( 0 < t < a_\eta(x, \bar{x}) \) we have

\[
(\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(\bar{x} + t\eta(x, \bar{x})) - (\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(\bar{x}) \geq 0
\]

which can be rewritten as,

\[
\bar{\lambda}^T (f(\bar{x} + t\eta(x, \bar{x})) - f(\bar{x})) + \bar{\mu}^T (g(\bar{x} + t\eta(x, \bar{x})) - g(\bar{x}))
\]

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\[ + \nabla^T (h(\bar{x} + t\eta(x,\bar{x})) - h(\bar{x})) \geq 0 \]

Dividing by \( t > 0 \) and taking limit as \( t \to 0^+ \) we get

\[ \bar{\lambda}^T (df)^+(\bar{x}, \eta(x,\bar{x})) + \bar{\mu}^T (dg)^+(\bar{x}, \eta(x,\bar{x})) + \bar{\nu}^T (dh)^+(\bar{x}, \eta(x,\bar{x})) \geq 0, \]

for all \( x \in S \). \hfill (2.37)

Next, let if possible \( \bar{\lambda} = 0 \), then (2.37) reduces to,

\[ \bar{\mu}^T (dg)^+(\bar{x}, \eta(x,\bar{x})) + \bar{\nu}^T (dh)^+(\bar{x}, \eta(x,\bar{x})) \geq 0 \quad \text{for all } x \in S. \hfill (2.38) \]

Since \((g, h)\) is \((Q \times \{0_{\bar{\mu}_k}\})\)-slpi at \( \bar{x} \), therefore we have for every \( x \in S \),

\[ g(x) - g(\bar{x}) - (dg)^+(\bar{x}, \eta(x,\bar{x})) \in Q \]

and \( h(x) - h(\bar{x}) - (dh)^+(\bar{x}, \eta(x,\bar{x})) = \{0_{\bar{\mu}_k}\} \)

\[ \Rightarrow \bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}) - \bar{\mu}^T (dg)^+(\bar{x}, \eta(x,\bar{x})) \geq 0 \hfill (2.39) \]

and \( \nabla^T h(x) - \nabla^T h(\bar{x}) - \nabla^T (dh)^+(\bar{x}, \eta(x,\bar{x})) = 0 \). \hfill (2.40)

Adding (2.39), (2.40) and using (2.36) and \( h(\bar{x}) = 0_{\bar{\mu}_k} \), we get

\[ \bar{\mu}^T g(x) + \nabla^T h(x) - \bar{\mu}^T (dg)^+(\bar{x}, \eta(x,\bar{x})) - \nabla^T (dh)^+(\bar{x}, \eta(x,\bar{x})) \geq 0, \]

for all \( x \in S \). \hfill (2.41)

On using (2.38) we obtain,

\[ \bar{\mu}^T g(x) + \nabla^T h(x) \geq 0, \quad \text{for all } x \in S. \hfill (2.42) \]

Now, by generalized Slater type constraint qualification, there exists \( x^* \in S \) such that,
\[-g(x^*) \in \text{int } Q \quad \text{and} \quad h(x^*) = \{0_{\mu^*}\}\]

which implies \(\bar{\mu}^T g(x^*) < 0\) and \(\nabla^T h(x^*) = 0\).

Adding the above we have

\[
\bar{\mu}^T g(x^*) + \nabla^T h(x^*) < 0
\]

which is a contradiction to (2.42).

Hence \(\bar{\lambda} \neq 0\). \(\square\)

The following theorem establishes sufficiency result for (VOP).

**Theorem 2.3.10 (Sufficient Optimality Conditions).** Let \(\bar{x} \in X_3\) and \(f\) be \(K\)-slppi, \(g\) be \(Q\)-slqpi and \(h\) be \(\{0_{\mu^*}\}\)-slnqpi at \(\bar{x}\) with respect to same \(\eta\). If there exist \(0 \neq \bar{\lambda} \in K^+, \bar{\mu} \in Q^+\) and \(\bar{v} \in R^k\) such that (2.33) holds \(\forall x \in X_3\) and (2.34) is satisfied, then \(\bar{x}\) is a weak minimum of (VOP).

**Proof.** Let \(x\) be feasible for (VOP) then \(-g(x) \in Q\).

On using \(\bar{\mu} \in Q^+\), we get \(\bar{\mu}^T g(x) \leq 0\). \(\quad \text{(2.44)}\)

In view of (2.34), (2.44) can be written as \(\bar{\mu}^T (g(x) - g(\bar{x})) \leq 0\) \(\quad \text{(2.45)}\)

If \(\bar{\mu} \neq 0\), then from (2.45), \(g(x) - g(\bar{x}) \notin \text{int } Q\)

Since \(g\) is \(Q\)-slqpi at \(\bar{x}\), we get

\[-(dg)^*(\bar{x}, \eta(x, \bar{x})) \in Q\]

which gives,

\[
\bar{\mu}^T (dg)^*(\bar{x}, \eta(x, \bar{x})) \leq 0 . \quad \text{(2.46)}
\]

If \(\bar{\mu} = 0\) then (2.46) holds trivially.
Again for $x \in X_3$, $h(x) = \{0_{R^+}\}$, therefore

$$-(h(x) - h(\overline{x})) = \{0_{R^+}\}$$

Since $h$ is $\{0_{R^+}\}$-slnqpi at $\overline{x}$ we have $-(dh)^+(\overline{x}, \eta(x, \overline{x})) = \{0_{R^+}\}$.

Therefore

$$\nabla^T (dh)^+(\overline{x}, \eta(x, \overline{x})) = 0$$

Adding (2.46) and (2.47) and using (2.33) we get $\overline{\lambda}^T (df)^+(\overline{x}, \eta(x, \overline{x})) \geq 0$.

Since $\overline{\lambda} \neq 0$, we obtain

$$-(df)^+(\overline{x}, \eta(x, \overline{x})) \notin \text{int} K.$$ 

As $f$ is $K$-slppi we get

$$-(f(x) - f(\overline{x})) \in \text{int} K,$$ 

that is $f(\overline{x}) - f(x) \notin \text{int} K$.

Since $x \in X_3$ is arbitrarily chosen, therefore $\overline{x}$ is a weak minimum of (VOP).

The following Mond-Weir type dual is associated with the primal problem (VOP).

(VOD) $K$-maximize $f(u)$

subject to

$$\lambda^T (df)^+(u, \eta(x,u)) + \mu^T (dg)^+(u, \eta(x,u)) + v^T (dh)^+(u, \eta(x,u)) \geq 0$$

$$\forall x \in X_3$$

$$\mu^T g(u) \geq 0$$

$$h(u) = 0_{R^+}$$
0 \neq \lambda \in K^+, \mu \in Q^+ \text{ and } \nu \in R^k, u \in S

**Theorem 2.3.11 (Weak Duality).** Let $x$ be feasible for (VOP) and $(u, \lambda, \mu, \nu)$ be feasible for (VOD). Let $f$ be $K$-slppi, $g$ be $Q$-slqpi and $h$ be $\{0_R\}$-slnqpi at $u$, with respect to same $\eta$. Then

$$f(u) - f(x) \not\in \text{int } K.$$

**Proof.** Since $x$ is feasible for (VOP) and $(u, \lambda, \mu, \nu)$ feasible for (VOD) therefore we have,

$$\mu^T g(x) \leq 0 \text{ and } \mu^T g(u) \geq 0$$

which implies

$$\mu^T (g(x) - g(u)) \leq 0.$$  

If $\mu \neq 0$, then the above inequality results in $g(x) - g(u) \not\in \text{int } Q$.

Since $g$ is $Q$-slqpi we have,

$$-(dg)^+(u, \eta(x, u)) \in Q$$

That is,

$$\mu^T (dg)^+(u, \eta(x, u)) \leq 0. \tag{2.51}$$

(2.51) also holds if $\mu = 0$.

Again using feasibility of $x$ and $u$, we have $-(h(\bar{x}) - h(u)) = \{0_R\}$.

As $h$ is $\{0_R\}$-slnqpi at $u$, we get

$$-(dh)^+(u, \eta(x, u)) = \{0_R\}$$
therefore, $v^T(dh)^*(u,\eta(x,u)) = 0$. \hfill (2.52)

Using (2.51), (2.52) in (2.48) we obtain

$$\lambda^T(df)^*(u,\eta(x,u)) \geq 0.$$ 

As, $0 \neq \lambda \in K^+$ we have

$$-(df)^*(u,\eta(x,u)) \notin \text{int } K,$$

Because $f$ is $K$-slppi at $u$, we get $-(f(x) - f(u)) \notin \text{int } K$ which gives \hfill (f(u) - f(x)) \notin \text{int } K.

**Theorem 2.3.12 (Strong Duality).** Let $f$ be $K$-slpi, $g$ be $Q$-slpi and $h$ be $\{0,_{\text{R}}\}$-slpi with respect to same $\eta$. Let $F_1(S) + (K \times Q \times \{0,_{\text{R}}\})$ be closed with nonempty interior. Suppose that the pair $(g, h)$ satisfies generalized Slater type constraint qualification. If $\bar{x}$ is a weak minimum of (VOP) and $\eta(\bar{x},\bar{x}) = 0$, then there exist $0 \neq \lambda^* \in K^+$, $\mu^* \in Q^+$ and $\nu^* \in R^k$ such that $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is a feasible solution of (VOD). Moreover if the conditions of Weak Duality Theorem 2.3.11 are satisfied for all feasible solutions of (VOP) and (VOD) then $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is a weak maximum of (VOD).

**Proof.** Since $\bar{x}$ is a weak minimum of (VOP), therefore by Theorem 2.3.9, there exist $0 \neq \lambda^* \in K^+$, $\mu^* \in Q^+$, $\nu^* \in R^k$ such that

$$\lambda^*^T(df)^*(\bar{x},\eta(x,\bar{x})) + \mu^*^T(dg)^*(\bar{x},\eta(x,\bar{x})) + \nu^*^T(dh)^*(\bar{x},\eta(x,\bar{x})) \geq 0, \quad \forall \ x \in S$$

and

$$\mu^*^Tg(\bar{x}) = 0.$$ 

Thus $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is feasible for (VOD). Let if possible, $(\bar{x},\lambda^*,\mu^*,\nu^*)$ be not a weak maximum of (VOD). Then there exists $(u,\lambda,\mu,\nu)$ feasible for
2.4 Optimality and Duality in Vector Optimization Involving Generalized Type-I Functions Over Cones

Based on the work of Craven [25], Reiland [117] extended the concept of invexity to nonsmooth Lipschitz functions. Yen and Sach [159] defined cone invex and cone invex in the limit. Suneja et al. [137] called these functions cone generalized invex and cone nonsmooth invex. They further introduced the concepts of cone nonsmooth quasi invex, cone nonsmooth pseudo invex and other related functions in terms of Clarke’s [22] generalized directional derivatives and used them to obtain optimality and duality results for a nonsmooth vector optimization problem.

In this section we introduce generalized type-I and nonsmooth type-I functions over cones and study optimality and duality results for the problem:

(VP) $K$-minimize $f(x)$

subject to $-g(x) \in Q$

where $f: S \to R^m$, $g: S \to R^p$, are locally Lipschitz functions on $S = R^n$, $K$ and $Q$ are closed convex pointed cones with nonempty interiors in $R^m$ and $R^n$ respectively. Let $X_0 = \{ x \in R^n : -g(x) \in Q \}$ be the feasible set of (VP).

We now give the definitions of cone generalized type-I and cone nonsmooth type-I functions.

**Definition 2.4.1.** $(f, g)$ is said to be $(K \times Q)$ **generalized type-I** at the point $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$ and $A \in \partial f(u), B \in \partial g(u)$,
\[ f(x) - f(u) - A\eta(x, u) \in K \]
\[ -g(u) - B\eta(x, u) \in Q. \]

**Definition 2.4.2.** \((f, g)\) is said to be \((K \times Q)\) nonsmooth type-I at \(u \in R^n\), if there exists \(\eta : X_0 \times R^n \to R^n\) such that for every \(x \in X_0\),
\[ f(x) - f(u) - f^0(u; \eta) \in K \]
\[ -g(u) - g^0(u; \eta) \in Q. \]

**Remark 2.4.1.** If \(f : R^n \to R\) and \(g : R^n \to R^m\), \(K = R_+\), \(Q = R_+^m\) then the above definition reduces to that of type-I invex functions given by Sach et al. [119].

**Lemma 2.4.1.** If \((f, g)\) is \((K \times Q)\) generalized type-I at \(u \in R^n\) with respect to \(\eta : X_0 \times R^n \to R^n\) then \((f, g)\) is \((K \times Q)\)-nonsmooth type-I with respect to same \(\eta\).

**Proof.** Let \((f, g)\) be \((K \times Q)\) generalized type-I at \(u \in R^n\), with respect to \(\eta : X_0 \times R^n \to R^n\), then for every \(x \in R^n\), for all \(A \in \partial f(u), B \in \partial g(u)\),
\[ f(x) - f(u) - A\eta(x, u) \in K \]
and \[ -g(u) - B\eta(x, u) \in Q. \]

For each index \(i \in \{1, 2, \ldots, m\}\), choose \(\bar{v}_i \in \partial f_i(u)\) such that
\[ \langle \bar{v}_i, \eta \rangle = \sup \{ \langle v_i, \eta \rangle : v_i \in \partial f_i(u) \} = f_i^0(u; \eta) \]
then \(\bar{A} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m) \in \partial f(u)\) and \(f(x) - f(u) - \bar{A}\eta(x, u) \in K\)
which implies that,
\[ f(x) - f(u) - f^0(u; \eta) \in K. \]
Similarly, for each \( j \in \{1, 2, \ldots, p\} \) we can choose \( \bar{w}_j \in \partial g_j(u) \) and obtain
\[
\bar{B} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_p) \in \partial g(u).
\]

Thus
\[
-g(u) - \bar{B}\eta(x,u) \in Q,
\]
which gives that
\[
-g(u) - g^0(u; \eta) \in Q.
\]

Hence \((f, g)\) is \((K \times Q)\) non smooth type-I at \( u \), with respect to same \( \eta \).

The following example shows that the converse of the above lemma is not true.

**Example 2.4.1.** Consider the problem

\[(VP_0)\quad \text{K-minimize } f(x)\]

subject to \(-g(x) \in Q\)

where \( f : \mathbb{R} \rightarrow \mathbb{R}^2, \ g : \mathbb{R} \rightarrow \mathbb{R}^2, \ f(x) = (f_1(x), f_2(x)), \ g(x) = (g_1(x), g_2(x)) \),

\[K = \{(x, y) : x \leq y, \ x \leq 0\} \quad \text{and} \quad Q = \{(x, y) : x \geq y, \ y \leq 0\};\]

\[f_1(x) = \begin{cases} x - 1, & x < 0 \\ -1, & x \geq 0 \end{cases} \quad f_2(x) = \begin{cases} x - x^3, & x < 0 \\ 0, & x \geq 0 \end{cases}\]
\[g_1(x) = \begin{cases} 1 + x, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad g_2(x) = \begin{cases} 2, & x < 0 \\ 2 - x^2, & x \geq 0 \end{cases}\]

then \(-g(x) \in Q \Rightarrow x \leq 1\).

Therefore the feasible set becomes \(X_0 = \{x \in \mathbb{R} : x \leq 1\}\).

Define \( \eta: X_0 \times \mathbb{R} \rightarrow \mathbb{R} \) as \( \eta(x, u) = (x - u)^3 \).

Now, \( \partial f_1(0) = [0,1], \ f_1^0(0,v) = \begin{cases} 0, & v < 0 \\ v, & v \geq 0 \end{cases} \)
\[ \partial f_2(0) = [0,1], \quad f_2^0(0,v) = \begin{cases} 0, & v < 0 \\ v, & v \geq 0 \end{cases} \]

dependent

\[ f(x) - f(0) - f^0(0; \eta) \in K. \]

Further \(-g(0) - g^0(0; \eta) \in Q\), for every \(x \in X_0\).

Hence \((f, g)\) is \((K \times Q)\)-nonsmooth type-I at \(u = 0\).

However, for \(A_1 = \frac{1}{4} \in \partial f_1(0), \quad A_2 = \frac{1}{2} \in \partial f_2(0)\) and \(x = 1 \in X_0\), \(A = (A_1, A_2)\)

\[ f(x) - f(0) - A\eta(x,0) = \left( \frac{-1}{4}, \frac{-1}{2} \right) \in K. \]

Thus, \((f, g)\) is not \((K \times Q)\) generalized type-I at \(u = 0\), with respect to \(\eta\) defined above.

**Remark 2.4.2.** If \(f\) is \(K\)-generalized invex and \(g\) is \(Q\)-generalized invex at \(u \in \mathbb{R}^n\) with respect to same \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\), then \((f, g)\) is \((K \times Q)\) generalized type-I at \(u\). However the converse is not true as can be seen from the following example.

**Example 2.4.2.** Consider the problem (VP\(_0\))

\[ f_i(x) = \begin{cases} x, & x < 0 \\ -x^3, & x \geq 0 \end{cases} \quad f_2(x) = \begin{cases} 1 + x, & x < 0 \\ 1, & x \geq 0 \end{cases} \]

\[ g_1(x) = \begin{cases} -x^2 + 3, & x < 0 \\ x + 3, & x \geq 0 \end{cases} \quad g_2(x) = \begin{cases} 0, & x < 0 \\ -\frac{x}{2}, & x \geq 0 \end{cases} \]

\(K = \{(x, y): x \leq 0, \ y \leq -x\}\) and \(Q = \{(x, y): x \leq y, \ y \geq 0\}\).

Then, \(-g(x) \in Q \Rightarrow x \geq -\sqrt{3}\) . Hence \(X_0 = \{x \in \mathbb{R}: x \geq -\sqrt{3}\}\)
Here \( \partial f_1(0) = [0, 1], \partial f_2(0) = [0, 1], \partial g_1(0) = [0, 1], \partial g_2(0) = [-1/2, 0]. \)

Define \( \eta : X_0 \times R \to R \) as \( \eta(x, u) = (x - u)^2 \).

Then, \((f, g)\) is \((K \times Q)\) generalized type-I at \( u = 0 \), with respect to \( \eta \) defined above.

As, \( f(x) - f(0) - A\eta(x, 0) \in K \), for every \( x \in X_0 \), \( A \in \partial f(0) \)
and \( -g(0) - B\eta(x, 0) \in Q \), for every \( x \in X_0 \) and \( B \in \partial g(0) \).

Further, for \( B_1 = \frac{1}{2} \in \partial g_1(0), B_2 = -\frac{1}{4} \in \partial g_2(0) \) and \( x = 1 \in X_0 \),

\[
g(x) - g(0) - B\eta(x, 0) = \left( \frac{1}{2}, -\frac{1}{4} \right) \notin Q.
\]

Therefore \((f, g)\) is not \((K \times Q)\) generalized invex at \( u = 0 \).

**Remark 2.4.3.** If \( K = R^m_+ \), \( Q = R^p_+ \) and \((f, g)\) is \((K \times Q)\) generalized type-I with respect to \( \eta \) then \((f, g)\) is generalized type-I \([79]\) with respect to same \( \eta \). However, the converse fails as can be viewed from the following example.

**Example 2.4.3.** The pair \((f, g)\) considered in Example 2.4.2, has been proved to be \((K \times Q)\) generalized type-I with respect to \( \eta(x, u) = (x - u)^2 \).

However \((f, g)\) is not generalized type-I with respect to same \( \eta \), because,
for \( x = 1 \in X_0 \), \( A_1 = \frac{1}{2}, f_1(x) - f_1(0) - A_1\eta(x, 0) = -\frac{3}{2} < 0 \) and for \( B_1 = \frac{1}{4}, \)

\[
-g_1(0) - B_1\eta(x, 0) = \frac{-13}{4} < 0.
\]

Based on the lines of Soleimani-damaneh \([125]\) we now introduce various generalizations of \((K \times Q)\) generalized type-I functions.
Definition 2.4.3. \((f, g)\) is said to be \((K \times Q)\) generalized pseudo type-I at \(u \in \mathbb{R}^n\) if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for all \(x \in X_0\), some \(A \in \partial f(u), B \in \partial g(u)\),

\[
-A\eta(x, u) \notin \text{int } K \quad \Rightarrow \quad -(f(x) - f(u)) \notin \text{int } K.
\]

\[
-B\eta(x, u) \notin \text{int } Q \quad \Rightarrow \quad g(u) \notin \text{int } Q.
\]

In other words, \((f, g)\) is said to be \((K \times Q)\) generalized pseudo-type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for every \(x \in X_0\) and for all \(A \in \partial f(u), B \in \partial g(u)\),

\[
-(f(x) - f(u)) \notin \text{int } K \quad \Rightarrow \quad -A\eta(x, u) \in \text{int } K
\]

\[
 g(u) \notin \text{int } Q \quad \Rightarrow \quad -B\eta(x, u) \in \text{int } Q.
\]

Definition 2.4.4. \((f, g)\) is said to be \((K \times Q)\) generalized quasi type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for every \(x \in X_0\), for all \(A \in \partial f(u), B \in \partial g(u)\)

\[
 f(x) - f(u) \notin \text{int } K \quad \Rightarrow \quad -A\eta(x, u) \in K
\]

\[
 g(u) \notin \text{int } Q \quad \Rightarrow \quad -B\eta(x, u) \in Q.
\]

Definition 2.4.5. \((f, g)\) is said to be \((K \times Q)\) generalized pseudo quasi type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for every \(x \in X_0\), for all \(A \in \partial f(u), B \in \partial g(u)\)

\[
-(f(x) - f(u)) \in \text{int } K \quad \Rightarrow \quad -A\eta(x, u) \in K
\]

\[
-g(u) \notin \text{int } Q \quad \Rightarrow \quad -B\eta(x, u) \in Q.
\]

Definition 2.4.6. \((f, g)\) is said to be \((K \times Q)\) generalized quasi pseudo type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for every \(x \in X_0\) for all \(A \in \partial f(u)\) and some \(B \in \partial g(u)\)
\[ f(x) - f(u) \notin \text{int } K \implies -A\eta(x, u) \in K \]
\[ -B\eta(x, u) \notin \text{int } Q \implies g(u) \notin \text{int } Q. \]

**Definition 2.4.7.** \((f, g)\) is said to be **strictly** \((K \times Q)\) generalized pseudo quasi type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\), such that for every \(x \in X_0\), for all \(A \in \partial f(u)\), \(B \in \partial g(u)\)

\[ -(f(x) - f(u)) \in K \implies -A\eta(x, u) \in \text{int } K \]
\[ -g(u) \notin \text{int } Q \implies -B\eta(x, u) \in Q. \]

**Definition 2.4.8.** \((f, g)\) is said to be **strongly** \((K \times Q)\) generalized pseudo quasi type-I at \(u \in \mathbb{R}^n\), if there exists \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) such that for every \(x \in X_0\), \(A \in \partial f(u)\)

\[ -A\eta(x, u) \notin \text{int } K \implies f(x) - f(u) \in K, \]
\[ -g(u) \notin \text{int } Q \implies -B\eta(x, u) \in Q, \quad \text{for all } B \in \partial g(u). \]

We now give an example of a function which is \((K \times Q)\)-generalized pseudo type-I but not \((K \times Q)\) generalized type-I.

**Example 2.4.4.** Consider the problem \((VP_0)\)

where \(f_1(x) = \begin{cases} -x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}\)
\(f_2(x) = \begin{cases} x, & x < 1 \\ x^3, & x \geq 1 \end{cases}\)

\(g_1(x) = \begin{cases} x, & x < 1 \\ 1, & x \geq 1 \end{cases}\)
\(g_2(x) = \begin{cases} 0, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}\)

\(K = \{(x, y) : y \leq x, x \geq 0\}, \quad Q = \{(x, y) : y \leq -x, y \geq 0\}\)

Now \(-g(x) \in Q \implies 0 \leq x \leq 2,\)

therefore the feasible set is \(X_0 = \{x \in \mathbb{R} : 0 \leq x \leq 2\}\).
Here \( \partial f_1(1) = [-2, -1], \partial f_2(1) = \{1\}, \partial g_1(1) = [0, 1] \) and \( \partial g_2(1) = [-1, 0] \).

Define \( \eta : X_0 \times R \to R \) as \( \eta(x, u) = x^2 - u^2 \).

Then \( (f, g) \) is \((K \times Q)\) generalized pseudo type-I at \( u = 1 \), with respect to \( \eta \), because for \( A \in \partial f(1) \)

\[-A\eta(x, 1) \notin \text{int } K \quad \Rightarrow \quad 0 \leq x \leq 1 \quad \Rightarrow \quad -(f(x) - f(1)) \notin \text{int } K\]

and for \( B \in \partial g(1), -B\eta(x, 1) \notin \text{int } Q \quad \Rightarrow \quad g(1) \notin \text{int } Q. \)

On the other hand,

for \( A = (-2, 1) \) and \( x = 0 \in X_0, \)

\[f(x) - f(1) - A\eta(x, 1) = (-1, 0) \notin K.\]

Hence \( (f, g) \) fails to be \((K \times Q)\) generalized type-I.

We now give an example to show that \((K \times Q)\) generalized pseudo quasi type-I function may fail to be \((K \times Q)\) generalized type-I.

**Example 2.4.5.** Consider the problem \((VP_0)\)

where \( f_1(x) = \begin{cases} 1, & x < 0 \\ -x + 1, & x \geq 0 \end{cases} \)

\[f_2(x) = \begin{cases} x - 2, & x < 0 \\ -2, & x \geq 0 \end{cases} \]

\[g_1(x) = \begin{cases} x, & x < 0 \\ x^2, & x \geq 0 \end{cases} \]

\[g_2(x) = \begin{cases} 0, & x < 0 \\ x^3 - x^2, & x \geq 0 \end{cases} \]

\[K = \{(x, y) : x \leq 0, y \leq x\}, \quad Q = \{(x, y) : x \leq 0, y \leq -x\}. \]

Then, \(-g(x) \in Q \quad \Rightarrow \quad x \geq 0\), therefore the feasible set is \( X_0 = \{x \in R : x \geq 0\}. \)

Here \( \partial f_1(0) = [-1, 0], \partial f_2(0) = [0, 1], \partial g_1(0) = [0, 1], \partial g_2(0) = \{0\} \)

Define \( \eta : X_0 \times R \to R \) as \( \eta(x, u) = x^3 - u^3. \)
Then \((f, g)\) is \((K \times Q)\) generalized pseudo quasi type-I, at \(u = 0\) with respect to \(\eta\) because, for \(A \in \partial f(0)\)

\[-A\eta(x, 0) \notin \text{int } K \Rightarrow -(f(x) - f(0)) \notin \text{int } K\]

and

\[-g(0) \notin \text{int } Q \Rightarrow -B\eta(x, 0) \in Q, \text{ for all } x \in X_0, B \in \partial g(0).\]

However, \((f, g)\) fails to be \((K \times Q)\) generalized type-I at \(u = 0\), because for \(A = \left(\frac{-1}{2}, \frac{1}{4}\right) \in \partial f(0)\) and \(x = 1 \in X_0\), we have

\[f(x) - f(0) - A\eta(x, 0) = \left(\frac{-1}{2}, \frac{-1}{4}\right) \notin K.\]

Suneja et al. [137] gave the following necessary optimality conditions for \((VP)\).

**Theorem 2.4.1 ([137], Fritz John type necessary optimality conditions).** Let \(f\) be \(K\)-generalized invex and \(g\) be \(Q\)-generalized invex at \(x_0 \in X_0\) with respect to same \(\eta : R^n \times R^n \rightarrow R^n\). If \((VP)\) attains a weak minimum at \(x_0\) then there exist \(\lambda \in K^+, \mu \in Q^+\) not both zero such that

\[0 \in \partial(\lambda^T f)(x_0) + \partial(\mu^T g)(x_0) \quad \text{(2.53)}\]

and

\[\mu^T g(x_0) = 0 \quad \text{(2.54)}\]

The following constraint qualification is used to establish Karush-Kuhn-Tucker type necessary optimality conditions.

**Definition 2.4.9.** The problem \((VP)\) is said to satisfy **generalized Slater constraint qualification**, if there exists \(\bar{x} \in R^n\) such that

\[-g(\bar{x}) \in \text{int } Q.\]
**Theorem 2.4.2** ([137], Karush-Kuhn-Tucker type necessary optimality conditions). Let \( f \) be \( K \)-generalized invex and \( g \) be \( Q \)-generalized invex at \( x_0 \in X_0 \) with respect to same \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \). Suppose that generalized Slater constraint qualification is satisfied. If (VP) attains a weak minimum at \( x_0 \), then there exist \( 0 \neq \lambda \in K^+, \mu \in Q^+ \) such that (2.53) and (2.54) hold.

We now obtain sufficient optimality conditions for (VP).

**Theorem 2.4.3.** Let \((f, g)\) be \((K \times Q)\)-generalized type-I at \( x_0 \in X_0 \) with respect to \( \eta : X_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) and suppose that there exist \( 0 \neq \lambda^* \in K^+, \mu^* \in Q^+ \), such that

\[
0 \in \partial (\lambda^* T f)(x_0) + \partial (\mu^* T g)(x_0) \quad (2.55)
\]

\[
\mu^* T g(x_0) = 0 \quad (2.56)
\]

Then \( x_0 \) is a weak minimum of (VP).

**Proof.** Let if possible \( x_0 \) be not a weak minimum of (VP). Then there exists \( x \in X_0 \) such that

\[
f(x_0) - f(x) \in \text{int } K \quad (2.57)
\]

Since \((f, g)\) is \((K \times Q)\)-generalized type-I at \( x_0 \in X_0 \), we have

\[
f(x) - f(x_0) - A \eta(x, x_0) \in K, \quad \text{for all } A \in \partial f(x_0) \quad (2.58)
\]

and

\[
-g(x_0) - B \eta(x, x_0) \in Q, \quad \text{for all } B \in \partial g(x_0) \quad (2.59)
\]

Adding (2.57) and (2.58) we get,

\[-A \eta(x, x_0) \in \text{int } K, \quad \text{for all } A \in \partial f(x_0). \quad (2.60)\]

Since \( 0 \neq \lambda^* \in K^+ \), we have from (2.60),
\[ \lambda^* A \eta(x, x_0) < 0, \text{ for all } A \in \partial f(x_0). \]

By virtue of (2.55), there exist \( x^* \in \partial (\lambda^* T f)(x_0), y^* \in \partial (\mu^* T g)(x_0) \) such that

\[ x^* + y^* = 0 \tag{2.61} \]

As \( x^* \in \partial (\lambda^* T f(x_0)) = \lambda^* \partial f(x_0), \) we get

\[ x^* \eta(x, x_0) < 0. \]

If \( \mu^* \neq 0, \) then we have \(- y^* \eta(x, x_0) < 0. \)

As \( y^* \in \partial (\mu^* T g)(x_0) = \mu^* \partial g(x_0), \) we get \( y^* = \mu^* B^* \), for some \( B^* \in \partial g(x_0). \)

Thus,

\[ -\mu^* B^* \eta(x, x_0) < 0, B^* \in \partial g(x_0). \tag{2.62} \]

Now \( \mu^* \in Q^+, \) therefore from (2.59) we have, \(- \mu^* T g(x_0) - \mu^* B \eta(x, x_0) \geq 0, \) for all \( B \in \partial g(x_0), \) which on using (2.56) gives

\[ -\mu^* B \eta(x, x_0) \geq 0, \text{ for all } B \in \partial g(x_0). \]

This is a contradiction to (2.62).

If \( \mu^* = 0, \) then also we arrive at a contradiction.

Hence \( x_0 \) is a weak minimum of (VP).

\[ \Box \]

**Theorem 2.4.4.** Let \( (f, g) \) be \( (K \times Q) \) generalized pseudo quasi type-I at \( x_0 \in X_0 \) with respect to \( \eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n \) and suppose there exist \( 0 \neq \lambda^* \in K^+, \mu^* \in Q^+, \) such that (2.55), (2.56) hold, then \( x_0 \) is a weak minimum of (VP).

**Proof.** Let if possible, \( x_0 \) be not a weak minimum of (VP) then there exists \( x \in X_0 \) such that

\[ f(x_0) - f(x) \in \text{int } K. \tag{2.63} \]
As (2.55) holds, therefore there exist $x^* \in \partial (\lambda^* T f)(x_0)$, $y^* \in \partial (\mu^* T g)(x_0)$ such that (2.61) is satisfied.

Since $(f, g)$ is $(K \times Q)$ generalized pseudo quasi type-I at $x_0 \in X_0$, therefore, from (2.63) we have

$$-A \eta(x, x_0) \in \text{int } K, \text{ for all } A \in \partial f(x_0).$$

Now, $0 \neq \lambda^* \in K^+$, gives

$$\lambda^* A \eta(x, x_0) < 0, \text{ for all } A \in \partial f(x_0)$$

which implies,

$$x^* \eta(x, x_0) < 0 \text{ as } x^* \in \partial (\lambda^* T f)(x_0).$$

(2.64)

From (2.56), $\mu^* T g(x_0) = 0$, which gives $-g(x_0) \notin \text{int } Q$, if $\mu^* \neq 0$.

As $(f, g)$ is $(K \times Q)$ generalized pseudo quasi type-I at $x_0$

$$-B \eta(x, x_0) \in Q, \text{ for all } B \in \partial g(x_0),$$

which implies $\mu^* B \eta(x, x_0) \leq 0$ as $\mu^* \in Q^+$.

The above inequality also holds for $\mu^* = 0$.

As $y^* \in \partial (\mu^* T g)(x_0)$, we have $y^* \eta(x, x_0) \leq 0$,

which on using (2.61) gives $x^* \eta(x, x_0) \geq 0$.

This contradicts (2.64).

Hence $x_0$ is a weak minimum of (VP).

**Theorem 2.4.5.** Let $(f, g)$ be $(K \times Q)$ generalized type-I at $x_0 \in X_0$ with respect to $\eta : X_0 \times R^n \to R^n$ Suppose there exist $\lambda^* \in K^{S+}$, $\mu^* \in Q^+$, such that (2.55) and (2.56) hold. Then $x_0$ is a minimum of (VP).
Proof. Let if possible, \( x_0 \) be not a minimum of (VP), then there exists \( x \in X_0 \) such that

\[
f(x_0) - f(x) \in K \setminus \{0\}\]

(2.65)

As (2.55) holds, there exist \( x^* \in \partial(\lambda^*Tf)(x_0) \), \( y^* \in \partial(\mu^*Tg)(x_0) \) such that (2.61) holds.

Since \((f, g)\) is \((K \times Q)\) generalized type-I at \( x_0 \in X_0 \), therefore proceeding on the similar lines as in proof of Theorem 2.4.3 and using (2.65) we have

\[-A\eta(x, x_0) \in K \setminus \{0\}.

As \( \lambda^* \in K^{S+} \), we have

\[\lambda^*A\eta(x, x_0) < 0, \text{ for all } A \in \partial f(x_0).\]

This leads to a contradiction as in Theorem 2.4.3. Hence \( x_0 \) is a minimum of (VP).

Theorem 2.4.6. Let \((f, g)\) be strictly \((K \times Q)\) generalized pseudo quasi type-I at \( x_0 \in X_0 \) with respect to \( \eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n \). Suppose that there exist \( 0 \neq \lambda^* \in K^*, \mu^* \in Q^* \), such that (2.55) and (2.56) hold. Then \( x_0 \) is a minimum of (VP).

Proof. It follows from (2.55) that there exist \( x^* \in \partial(\lambda^*Tf)(x_0) \), \( y^* \in \partial(\mu^*Tg)(x_0) \) such that (2.61) holds. Let if possible \( x_0 \) be not a minimum of (VP) then there exists \( x \in X_0 \) such that (2.65) holds.

Since \((f, g)\) is strictly \((K \times Q)\) generalized pseudo quasi type-I, therefore, we have

\[-A\eta(x, x_0) \in \text{int} K, \text{ for all } A \in \partial f(x_0).\]
Proceeding on the same lines as in proof of Theorem 2.4.4, we arrive at a contradiction. Hence $x_0$ is a minimum of (VP).

**Theorem 2.4.7.** Let $(f, g)$ be $(K \times Q)$ generalized type-I at $x_0 \in X_0$ with respect to $\eta : X_0 \times R^n \to R^n$ and suppose that there exist $\lambda^* \in K^+, \mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ such that (2.55) and (2.56) hold. Then $x_0$ is a minimum solution of the scalarized problem

$$\text{(VP)}_{A^*} \begin{array}{ll}
\text{minimize} & (\lambda^*^T f)(x) \\
\text{subject to} & -g(x) \in Q \\
& \lambda^* \in K^{++}
\end{array}$$

and is hence a Benson proper minimizer of (VP).

The proof of the above theorem follows on the lines of Theorem 2.4.3.

**Theorem 2.4.8.** Let $(f, g)$ be strongly $(K \times Q)$ generalized pseudo quasi type-I at $x_0 \in X_0$ with respect to $\eta : X_0 \times R^n \to R^n$ and suppose that there exist $0 \neq \lambda^* \in K^+, \mu^* \in Q^+$ such that (2.55) and (2.56) hold. Then $x_0$ is a strong minimum of (VP).

**Proof.** By virtue of (2.55), there exist $x^* \in \partial (\lambda^*^T f)(x_0)$, $y^* \in \partial (\mu^*^T g)(x_0)$ such that (2.61) holds.

Let if possible $x_0$ be not a strong minimum of (VP), then there exists $x \in X_0$ such that $f(x) - f(x_0) \notin K$.

Since $(f, g)$ is strongly $(K \times Q)$ generalized pseudo quasi type-I at $x_0$, we have

$$-A\eta(x, x_0) \in \text{int } K, \text{ for all } A \in \partial f(x_0)$$

Now proceeding as in Theorem 2.4.4 we arrive at a contradiction, thus proving that $x_0$ is a strong minimum of (VP).
We now consider the Mond-Weir type dual of the vector optimization problem (VP).

(VD) K-maximize $f(u)$

subject to

$$0 \in \partial(\lambda^T f)(u) + \partial(\mu^T g)(u)$$  \hspace{1cm} (2.66)

$$\mu^T g(u) \geq 0$$

$$0 \neq \lambda \in K^+, \mu \in Q^+$$

We now establish weak duality result for the pair (VP) and (VD).

**Theorem 2.4.9 (Weak duality).** Let $x$ be feasible for (VP) and $(u, \lambda, \mu)$ be feasible for (VD). Let $(f, g)$ be $(K \times Q)$ generalized type-I at $u$ with respect to $\eta : X_0 \times R^n \rightarrow R^n$ then $f(u) - f(x) \notin int K$.

**Proof.** Since $(u, \lambda, \mu)$ is feasible for (VD) therefore by (2.66) there exist $x^* \in \partial(\lambda^T f)(u), y^* \in \partial(\mu^T g)(u)$ such that

$$x^* + y^* = 0. \hspace{1cm} (2.67)$$

Let if possible $f(u) - f(x) \in int K \hspace{1cm} (2.68)$

Now $(f, g)$ is $(K \times Q)$ generalized type-I at $u$, therefore we have

$$f(x) - f(u) - A\eta(x, u) \in K, \text{ for all } A \in \partial f(u) \hspace{1cm} (2.69)$$

$$-g(u) - B\eta(x, u) \in Q \hspace{1cm} \text{ for all } B \in \partial g(u). \hspace{1cm} (2.70)$$

Adding (2.68) and (2.69) we have

$$-A\eta(x, u) \in int K, \hspace{1cm} \text{ for all } A \in \partial f(u).$$

As $0 \neq \lambda \in K^+$, we get
\( \lambda A \eta(x, u) < 0 \)

which implies

\( x^* \eta(x, u) < 0, \) as \( x^* \in \partial(\lambda^T f)(u). \)

By (2.67) we have \(-y^* \eta(x, u) < 0.\)

Since \( y^* \in \partial(\mu^T g)(u), \) we obtain \( y^* = \mu B^*, \) \( B^* \in \partial g(u).\)

Thus we have \(-\mu B^* \eta(x, u) < 0, \) for \( \mu \neq 0. \) \( \text{(2.71)} \)

From (2.70) we get

\[-\mu^T g(u) - \mu B^* \eta(x, u) \geq 0, \) as \( \mu \in Q^+.\)

Since \( u \) is feasible for (VD), we get

\[-\mu B^* \eta(x, u) \geq 0,\)

which contradicts (2.71).

If \( \mu = 0, \) then also we arrive at a contradiction.

Hence \( f(u) - f(x) \notin \text{int} K. \)

**Theorem 2.4.10 (Weak Duality).** Let \( x \in X_0 \) and \( (u, \lambda, \mu) \) be feasible for (VD), \((f, g)\) be \((K \times Q)\) generalized pseudo quasi type-I at \( u \) with respect to \( \eta : X_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( (\lambda, \mu) \neq 0 \) then \( f(u) - f(x) \notin \text{int} K. \)

**Proof.** Since \( (u, \lambda, \mu) \) is feasible for (VD), by (2.66), there exists \( x^* \in \partial(\lambda^T f)(u), y^* \in \partial(\mu^T g)(u) \) such that (2.67) holds.

Let if possible \( f(u) - f(x) \in \text{int} K.\)

Since \( (f, g) \) is \((K \times Q)\) generalized pseudo quasi type-I at \( u \) we get

\[-A \eta(x, u) \in \text{int} K, \) for all \( A \in \partial f(u).\)
Now proceeding as in Theorem 2.4.9, we get

$$-\mu B^* \eta(x, u) < 0. \quad (2.72)$$

Also, \((u, \lambda, \mu)\) is feasible for (VD), so we have

$$\mu^T g(u) \geq 0$$

$$\Rightarrow -g(u) \notin \text{int} \ Q \text{ if } \mu \neq 0$$

$$\Rightarrow -B \eta(x, u) \in Q \text{ for all } B \in \partial g(u), \text{ as } (f, g) \text{ is } (K \times Q) \text{ generalized pseudo quasi type-I at } u.$$  

$$\Rightarrow \text{ in particular } -\mu B^* \eta(x, u) \geq 0.$$  

The above inequality also holds for \(\mu = 0\).

But this contradicts (2.72), hence \(f(u) - f(x) \notin \text{int} \ K\).

We next prove strong duality result.

**Theorem 2.4.11 (Strong Duality).** Suppose that (VP) attains a weak minimum at \(x_0\) and generalized Slater constraint qualification is satisfied. Let \((f, g)\) be \((K \times Q)\) generalized invex at \(x_0\) with respect to \(\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\), then there exist \(0 \neq \lambda_0 \in K^+, \mu_0 \in Q^+\) such that \((x_0, \lambda_0, \mu_0)\) is a feasible solution of (VD). Further if the conditions of Weak Duality Theorem 2.4.10 hold for all feasible solution of (VP) and of (VD), then \((x_0, \lambda_0, \mu_0)\) is a weak maximum of (VD).

**Proof.** As \(x_0\) is a weak minimum of (VP), by Theorem 2.4.2, there exist \(0 \neq \lambda_0 \in K^+, \mu_0 \in Q^+\) such that

$$0 \in \partial (\lambda_0^T f)(x_0) + \partial (\mu_0^T g)(x_0)$$

and $$\mu_0^T g(x_0) = 0,$$
which clearly shows that \((x_0, \lambda_0, \mu_0)\) is a feasible solution of \((VD)\) and the values of two objectives functions are equal. Further if \((x_0, \lambda_0, \mu_0)\) is not a weak maximum of \((VD)\), then there exists a feasible solution \((u, \lambda, \mu)\) of \((VD)\) such that \(f(u) - f(x_0) \in \text{int } K\), which contradicts Theorem 2.4.10. Hence \((x_0, \lambda_0, \mu_0)\) is a weak maximum of \((VD)\).