Chapter 1

INTRODUCTION

Life inevitably involves decision making, choices and searching for compromises. It is only natural to want them to be as good as possible, in other words, optimal. The major quantitative tool in the machinery of decision making is optimization, in which decisions have to be taken to optimize one or more objectives under some prescribed set of circumstances. The process of simultaneously optimizing two or more objectives subject to certain constraints is called Multiobjective Optimization. Such optimization problems can be found in various real life problems such as finance, aircraft design, oil and gas industry, automobile industry and so on.

Vector Optimization deals with investigation of optimal elements of a nonempty subset of a partially ordered linear space. Problems of this type can be found in engineering, economics and in the various fields of mathematics such as functional analysis, statistics, game theory etc. The multiobjective optimization problems are finite dimensional vector optimization problems with the natural partial ordering in the image space of the vector valued objective function.

The present thesis studies certain optimality and duality aspects in vector optimization and is divided into four chapters. The first chapter comprises of two sections. Section 1.1 gives a brief survey of the work related to optimization problems and some related concepts. Section 1.2 summarizes the work carried out in the thesis.
The following conventions for vectors in $\mathbb{R}^n$, the $n$-dimensional Euclidean space, will be used:

\[
\begin{align*}
    x < y & \iff x_i < y_i, \quad \forall \ i = 1, 2, \ldots, n \\
    x \leq y & \iff x_i \leq y_i, \quad \forall \ i = 1, 2, \ldots, n \\
    x \leq y & \iff x_i \leq y_i, \quad \forall \ i = 1, 2, \ldots, n, \ x \neq y \\
    x \not\leq y & \text{ is the negation of } x \leq y.
\end{align*}
\]

For $x, y \in \mathbb{R}$, $x \leq y$ and $x < y$ have usual meaning.

1.1 Optimization Problems and Some Related Concepts

Optimization problems which seek to minimize or maximize a function of one or more variables subject to certain constraints are called Mathematical Programming Problems. The first such problem was formulated by Hitchcock and Koopmans in 1941 as a Transportation problem. However a good solution technique became available in 1947 during Second World War, when Dantzig [29] suggested an efficient way to solve the problem. Later he generalized the Transportation Problem to what is now known as Linear Programming Problem. The beginning of another branch of mathematical programming called Nonlinear Programming can be dated back to the important paper published by Kuhn and Tucker [80] in 1951 which laid the foundation of this field. Mathematical programming problems have applications in vast areas of research such as economics, physics, management science, computer science and operational research. Floudas and Pardalos [40] in their book entitled “Encyclopedia of Optimization” introduce various spectrums of research and applications that have come from this field.

We now discuss various optimization problems and related concepts.
1.1.1. Mathematical Programming

The general **Mathematical Programming Problem** can be expressed as:

maximize (minimize) \( f(x) \)

subject to \( g_j(x) (\leq, =, \geq) 0, \ j = 1,2,\ldots,m \)

\[ x \in S, \]

where \( f \) and \( g_j, j = 1,2,\ldots,m \) are real valued functions defined on \( S \subseteq R^n \).

The function \( f(x) \) is called the **objective function** and \( g_j(x), j = 1,2,\ldots,m \) are called **constraint functions**. The various constraint functions may have different equality or inequality signs, however one and only one sign holds for each constraint.

If both, the objective function \( f(x) \) and the constraint functions \( g_j(x) \) are linear then the above problem is called a **Linear Programming Problem**. However if any one of the objective function or the constraints are nonlinear then the problem is called a **Nonlinear Programming Problem**. Problems of this type are found in economics, management sciences such as forecasting, production scheduling and many other disciplines.

A general **Nonlinear Programming Problem** is of the form

\([NP] \quad \text{minimize } f(x)\]

subject to \( g_j(x) \leq 0, \ j = 1,2,\ldots,m \)

\[ x \in S \subseteq R^n \]

where one or more of the functions \( f \) and \( g_j, j = 1,2,\ldots,m \) are nonlinear.

If \( f(x) \) is the ratio of two functions \( q(x) \) and \( r(x) \), the problem \([NP]\) is called a **Fractional Programming Problem**. If \( q(x), \ r(x) \) and
the constraints are linear then it is a linear fractional programming problem. If either \( q(x) \) or \( r(x) \) or the constraint functions or all are nonlinear then it is a nonlinear fractional programming problem. The bibliography by Schaible [121] was an attempt to include all publications in this area of fractional programming as they have appeared. Fractional programming problems are frequently encountered in finance, marine transportation, health care and various other fields. Many applications of fractional programming have been discussed by Schaible [120].

1.1.2. Multiobjective Programming

In the world around us it is rare for any problem to have a single objective. Most realistic optimization problems require the simultaneous optimization of more than one objective function. For example, in a transportation problem we wish to minimize the transportation cost, average shipping time and fuel consumption while maximizing the profit using the given process. Some other examples of multiobjective optimization problems are minimizing the cost and maximizing the profit of a product, maximizing performance and minimizing fuel consumption of a vehicle and minimizing weight while maximizing the strength of a particular component. Multiobjective optimization has a wide range of applications, and is an important tool for problems in welfare economics, resource planning and management, mathematical biology and engineering [128].

A general multiobjective programming problem having \( k \ (\geq 2) \) objectives is of the form :

\[
\text{(MOP) } \begin{align*}
\text{minimize} & \quad f(x) = (f_1(x), \ldots, f_k(x)) \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, 2, \ldots, m \\
& \quad x \in S,
\end{align*}
\]
where $f_i, i = 1,2,\ldots,k$ and $g_j, j = 1,2,\ldots,m$ are real valued functions defined on $S \subseteq \mathbb{R}^n$. Let $X_1 = \{x \in S: g_j(x) \leq 0, j = 1,2,\ldots,m\}$ denote the set of all feasible solutions of (MOP).

Multiobjective optimization problems have been studied by several authors like Craven [27], Chankong and Haimes [18], Chen [20], Singh [123], Bector et al. [10], Preda [112], Coladas et al. [23], Li and Wang [83], Antczak [1] and Minami [90]. A survey of recent developments in multiobjective optimization has been given by Chinchuluun and Pardalos [21], while Pardalos et al. [111] and Zopounidis and Pardalos [165] have studied recent advances in multicriteria analysis which includes multiobjective optimization.

Unlike single objective mathematical programming problem, multiobjective programming problem does not necessarily have an optimal solution in the sense that minimizes all the objective functions simultaneously, and hence the decision maker has to face conflicting objectives, for instance in the field of performances, gains, errors, pay off etc. This led to the notion of efficiency.

**Definition 1.1.1.** $x^\circ \in X_1$ is an efficient (pareto optimal) solution of (MOP), if there does not exist any other $x \in X_1$ such that

$$f_i(x) \leq f_i(x^\circ), \quad \text{for all } i = 1,2,\ldots,k,$$

$$f_r(x) < f_r(x^\circ) \quad \text{for some } r.$$

The following result is due to Chankong and Haimes [18].

**Lemma 1.1.1.** $x^\circ \in X_1$ is an efficient solution of (MOP) if and only if $x^\circ$ is an optimal solution of $P_r(x^\circ)$ for each $r = 1,2,\ldots,k$ where $P_r(x^\circ)$ is defined as:
\[ P_r(x^o) \quad \text{minimize } f_i(x) \]

subject to \( f_i(x) \leq f_i(x^o), \) for all \( i = 1,2,\ldots,k, \ i \neq r \)

\[ g_j(x) \leq 0, \ j = 1,2,\ldots,m \]

\( x \in S \)

Weir and Mond [148] considered following definition of weak efficiency.

**Definition 1.1.2.** \( x^o \in X_1 \) is a **weakly efficient solution** of (MOP), if there does not exist any \( x \in X_1 \) such that

\[ f_i(x) < f_i(x^o), \ i = 1,2,\ldots,k. \]

Kuhn and Tucker [80], Klinger [78], and White [151] observed that some efficient points exhibit some abnormal properties. To eliminate such anomalous solutions in large sets of efficient solutions, a number of proper efficiencies have been introduced. The following definition of proper efficiency has been given by Geoffrion [44].

**Definition 1.1.3 ([44]).** A point \( x^e \in X_1 \) is said to be a **properly efficient solution** of (MOP), if it is efficient and if there exists a scalar \( M > 0 \) such that for each \( i \) and \( x \in X_1 \) satisfying \( f_i(x) < f_i(x^e) \) we have

\[ \frac{f_i(x^e) - f_i(x)}{f_j(x) - f_j(x^e)} \leq M \]

for some \( j \), satisfying \( f_j(x) > f_j(x^e) \).

An efficient solution is said to be improperly efficient if it is not properly efficient.

The following lemma has been established by Geoffrion [44] which gives a relationship between the multiobjective programming problem (MOP) and the scalarized problem
(MOP_λ) \quad \text{minimize } \sum_{i=1}^{k} \lambda_i f_i(x) \\
\text{subject to } x \in X_i \\
\text{where } \lambda_i \text{'s are strictly positive parameters often normalized according to } \sum_{i=1}^{k} \lambda_i = 1.

Lemma 1.1.2 ([44]). (a) Let \( \lambda_i > 0, \ i = 1,2,...,k \) be fixed. If \( x^\circ \) is an optimal solution of (MOP_λ), then \( x^\circ \) is a properly efficient solution of (MOP).

(b) Let \( X_i \) be convex and \( f_i, \ i = 1,2,...,k \) be convex on \( S \). If \( x^\circ \) is a properly efficient solution of (MOP), then there exists some \( \lambda \in R^k \) with strictly positive components such that \( x^\circ \) is an optimal solution of (MOP_λ).

1.1.3. Multiobjective Fractional Programming

The class of multiobjective fractional programming problems finds its uses in a variety of contexts such as transportation, information theory, production, numerical analysis etc. Multiobjective fractional criteria are also encountered in finance, corporate planning and bank balance sheet management. A general multiobjective fractional programming problem can be expressed as:

maximize (minimize) \( \frac{f(x)}{g(x)} = \left[ \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)} \right] \)

subject to \( h_j(x) \leq 0, \ j = 1,2,...,m \)

\( x \in S, \)

where \( f_i, g_i, \ i = 1,2,...,k \) and \( h_j, \ j = 1,2,...,m \) are real valued functions defined on \( S \subseteq R^n \) and \( g_i(x) > 0, \ i = 1,2,...,k, \) for all \( x \in S. \)
The fractional optimization problems with multiple objective functions have been studied by several authors such as Kaul et al. [73], Bhatia and Gupta [13], Weir [143], Egudo [39], Singh [124], Suneja and Gupta [133], Bector et al. [8], Liang et al. [86], Shashi [122], Preda [114], Zalmai [161] and Frenk and Schaible [41].

1.1.4. Optimality

In the study of optimization problems one is interested in locating optimal solutions. The problem of identifying an optimal solution of a nonlinear programming problem led to the development of various types of optimality conditions. John [66] gave necessary optimality criteria for a nonlinear programming problem without using any constraint qualification. Kuhn and Tucker [80] developed necessary optimality conditions by imposing constraint qualifications which are restrictions on the constraints and sufficient optimality conditions by assuming the functions to be convex. Due to the contributions by Karush [67], who developed optimality conditions similar to those of Kuhn and Tucker, these conditions are usually known as the Karush-Kuhn-Tucker conditions. Some of the other authors to contribute to this area are Mangasarian [88], Bector et al. [7], Antczak [2] etc. Optimality conditions for multiobjective optimization problems for both differentiable and nondifferentiable functions have been investigated by several authors, for example, Chankong and Haimes [18], Singh [123], Gulati and Islam [47], Li [84], Lalitha [81], Suneja et al. [135].

1.1.5. Duality

Duality is one of the most fundamental issues in optimization. The notion of duality was first introduced in linear programming by Von Neumann [141] and was subsequently formulated in a precise form by Gale, Kuhn and Tucker [43]. An extensive use of duality in
mathematical programming has been found not only in its theoretical and computational development but also in the fields of engineering, operational research, management sciences and economics. The idea of duality is to associate with a given minimization (maximization) problem called primal problem, another maximization (minimization) problem called dual problem such that the existence of optimal solution for one of them guarantees an optimal solution to the other and optimal values of the two problems are equal. Sometimes the dual problem has a better structure than the primal and it might be easier to work with the dual.

Duality in nonlinear programming originated with the duality results in quadratic programming given by Dennis [32] and Dorn [34]. Wolfe [150] formulated a dual for a nonlinear programming problem involving convex functions and proved various duality results. Mahajan and Vartak [87] established duality results for nonlinear programming problems whose feasible region is defined by both equality and inequality constraints. Mond and Weir [103] associated a different type of dual with the nonlinear programming problem and proved duality results by relaxing the convexity assumptions. Bector et al. [7] studied Mond-Weir type dual for a nonlinear programming problem under quasi convexity assumption. Many researchers have studied duality results for multiobjective programming problems. Egudo [38] and Tanino and Swaragi [139] established duality results under generalized convexity assumptions, while Weir [144] studied them under invexity assumptions. Wang and Li [142] discussed scalarization and Lagrange duality results for multiobjective programs. Preda [112] formulated Wolfe and Mond-Weir type duals for a multiobjective programming problem and studied duality results using generalized \((F,\rho)\)-convexity. Preda [113] and Preda and Stancu-Minasian [115] proved duality results for problems involving semilocally convex and semilocally preinvex functions respectively.
Suneja et al. [138] established duality results for multiobjective programs involving ρ-semilocally preinvex and related functions. Duality results for fractional programming problems have also been studied by various authors which include Egudo [39], Preda [114] and Liang et al. [85, 86].

1.1.6 Second and Higher Order Duality


Higher order duality has been studied in past few years by many researchers which include Mond and Zhang [106], Yang et al. [154], Mishra and Rueda [96] and Zhang [162].
1.1.7 Symmetric Duality

A pair of mathematical programming problem and its dual are said to be symmetric, if the dual of the dual is the original problem. The first symmetric dual formulation for a quadratic programming problem was proposed by Dorn [35]. Dantzig et al. [30] and Mond [99] established symmetric duality results for a nonlinear programming problem under the conditions of convexity/concavity. Mond and Weir [105] weakened the conditions to pseudoconvexity/pseudoconcavity and established duality results for a pair of multiobjective symmetric dual problems. Chandra et al. [16] and Lalitha et al. [82] studied symmetric duality in fractional programming problems. Mond and Schechter [102] studied symmetric dual programs for nonlinear programming problems containing support functions. Second order symmetric duality for mathematical programming problems with support functions have been also explored by Hou and Yang [55] and Yang et al. [155, 156] etc. Bector and Chandra [6] established second order symmetric duality results under pseudobonvexity and pseudoboncavity assumptions. Gulati et al. [45] studied second order symmetric duality results with \( \eta \)-convexity, \( \eta \)-pseudoconvexity assumptions. Yang et al. [157] and Mishra [93] proved second order symmetric duality results under \( F \)-convexity assumption.

and Gulati et al. [48] studied second order symmetric duality results for multiobjective programs with cone constraints under the assumptions of second order $\eta$-convexity and second order $\eta$-pseudoconvexity respectively.

Suneja et al. [130] formulated a pair of symmetric dual multiobjective programs of Wolfe type over arbitrary cones and established weak, strong, converse and self duality theorems by using cone-convexity. Khurana [75] formulated a pair of Mond-Weir type symmetric dual multiobjective programming problems and established duality results by using cone (strong) pseudoinvexity of the functions involved. Kim et al. [77] studied nondifferentiable multiobjective symmetric dual programs over arbitrary cones, while Mishra and Lai [94] formulated a pair of second order symmetric dual programs over cones and studied duality results using second order pseudoinvexity over cones.

Much work is being done in the area of higher order symmetric duality. Mishra [91] proved higher order symmetric duality results for mathematical programming problems with generalized invexity. Chen [19] established higher order symmetric duality results for multiobjective programming problems with support functions under higher order F-convexity assumptions. Gualti and Gupta [46] studied these results under generalized F-convexity.

1.1.8. Generalized Convex and Related Functions

The notion of convexity is a key factor responsible for the development of the subject of optimization. Convex functions occupy an important position in economics, engineering, management sciences and applied optimization theory and are defined over convex sets.
**Definition 1.1.4.** A nonempty subset \( S \) of \( \mathbb{R}^n \) is said to be **convex** if for \( x, x^* \in S \) and \( t \in [0, 1] \) we have

\[
t x + (1 - t) x^* \in S
\]

Geometrically, \( S \) is convex if the line segment joining any two points in \( S \), also lies in \( S \).

**Definition 1.1.5.** A function \( f : S \rightarrow \mathbb{R} \) defined on a convex set \( S \subseteq \mathbb{R}^n \) is said to be **convex** if for \( x, x^* \in S \) and \( t \in [0, 1] \) we have

\[
f(t x + (1 - t)x^*) \leq t f(x) + (1 - t) f(x^*).
\]

Geometrically, a real valued function \( f \) defined on a convex set \( S \subseteq \mathbb{R}^n \) is **convex** if the line segment joining any two points on the graph of the function lies on or above the portion of the graph between these points.

A differentiable function \( f \) defined on an open convex set \( S \subseteq \mathbb{R}^n \) is convex if and only if

\[
f(x) - f(x^*) \geq \nabla f(x^*)^T (x - x^*) \quad \text{for all } x, x^* \in S.
\]

A function \( f \) is said to be **concave** if \((-f)\) is convex.

A vector valued function \( f : S \rightarrow \mathbb{R}^k \) where \( f = (f_1, f_2, \ldots, f_k) \), is said to be convex if each \( f_i, i = 1, 2, \ldots, k \) is convex.

Various generalizations of convex functions have appeared in the literature as given below.

**Definition 1.1.6.** A function \( f : S \rightarrow \mathbb{R} \) defined on a convex set \( S \subseteq \mathbb{R}^n \) is said to be **quasiconvex** if, for \( x, y \in S \) and \( t \in [0, 1] \) we have

\[
f(t x + (1 - t) y) \leq \max\{f(x), f(y)\}.
\]
A differentiable function $f$ defined on an open convex set $S$ is quasiconvex if and only if for every $x, y \in S$

$$f(y) \leq f(x) \implies \nabla f(x)^T (y - x) \leq 0.$$  

Another generalization of convex function is pseudoconvex function defined as follows:

**Definition 1.1.7.** A real valued differentiable function $f : S \to R$ defined on an open set $S \subseteq R^n$ is said to be **pseudoconvex** on $S$ if for every $x, y \in S$ we have

$$\nabla f(x)^T (y - x) \geq 0 \implies f(y) \geq f(x).$$

Yang [153] defined semistrictly convex functions as follows:

**Definition 1.1.8.** Let $f : S \to R$, where $S$ is an open convex subset of $R^n$. Then $f$ is said to be **semistrictly convex** on $S$, if for every pair of points $x, y \in S$ $f(x) \neq f(y)$, $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y), \ t \in (0, 1).$$

Vial [140] generalized convex functions by introducing $\rho$-convex functions.

**Definition 1.1.9.** Let $f : S \to R$, be a real valued function defined on a convex subset $S$ of $R^n$, $f$ is said to be **$\rho$-convex** if there exists some real number $\rho$ such that for every $x, y \in S$, $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y), \ t \in (0, 1).$$
Hanson [51] observed that the term \((x - x^*)\), in the definition of differentiable convex function, does not play any role, while developing sufficiency and weak duality results for the nonlinear programming problem (NP). He thus replaced \((x - x^*)\) by \(\eta(x, x^*) : S \times S \rightarrow R^n\) and introduced new functions, which were named **invex** by Craven [26] and \(\eta\text{-convex}\) by Kaul and Kaur [68].

**Definition 1.1.10.** The differentiable function \(f\) is said to be **invex** at \(x^* \in S\), if there exists a vector function \(\eta(x, x^*)\) on \(S \times S\), such that for every \(x \in S\),

\[
f(x) - f(x^*) \geq \nabla f(x^*)^T \eta(x, x^*).
\]

Ben Israel and Mond [11] extended the definition of invex functions and introduced quasi invex and pseudo invex functions. Kaul and Kaur [68] called these generalizations \(\eta\)-quasiconvex and \(\eta\)-pseudoconvex respectively.

**Definition 1.1.11.** The differentiable function \(f\) is said to be **quasi invex** at \(x^* \in S\), if there exists a vector function \(\eta(x, x^*)\) on \(S \times S\), such that for every \(x \in S\),

\[
f(x) \leq f(x^*) \Rightarrow \nabla f(x^*)^T \eta(x, x^*) \leq 0.
\]

**Definition 1.1.12.** The differentiable function \(f\) is said to be **pseudo invex** at \(x^* \in S\), if there exists a vector function \(\eta(x, x^*)\) on \(S \times S\), such that for every \(x \in S\),

\[
\nabla f(x^*)^T \eta(x, x^*) \geq 0 \Rightarrow f(x) \geq f(x^*).
\]

The class of preinvex functions are defined on invex sets given as follows:
**Definition 1.1.13.** The set $S \subseteq \mathbb{R}^n$ is said to be **invex** at $x^* \in S$ if for every $x \in S$, there exists a vector function $\eta (x, x^*)$ on $S \times S$, such that

$$x^* + t \eta (x, x^*) \in S, \quad t \in [0, 1].$$

The set $S$ is said to be invex if it is invex at each $x^* \in S$.

**Remark 1.1.1.** Every convex set is invex with respect to $\eta (x, x^*) = x - x^*$, but an invex set need not be convex as viewed by the following example.

**Example 1.1.1.** The set $S = \{ x \in \mathbb{R} \mid x \sim -1, 1 \}$ is invex with respect to $\eta : S \times S \to \mathbb{R}$ given by

$$\eta (x, x^*) = \begin{cases} x - x^*, & \text{if } x > 1, x^* > 1, \quad \text{or } x < -1, x^* < -1 \\ x^* - x, & \text{if } x \geq 1, x^* \leq -1, \quad \text{or } x \leq -1, x^* \geq 1 \end{cases}$$

However, $S$ is not convex because for $x = -1, \ x^* = 1, \ t = \frac{1}{2}$

$$tx + (1 - t)x^* = \frac{1}{2} - \frac{1}{2} = 0 \notin S.$$ 

**Definition 1.1.14 ([147]).** The function $f : S \to \mathbb{R}$ defined on an invex set $S \subseteq \mathbb{R}^n$ is said to be **preinvex** if there exists a vector function $\eta (x, x^*)$ on $S \times S$, such that for every $x, x^* \in S$ and $t \in [0, 1]$,

$$f(x^* + t \eta (x, x^*)) \leq t f (x) + (1 - t) f(x^*).$$

Mohan and Neogy [98] established that preinvex and invex functions are equivalent under certain assumption.

Jeyakumar [65] introduced the class of $\rho$-invex functions.

**Definition 1.1.15 ([65]).** Let $f$ be a differentiable real valued function defined on a nonempty open set $S \subseteq \mathbb{R}^n$, $\rho$ be a real number and
\( \eta, \theta : S \times S \rightarrow \mathbb{R}^n \) be two functions; \( f \) is said to be \( \rho \text{-invex} \) at \( x^* \), with respect to \( \eta \) and \( \theta \), if for every \( x \in S \);

\[
f(x) - f(x^*) \geq \nabla f(x^*)^T \eta (x, x^*) + \rho \theta (x, x^*)^2
\]

Hanson and Mond [53] generalized invex functions to introduce the class of \( F \)-convex functions.

**Definition 1.1.16.** A functional \( F : S \times S \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be **sublinear**, if for any \( x, x^* \in S \)

\[
F(x, x^*; a_1 + a_2) \leq F(x, x^*; a_1) + F(x, x^*; a_2) \quad \text{for any } a_1, a_2 \in \mathbb{R}^n
\]

and

\[
F(x, x^*; \alpha a) = \alpha F(x, x^*; a), \quad \text{for any } \alpha \in \mathbb{R}, \alpha > 0 \text{ and } a \in \mathbb{R}^n.
\]

Clearly, \( F(x, x^*; 0) = 0 \).

**Definition 1.1.17** ([53]). The differentiable function \( f \) is said to be **\( F \)-convex** at \( x^* \in S \), if there exists a sublinear functional \( F : S \times S \times \mathbb{R}^n \rightarrow \mathbb{R} \), such that for each \( x \in S \),

\[
f(x) - f(x^*) \geq F(x, x^*; \nabla f(x^*))
\]

Preda [112] extended \( F \)-convex, \( \rho \)-convex and \( \rho \)-invex functions to \( (F, \rho) \)-convex functions.

**Definition 1.1.18.** A function \( d : S \times S \rightarrow \mathbb{R} (S \subseteq \mathbb{R}^n) \) is a **pseudo metric**, if for all \( x, y, z \in S \)

(i) \( d(x, y) \geq 0 \)

(ii) \( x = y \Rightarrow d(x, y) = 0 \)

(iii) \( d(x, y) = d(y, x) \)

(iv) \( d(x, z) \leq d(x, y) + d(y, z) \).
Definition 1.1.19. Let $f : S \to R$, $F$ be a sublinear functional, $d(.,.)$ be a pseudometric on $S \times S$ and $\rho$ be a real number, then $f$ is said to be $(F, \rho)$-convex at $x^*$ if for all $x \in S$,

$$f(x) - f(x^*) \geq F(x, x^*; \nabla f(x^*)) + \rho d^2(x, x^*).$$

Hanson and Mond [52] introduced a generalization of the class of invex functions to a class of Type-I functions for the nonlinear programming problem (NP).

Definition 1.1.20. $f(x)$ and $g(x)$ are type-I objective and constraint functions respectively at $x^*$ if there exists a vector function $\eta(x, x^*)$ defined on $X_1$, the set of all feasible solutions of (NP), such that for each $x \in X_1$,

$$f(x) - f(x^*) \geq \nabla f(x^*)^T \eta (x, x^*) - g(x^*) \geq \nabla g(x^*) \eta (x, x^*).$$

Rueda and Hanson [118] extended these type-I functions to quasi type-I and pseudo type-I functions.

Definition 1.1.21. $f(x)$ and $g(x)$ are quasi type-I objective and constraint functions respectively at $x^*$ if there exists a vector function $\eta (x, x^*)$ defined on $X_1$ such that for each $x \in X_1$,

$$f(x) \leq f(x^*) \quad \Rightarrow \quad \nabla f(x^*)^T \eta (x, x^*) \leq 0$$

$$-g(x^*) \leq 0 \quad \Rightarrow \quad \nabla g(x^*) \eta (x, x^*) \leq 0.$$

Definition 1.1.22. $f(x)$ and $g(x)$ are pseudo type-I objective and constraint functions respectively at $x^*$ if there exists a vector function $\eta (x, x^*)$ defined on $X_1$, such that for each $x \in X_1$,
\[ \nabla f(x^*)^T \eta (x, x^*) \geq 0 \implies f(x) \geq f(x^*) \]
\[ \nabla g(x^*) \eta (x, x^*) \geq 0 \implies -g(x^*) \geq 0 \]

1.1.9. Cone Convex and Related Functions

We first recall the preliminary concepts related to cones.

**Definition 1.1.23.** Let \( K \) be a nonempty subset of \( \mathbb{R}^m \) then \( K \) is said to be a **cone** if \( t k \in K, \) whenever \( k \in K, t \geq 0. \)

**Definition 1.1.24.** The cone generated by \( K \) is defined by

\[
\text{Cone} \, K = \{ t k : t \geq 0, k \in K \}.
\]

**Definition 1.1.25.** Cone \( K \) is said to be **pointed** if \( K \cap (-K) = \{0\}. \)

**Definition 1.1.26.** Cone \( K \) is called a **convex cone** if \( K + K \subseteq K. \)

Let \( K \) be a closed convex cone with nonempty interior and vertex at the origin. Let \( \text{int} K \) and \( \text{Cl} K \) denote the interior and closure of cone \( K \) respectively. The positive dual cone \( K^+ \) of \( K \) is defined as

\[
K^+ = \{ y^* \in \mathbb{R}^m : y^*^T y \geq 0, \text{ for all } y \in K \}.
\]

The strict positive dual cone \( K^{S+} \) is given by

\[
K^{S+} = \{ y^* \in \mathbb{R}^m : y^*^T y > 0, \text{ for all } y \in K \setminus \{0\} \}.
\]

Weir et al. [149] considered the following generalization of convex functions.

**Definition 1.1.27.** The function \( f \) is said to be **\( K \)-convex** on \( S \), if

\[
t f(x) + (1-t) f(y) - f(tx + (1-t)y) \in K, \text{ for all } x, y \in S, t \in (0, 1).
\]
Definition 1.1.28 ([146]). The function $f$ is said to be $K$-preinvex on $S$, where $S \subseteq R^n$ is a nonempty invex subset, if
\[
t f(x) + (1-t) f(y) - f(y + t \eta (x, y)) \in K, \text{ for all } x, y \in S, \ t \in (0, 1).
\]

Definition 1.1.29 ([62]). The function $f$ is $K$-subconvexlike on $S$, where $S \subseteq R^n$, if, there exists $v \in \text{int} K$ and for any $t \in (0, 1), \ e > 0$, there exists $u \in S$ such that
\[
e v + t f(x) + (1-t) f(y) - f(u) \in K.
\]

Cambini [15] generalized the class of scalar generalized concave functions by using three order relations, generated by a cone $K$, interior of a cone $K^{(0)}$ and the cone $K$ without origin ($K^0$).

Let $f : S \rightarrow R^m$, where $S \subseteq R^n$ is a convex set, be a vector valued function.

Definition 1.1.30. Set $K^* \in \{K, K^0, K^{(0)}\}$; the function $f$ is $K^*$-concave if and only if for every $x, y \in S, x \neq y,$
\[
f(x + t(y - x)) - t(f(y) - f(x)) \in f(x) + K^* \text{ for } t \in (0,1).
\]

Definition 1.1.31. Set $K^* \in \{K, K^0, K^{(0)}\}$ and $K^\# \in \{K^0, K^{(0)}\}$; the function $f$ is $(K^*, K^\#)$-quasiconcave if and only if for every $x, y \in S, x \neq y,$
\[
f(y) \in f(x) + K^* \Rightarrow f(x + t(y - x)) \in f(x) + K^\# \text{ for } t \in (0,1).
\]

Definition 1.1.32. Set $K^* \in \{K, K^0, K^{(0)}\}$ and $K^\# \in \{K^0, K^{(00)}\}$; the function $f$ is $(K^*, K^\#)$-pseudoconcave if and only if for every $x, y \in S, x \neq y,$
\[
f(y) \in f(x) + K^* \Rightarrow J_f(x)(y - x) \in K^\#
\]
where $J_f(x)$ denotes the Jacobian of $f$ at $x.$
1.1.10. Vector Optimization Over Cones

Vector optimization problems have been widely developed in recent years. Various solution approaches have been considered by using the idea of cone for minimizing a function.

A general vector minimization problem with respect to cones is given by

**(VP)** $K$-minimize $f(x)$

subject to $-g(x) \in Q$

where $f : S \rightarrow \mathbb{R}^m$ and $g : S \rightarrow \mathbb{R}^p$, $S$ is a nonempty subset of $\mathbb{R}^n$, $K$ and $Q$ are closed convex pointed cones in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively with nonempty interiors. The feasible set of (VP) is given by

$$X_0 = \{x \in S : -g(x) \in Q\}$$

Yu [160] generalized the concept of efficiency to cone efficiency. Coladas et al. [23] considered the following types of solutions to (VP).

**Definition 1.1.33.** A point $\bar{x} \in X_0$ is called

(i) a weak minimum of (VP), if for all $x \in X_0$, $f(\bar{x}) - f(x) \notin \text{int} \ K$

(ii) a minimum or efficient solution of (VP), if for all $x \in X_0$,

$$f(\bar{x}) - f(x) \notin K \setminus \{0\}.$$  

(iii) a strong minimum of (VP), if for all $x \in X_0$, $f(x) - f(\bar{x}) \in K$.

Borwein [14] introduced proper efficiency over cones. The definition was strengthened by Benson [12] to assure equivalence to the Geoffrion definition even when the decision set is non convex.

Benson [12] proposed the following definition of proper efficiency.
Definition 1.1.34. $\bar{x} \in X_0$ is called a Benson proper minimum of (VP) if

$$(-K) \cap \text{Cl} \text{ cone}(f(X_0) + K - f(\bar{x})) = \{0\}$$

where Cl stands for the closure of the set.

Optimality conditions for vector optimization problems have been studied extensively in the literature by various researchers like Hayashi and Komiya [54], Weir et al. [149], Weir and Jeyakumar [146], Jeyakumar [62], Wang and Li [142], Khanh and Nuong [74], Fulga and Preda [42], Dutta [31] and Jahn [61].

1.1.11. Nonsmooth Optimization

Nonsmooth optimization deals with the minimization or maximization of functions which do not have the differentiability properties required by classical methods. The field of nonsmooth optimization is significant, not only because of the existence of non differentiable functions arising directly in applications, but also because several important methods for solving difficult smooth problems lead directly to the need to solve nonsmooth problems, which are either smaller in dimension or simpler in structure. Nonsmooth optimization problems occur in many fields, including engineering, mathematics and economics.

Several authors have studied optimization problems involving nonsmooth functions. Based on the work of Craven [25], Reiland [117] extended the concept of invexity to nonsmooth Lipschitz functions. Nobakhtian [110] studied sufficiency results for nonsmooth multiobjective programming problem involving generalized $(F, \rho)$-convex functions. Kuk and Tanino [79] studied optimality and duality results for a nonsmooth multiobjective optimization problem involving generalized type-I functions while Suneja et al. [135] investigated optimality and
duality results for nonsmooth multiobjective fractional programming problems under generalized type-I assumptions. Zhao [164] gave Karush-Kuhn-Tucker type sufficiency conditions and duality results in non differentiable scalar optimization problems under type-I functions. Yen and Sach [159] studied the classes of nonsmooth functions, namely, cone invex and cone invex in the limit. Suneja et al. [137] named these functions cone generalized invex and cone nonsmooth invex. They further studied optimality and duality results for nonsmooth vector optimization problems by using Generalized Alternative Theorem given by Craven and Yang [28].

Directional derivatives and subgradients are used to study nonsmooth optimization as compared to gradients in the case of smooth functions. Clarke [22] introduced generalized gradients by assuming the functions to be Lipschitz.

**Definition 1.1.35.** A real valued function \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) is **Lipschitz** if there exists a positive constant \( \ell \) such that for all \( x, \bar{x} \in \mathbb{R}^n \),

\[
|f(x) - f(\bar{x})| \leq \ell \|x - \bar{x}\|, \quad \text{where } \| . \| \text{ denotes the Eucledian norm.}
\]

**Definition 1.1.36.** \( \varphi \) is said to be **locally Lipschitz** at a point \( u \in \mathbb{R}^n \) if there exists a number \( \ell > 0 \) such that

\[
|f(x) - f(\bar{x})| \leq \ell \|x - \bar{x}\|,
\]

for all \( x, \bar{x} \) in a neighbourhood of \( u \).

The function \( \varphi \) is said to be locally Lipschitz, if it is locally Lipschitz at each point of \( \mathbb{R}^n \).
Definition 1.1.37 ([22]). Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be locally Lipschitz, then \( \varphi^0(u; v) \) denotes the **Clarke's generalized directional derivative** of \( \varphi \) at \( u \in \mathbb{R}^n \) in the direction \( v \) and is defined as

\[
\varphi^0(u; v) = \limsup_{y \to u, t \to 0^+} \frac{\varphi(y + tv) - \varphi(y)}{t}.
\]

Clarke’s generalized gradient of \( \varphi \) at \( u \) is denoted by \( \partial \varphi(u) \) and is defined as

\[
\partial \varphi(u) = \{ \xi \in \mathbb{R}^n : \varphi^0(u; v) \geq \langle \xi, v \rangle, \text{ for all } v \in \mathbb{R}^n \}.
\]

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a vector valued function given by \( f = (f_1, f_2, \ldots, f_m) \), \( f_i : \mathbb{R}^n \to \mathbb{R} \). Then \( f \) is said to be locally Lipschitz on \( \mathbb{R}^n \) if each \( f_i \) is locally Lipschitz on \( \mathbb{R}^n \). The generalized directional derivative of a locally Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R}^m \) at \( u \in \mathbb{R}^n \) in the direction \( v \) is given by

\[
f^0(u; v) = \{ f_1^0(u; v), f_2^0(u; v), \ldots, f_m^0(u; v) \}.
\]

The generalized gradient of \( f \) at \( u \) is the set

\[
\partial f(u) = \partial f_1(u) \times \ldots \times \partial f_m(u),
\]

where \( \partial f_i(u) \) is the generalized gradient of \( f_i \) at \( u, i = 1, 2, \ldots, m \).

Every element \( A = (r_1, r_2, \ldots, r_m) \in \partial f(u) \) is a continuous linear operator from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) and

\[
Au = (\langle r_1, u \rangle, \ldots, \langle r_m, u \rangle) \in \mathbb{R}^m, \text{ for all } u \in \mathbb{R}^n.
\]

The following are some properties of Clarke’s generalized gradient:

**Lemma 1.1.3.** (i) If \( f_i : \mathbb{R}^n \to \mathbb{R} \) is locally Lipschitz then, for each \( u \in \mathbb{R}^n \),

\[
f_i^0(u; v) = \max\{ \langle \xi, v \rangle : \xi \in \partial f_i(u) \}.
\]
(ii) Let $f_i (i = 1, 2, ..., m)$ be a finite family of locally Lipschitz functions on $\mathbb{R}^n$, then $\sum_{i=1}^{m} f_i$ is also locally Lipschitz and

$$\partial \left( \sum_{i=1}^{m} f_i \right)(u) \subseteq \sum_{i=1}^{m} \partial f_i(u), \text{ for every } u \in \mathbb{R}^n.$$ 

Yen and Sach [159] studied $K$-invex and $K$-invex in the limit functions as given below:

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a locally Lipschitz function on $\mathbb{R}^n$.

**Definition 1.1.38 ([159]).** $f$ is said to be $K$-invex at the point $u \in \mathbb{R}^n$ if there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$, $A \in \partial f(u)$

$$f(x) - f(u) - A\eta(x, u) \in K.$$ 

**Definition 1.1.39 ([159]).** $f$ is said to be $K$-invex in the limit at $u \in \mathbb{R}^n$, if there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$,

$$f(x) - f(u) - f^\circ(u; \eta) \in K.$$ 

The above defined functions are respectively called $K$-generalized invex and $K$-nonsmooth invex by Suneja et al. [137].

### 1.1.12. Semilocusly Convex and Related Concepts

The concept of semilocally convex functions was introduced by Ewing [36] who applied the notion to provide sufficient optimality conditions in variational and control problems. These functions have some important properties such as local minimum of a semilocally convex function defined on a locally star shaped set is a global minimum and non-negative linear combination of semilocally convex functions is also semilocally convex. Kaul and Kaur [70] defined semilocally quasiconvex and semilocally pseudoconvex functions. Suneja and Gupta [132] defined the (strict) semilocally pseudoconvexity at a point with respect to a set.
By using these concepts Kaul and Kaur [69, 71] and Suneja and Gupta [132, 134] obtained optimality conditions and duality results respectively for a class of nonlinear programming problems. Gupta and Vartak [49] defined \( \rho \)-semilocally convex and related functions and studied sufficient optimality conditions for a nonlinear program involving these functions. Mukherjee and Mishra [107] and Preda [113] discussed optimality results for a multiobjective programming problem using semilocally convex functions. Weir [145] introduced cone-semilocally convex functions and studied optimality conditions and duality theorems for vector optimization problems over cones.

Preda and Stancu-Minasian [115] discussed the duality results for a multiobjective programming problem and Stancu-Minasian [129] established optimality and duality results for a nonlinear fractional programming problem using semilocally preinvex and related functions. Suneja et al. [138] studied optimality and duality results for a multiobjective programming problem by introducing \( \rho \)-semilocally preinvex, pseudo preinvex and quasi preinvex functions Mishra et al. [97] and Niculescu [109] studied optimality and duality results for multiobjective fractional programming problems involving semilocally type-I preinvex and \( \rho \)-semilocally type-I preinvex functions respectively defined on \( \eta \)-locally starshaped set.

Let \( S \) be a nonempty subset of \( R^n \) and \( \eta : S \times S \rightarrow R^n \) be a vector valued function.

**Definition 1.1.40 ([114]).** The set \( S \) is said to be \( \eta \)-locally star shaped at \( \bar{x} \in S \) if for each \( x \in S \), there exists a positive number \( a_{\eta}(x,\bar{x}) \leq 1 \) such that \( \bar{x} + \tau \eta(x,\bar{x}) \in S \) for \( 0 < \tau < a_{\eta}(x,\bar{x}) \).

If \( \eta(x,\bar{x}) = x - \bar{x} \), then \( S \) is said to be locally star shaped at \( \bar{x} \in S \) [36].
**Definition 1.1.41 ([36]).** Let \( \psi \) be a numerical function defined on a set \( S \subseteq \mathbb{R}^n \), where \( S \) is locally star shaped at \( \bar{x} \in S \). The right differential of \( \psi \) at \( \bar{x} \in S \) in the direction \( \bar{x} - x \), denoted by \( (d\psi)^+ (\bar{x}, x - \bar{x}) \) is defined as

\[
(d\psi)^+ (\bar{x}, x - \bar{x}) = \lim_{t \to 0^+} \frac{\psi(\bar{x} + t(x - \bar{x})) - \psi(\bar{x})}{t}
\]

provided the limit exists.

**Remark 1.1.2.** \( (d\psi)^+ (\bar{x}, v) = \lim_{t \to 0^+} \frac{\psi(\bar{x} + tv) - \psi(\bar{x})}{t} \) is the right differential of \( \psi \) at \( x^* \) in the direction \( v \).

Let \( f : S \to \mathbb{R}^m \) be a vector valued function. Then \( f = (f_1, f_2, \ldots, f_m) \) and \( (df)^+ (\bar{x}, v) = ((df_1)^+ (\bar{x}, v), (df_2)^+ (\bar{x}, v), \ldots, (df_m)^+ (\bar{x}, v)) \).

### 1.1.13. Second and Higher Order Convex and Related Functions

Mond [100] defined second order convex functions which were called bonvex by Bector and Chandra [5] and are defined as follows:

**Definition 1.1.42.** A twice differentiable function \( f : S \subseteq \mathbb{R}^n \to \mathbb{R} \) is called **second order convex** at \( x^* \in S \), if for all \( x \in S \), \( p \in \mathbb{R}^n \)

\[
f(x) - f(x^*) \geq (x - x^*)^T [\nabla_x f(x^*) + \nabla_{xx} f(x^*) p] - \frac{1}{2} p^T \nabla_{xx} f(x^*) p.
\]

Mond and Weir [103, 104] extended second order convex functions to define second order pseudoconvex and second order quasiconvex functions.

Mishra and Rueda [95] introduced the concept of second order \((F, \rho)\)-convexity. Hanson [50] defined second order type-I functions. Srivastava and Govil [126] introduced second order \((F, \rho, \sigma)\)-type-I functions for the problem (MOP) as follows:
Let $F$ be a sublinear functional, $\rho \in R$ be a scalar, $d : S \times S \to R$ be a pseudometric and $p, q, r$ be vector valued functions defined from $X_1 \times S$ to $R^n$.

**Definition 1.1.43 ([126]).** For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, $(f_i, g_j)$ is said to be **second order** $(F, \rho, \sigma_j)$-type-I with respect to vector functions, $p, q, r$ at $x^* \in S$, if for all $x \in X_1$,

$$f_i(x) - f_i(x^*) \geq F(x, x^*; \nabla_x f_i(x^*) + \nabla_{xx} f_i(x^*)p(x, x^*) + \rho_i d^2(x, x^*)$$

$$-\frac{1}{2} q(x, x^*)^T \nabla_{xx} f_i(x^*)r(x, x^*)$$

and

$$-g_j(x^*) \geq F(x, x^*; \nabla_x g_j(x^*) + \nabla_{xx} g_j(x^*)p(x, x^*) + \sigma_j d^2(x, x^*)$$

$$-\frac{1}{2} q(x, x^*)^T \nabla_{xx} g_j(x^*)r(x, x^*)$$

Suneja et al. [131] introduced second order $K$-convex and other related functions where $K$ is a closed convex pointed cone in $R^m$.

Let $f_i, \ i = 1, \ldots, m$ be twice continuously differentiable real valued functions defined on a nonempty subset $S$ of $R^n$ and $f = (f_1, f_2, \ldots, f_m)$.

**Definition 1.1.44.** $f$ is said to be **second order $K$-convex** at $x^* \in S$ with respect to $p \in R^n$ if for every $x \in S$

$$\left[f_i(x) - f_i(x^*) - (x - x^*)^T (\nabla_x f_i(x^*) + \nabla_{xx} f_i(x^*)p) + \frac{1}{2} p^T \nabla_{xx} f_i(x^*)p, \ldots, \right.$$\n
$$\left. f_m(x) - f_m(x^*) - (x - x^*)^T (\nabla_x f_m(x^*) + \nabla_{xx} f_m(x^*)p) + \frac{1}{2} p^T \nabla_{xx} f_m(x^*)p \right] \in K.$$

**Definition 1.1.45.** $f$ is said to be **second order $K$-pseudoconvex** at $x^* \in S$ with respect to $p \in R^n$ if for every $x \in S$
Definition 1.1.46. \( f \) is said to be **second order strongly\) \( K\)-pseudoconvex at \( x^* \in S \) with respect to \( p \in \mathbb{R}^n \) if for every \( x \in S \),

\[
-(x-x^*)^T(\nabla_x f_1(x^*) + \nabla_{xx} f_1(x^*)p), \ldots, -(x-x^*)^T(\nabla_x f_m(x^*) + \nabla_{xx} f_m(x^*)p) \notin \text{int } K
\]

\[
\Rightarrow \left[ f_1(x) - f_1(x^*) + \frac{1}{2} p^T \nabla_{xx} f_1(x^*)p, \ldots, f_m(x) - f_m(x^*) + \frac{1}{2} p^T \nabla_{xx} f_m(x^*)p \right] \notin \text{int } K.
\]

Definition 1.1.47. \( f \) is said to be **second order \( K\)-quasiconvex at \( x^* \in S \) with respect to \( p \in \mathbb{R}^n \) if for every \( x \in S \),

\[
f_1(x) - f_1(x^*) + \frac{1}{2} p^T \nabla_{xx} f_1(x^*)p, \ldots, f_m(x) - f_m(x^*) + \frac{1}{2} p^T \nabla_{xx} f_m(x^*)p \notin \text{int } K
\]

\[
\Rightarrow \left[ -(x-x^*)^T(\nabla_x f_1(x^*) + \nabla_{xx} f_1(x^*)p), \ldots, -(x-x^*)^T(\nabla_x f_m(x^*) + \nabla_{xx} f_m(x^*)p) \right] \notin \text{int } K.
\]

The concept of second order convexity was extended to higher order convexity by Zhang [162] and Mishra and Rueda [96] introduced higher order generalized invexity.

Chen [19] defined the class of higher order \( F\)-convex functions as given below:

**Definition 1.1.48.** Let \( h : S \times \mathbb{R}^n \to \mathbb{R} \) be a differentiable function, \( F \) be a sublinear functional. \( f : S \to \mathbb{R} \) is said to be **higher order \( F\)-convex** at \( x^* \in S \) with respect to \( h \), for all \((x, p) \in S \times \mathbb{R}^n\).

\[
f(x) - f(x^*) \geq F(x, x^*; \nabla_x f(x^*) + \nabla_p h(x^*, p)) + h(x^*, p) - p^T[\nabla_p h(x^*, p)]
\]

He further introduced higher order \( F\)-pseudoconvex and higher order \( F\)-quasiconvex functions.
1.2 Summary of the Thesis

The research work carried out by the author is presented in Chapters 2 to 4.

In Chapter 2 we discuss some important generalizations of convex functions over cones and study optimality conditions and duality results for vector optimization problems. This chapter is divided into four sections.

In Section 2.1 we first introduce the concept of cone semistrictly convex functions on topological vector spaces as an extension of the class of semistrictly convex functions. The following result gives a characterization of cone convex functions in terms of Gâteaux derivative

\[\text{Theorem 1.2.1.} \quad \text{Let} \, f : S \to Y, \text{ be Gâteaux differentiable then } f \text{ is } C\text{-convex on } S \text{ if and only if} \]

\[f(x) - f(y) - f_y'(x - y) \in C, \quad \forall \, x, y \in S\]

where \( f_y'(x - y) \) is the Gâteaux derivative of \( f \) at \( y \) in the direction \( x - y \).

An interrelation between cone convex and cone semistrictly convex functions is given by the following theorem:

\[\text{Theorem 1.2.2.} \quad \text{Let} \, f : S \to Y \text{ be } C\text{-semistrictly convex. If there exists } \alpha \in (0, 1) \text{ such that for every } x, y \in S, \]

\[\alpha f(x) + (1 - \alpha) f(y) - f(\alpha x + (1 - \alpha) y) \in C, \]

then \( f \) is \( C\)-convex.

In Section 2.2, sufficiency and duality results are established for the following problem
(MP) C-minimize $f(x)$
subject to $-g(x) \in D$,

where $f : X \to Y$ and $g : X \to Z$; $X$ is a topological vector space and $Y$ and $Z$ are locally convex linear spaces. $C \subseteq Y$ and $D \subseteq Z$ are closed convex pointed cones with nonempty interiors.

Let $X_2 = \{ x \in X : -g(x) \in D \}$ be the feasible set of (MP).

**Theorem 1.2.3.** Let $(f, g)$ be Gâteaux differentiable at $\bar{x} \in X_2$ and $(C \times D)$-subconvex on $X$. If there exist $\bar{\lambda} \in C^* \setminus \{0\}$, $\bar{\mu} \in D^*$ such that

$$
\langle \bar{\lambda}, f'_x(x) \rangle + \langle \bar{\mu}, g'_x(x) \rangle = 0, \quad \forall \ x \in X_2,
$$

$$
\langle \bar{\mu}, g(\bar{x}) \rangle = 0
$$

then $\bar{x}$ is a weak minimum solution of (MP).

**Theorem 1.2.4.** Let $(f, g)$ be Gâteaux differentiable at $\bar{x} \in X_2$ and $(C \times D)$ subconvex on $X$. If there exist $\bar{\lambda} \in C^*$, $\bar{\mu} \in D^*$, $(\bar{\lambda}, \bar{\mu}) \neq 0$ such that (1.1) and (1.2) hold then $\bar{x}$ is an efficient solution of (MP).

The following dual is associated with (MP) and duality results are proved

(MD) C-maximize $f(u)$
subject to $\langle \lambda, f'_u(x-u) \rangle + \langle \mu, g'_u(x-u) \rangle \geq 0$, $\forall x \in X_2$.
$$
\langle \mu, g(u) \rangle \geq 0,
$$
$$
\mu \in D^*, \quad 0 \neq \lambda \in C^*, \quad u \in X.
$$

The following weak and strong duality results are proved for the pair (MP) and (MD).
**Theorem 1.2.5 (Weak Duality Theorem).** Let $\overline{x}$ be a feasible solution for (MP) and $(\overline{u}, \overline{\lambda}, \overline{\mu})$ be feasible for (MD). Let $(f, g)$ be Gâteaux differentiable at $\overline{x} \in X_2$ and $(C \times D)$-subconvex on $X$, then,

$$f(\overline{u}) - f(\overline{x}) \notin \text{int} C.$$ 

**Theorem 1.2.6 (Strong Duality theorem).** Let $(f, g)$ be Gâteaux differentiable and $(C \times D)$-subconvex on $X$. Suppose that $g$ satisfies the generalized Slater constraint qualification. If $\overline{x}$ is a weak minimum of (MP), then there exist $0 \neq \overline{\lambda} \in C^*$, $\overline{\mu} \in D^*$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weak maximum solution of (MD).

In **Section 2.3** we introduce cone semilocally preinvex, cone semilocally quasi preinvex and cone semilocally pseudo preinvex functions over an $\eta$-locally star shaped set and study their properties and relations with some functions already present in the literature.

The section considers the vector optimization problem

**(VOP)** $K$-minimize $f(x)$

subject to $-g(x) \in Q$

$h(x) = \{0_{R^k}\}$

where $f : S \to R^m$, $g : S \to R^p$ and $h : S \to R^k$ are $\eta$-semi differentiable functions with respect to same $\eta$ and $S \subseteq R^n$ is a nonempty $\eta$-locally star shaped set.

Let $K \subseteq R^m$ and $Q \subseteq R^p$ be closed convex cones having nonempty interiors and let $X_3 = \{x \in S : -g(x) \in Q, h(x) = \{0_{R^k}\}\}$ be the set of all feasible solutions of (VOP).
An alternative theorem is given on the lines of Illés and Kassay [60] and necessary and sufficient optimality conditions are obtained for the above problem.

**Theorem 1.2.7 (Theorem of Alternative).** Let \( F_0 = (f,g,h) : S \rightarrow S' \) where \( S \subseteq \mathbb{R}^n \) is a nonempty set and \( S' = \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^k \) and \( K_0 = (K \times Q \times \{0_{\mu^*}\}) \). If \( F_0 \) is \( K_0 \)-slpi on \( S \), that is, \( f \) is \( K \)-slpi, \( g \) is \( Q \)-slpi and \( h \) is \( \{0_{\mu^*}\} \)-slpi with respect to same \( \eta \), such that \( F_0(S) + K_0 \) is closed with nonempty interior, then the following assertions hold:

(i) if there is no \( x \in S \) such that \( f(x) \in \text{int} K \), \( g(x) \in -Q \) and \( h(x) = \{0_{\mu^*}\} \), then there exist \( \lambda \in K^+ \), \( \mu \in Q^+ \) and \( v \in R^k \) with \((\lambda,\mu,v) \neq 0\) such that \( \lambda^T f(x) + \mu^T g(x) + v^T h(x) \geq 0 \), \( \forall \ x \in S \).

(ii) if there exist \( \lambda \in K^+ \setminus \{0\} \), \( \mu \in Q^+ \) and \( v \in R^k \) such that

\[
\lambda^T f(x) + \mu^T g(x) + v^T h(x) \geq 0 , \ \forall \ x \in S ,
\]

then there is no \( x \in S \) such that \( f(x) \in \text{int} K \), \( g(x) \in -Q \) and \( h(x) = \{0_{\mu^*}\} \).

**Theorem 1.2.8 (Necessary optimality Conditions).**

Let \( F_1(x) = (f(x) - f(\bar{x}), g(x), h(x)) \ \forall x \in S \) and \( F_1(S) + (K \times Q \times \{0_{\mu^*}\}) \) be closed, with nonempty interior. Let \( \bar{x} \in X_3 \) be a weak minimum of (VOP), \( f \) be \( K \)-slpi, \( g \) be \( Q \)-slpi and \( h \) be \( \{0_{\mu^*}\} \)-slpi with respect to same \( \eta \).

Suppose that \( (g,h) \) satisfies generalized Slater type constraint qualification and \( \eta(\bar{x},\bar{x}) = 0 \), then there exist \( 0 \neq \lambda \in K^+ \), \( \mu \in Q^+ \) and \( v \in R^k \) such that

\[
\lambda^T (df)^+(\bar{x},\eta(x,\bar{x}))+\mu^T (dg)^+(\bar{x},\eta(x,\bar{x}))+v^T (dh)^+(\bar{x},\eta(x,\bar{x})) \geq 0 , \ (1.3)
\]

for all \( x \in S \),
and \( \bar{\mu}^T g(\bar{x}) = 0 \).

(1.4)

**Theorem 1.2.9 (Sufficient Optimality Conditions).** Let \( \bar{x} \in X_3 \), \( f \) be \( K \)-slppi, \( g \) be \( Q \)-slqpi and \( h \) be \( \{0_{R^k}\} \)-slnqpi at \( \bar{x} \) with respect to same \( \eta \).

If there exist \( 0 \neq \bar{\lambda} \in K^+ \), \( \bar{\mu} \in Q^+ \) and \( \bar{\nu} \in R^k \) such that (1.3) holds, \( \forall x \in X_3 \) and (1.4) is satisfied, then \( \bar{x} \) is a weak minimum of (VOP).

At the end we associate a Mond-Weir type dual and obtain the duality results.

(VOD) \( K \)-maximize \( f(u) \)

subject to

\[
\lambda^T (df)^+(u,\eta(x,u)) + \mu^T (dg)^+(u,\eta(x,u)) + \nu^T (dh)^+(u,\eta(x,u)) \geq 0 \quad (1.5)
\]

\[ \forall x \in X_3 \]

\[
\mu^T g(u) \geq 0 \quad (1.6)
\]

\[
h(u) = 0_{R^k} \quad (1.7)
\]

\( 0 \neq \lambda \in K^+, \mu \in Q^+ \) and \( \nu \in R^k \), \( u \in S \)

**Theorem 1.2.10 (Weak Duality).** Let \( x \) be feasible for (VOP) and \( (u,\lambda,\mu,\nu) \) be feasible for (VOD). Let \( f \) be \( K \)-slppi, \( g \) be \( Q \)-slqpi and \( h \) be \( \{0_{R^k}\} \)-slnqpi at \( u \), with respect to same \( \eta \). Then

\[
f(u) - f(x) \in \text{int } K.
\]

**Theorem 1.2.11 (Strong Duality).** Let \( f \) be \( K \)-slpi, \( g \) be \( Q \)-slpi and \( h \) be \( \{0_{R^k}\} \)-slpi with respect to same \( \eta \). Let \( F_1(S) + (K \times Q \times \{0_{R^k}\}) \) be closed with nonempty interior. Suppose that the pair \((g, h)\) satisfies generalized Slater type constraint qualification. If \( \bar{x} \) is a weak minimum of (VOP)
and \( \eta(\bar{x}, \bar{x}) = 0 \), then there exist \( 0 \neq \lambda^* \in K^+ \), \( \mu^* \in Q^+ \) and \( \nu^* \in R^k \) such that \( (\bar{x}, \lambda^*, \mu^*, \nu^*) \) is a feasible solution of (VOD). Moreover if the conditions of Weak Duality Theorem 1.2.10 are satisfied for all feasible solutions of (VOP) and (VOD) then \( (\bar{x}, \lambda^*, \mu^*, \nu^*) \) is a weak maximum of (VOD).

**Section 2.4** introduces generalized type-I, generalized quasi type-I, generalized pseudo type-I, generalized quasi pseudo type-I and generalized pseudo quasi type-I functions over cones, for the following nonsmooth vector optimization problem using Clarke’s generalized gradients of locally Lipschitz functions.

**\( (VP) \)** \( K \)-minimize \( f(x) \)

subject to \(-g(x) \in Q\)

where \( f : R^n \to R^m \), \( g : R^n \to R^p \) are locally Lipschitz functions on \( R^n \), \( K \) and \( Q \) are closed convex pointed cones with nonempty interiors in \( R^m \) and \( R^p \) respectively.

Let \( X_0 = \{x \in R^n : -g(x) \in Q\} \) be the feasible set of (VP).

The following sufficient optimality conditions are proved for (VP).

**Theorem 1.2.12.** Let \((f, g)\) be \((K \times Q)\) generalized type-I at \( x_0 \in X_0 \) with respect to \( \eta : X_0 \times R^n \to R^n \) and suppose that there exist \( 0 \neq \lambda^* \in K^+ \), \( \mu^* \in Q^+ \), such that

\[
0 \in \partial(\lambda^* T f)(x_0) + \partial(\mu^* T g)(x_0) \tag{1.8}
\]

\[
\mu^* T g(x_0) = 0 \tag{1.9}
\]

Then \( x_0 \) is a weak minimum of (VP).
Theorem 1.2.13. Let \((f, g)\) be \((K \times Q)\) generalized pseudo quasi type-I at \(x_0 \in X_0\) with respect to \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) and suppose there exist \(0 \neq \lambda^* \in K^+, \mu^* \in Q^+\), such that (1.8), (1.9) hold, then \(x_0\) is a weak minimum of (VP).

Theorem 1.2.14. Let \((f, g)\) be \((K \times Q)\) generalized type-I at \(x_0 \in X_0\) with respect to \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\). Suppose there exist \(\lambda^* \in K^{S^+}, \mu^* \in Q^+\), such that (1.8), (1.9) hold. Then \(x_0\) is a minimum of (VP).

Theorem 1.2.15. Let \((f, g)\) be strictly \((K \times Q)\) generalized pseudo quasi type-I at \(x_0 \in X_0\) with respect to \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\). Suppose that there exist \(0 \neq \lambda^* \in K^+, \mu^* \in Q^+\), such that (1.8) and (1.9) hold. Then \(x_0\) is a minimum of (VP).

Theorem 1.2.16. Let \((f, g)\) be strongly \((K \times Q)\) generalized pseudo quasi type-I at \(x_0 \in X_0\) with respect to \(\eta : X_0 \times \mathbb{R}^n \to \mathbb{R}^n\) and suppose that there exist \(0 \neq \lambda^* \in K^+, \mu^* \in Q^+\) such that (1.8) and (1.9) hold. Then \(x_0\) is a strong minimum of (VP).

The following Mond-Weir type dual is associated with (VP) and weak and strong duality results are established

(VD) \(K\)-maximize \(f(u)\)

subject to \(0 \in \partial(\lambda^T f)(u) + \partial(\mu^T g)(u)\)

\[\mu^T g(u) \geq 0\]

\[0 \neq \lambda \in K^+, \mu \in Q^+\]

Chapter 3 studies second order symmetric duality for a multiobjective programming problem and a vector optimization problem over cones. This chapter is divided into two sections.
Section 3.1 begins with defining second order \((F, \rho)\) convex and second order \((F, \rho)\) pseudoconvex functions in two variables. In this section we consider two pairs of second order symmetric dual problems and study duality results.

The first pair of second order multiobjective programming problems is of Wolfe type:

(WMP) \(\begin{align*}
&\text{minimize } (M_1(x, y, \lambda, p_1), \ldots, M_k(x, y, \lambda, p_k)) \\
&\text{subject to } \\
&\sum_{i=1}^b \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) p_i) \leq 0 \\
&\lambda > 0, \sum_{i=1}^b \lambda_i = 1
\end{align*}\)

(WMD) \(\begin{align*}
&\text{maximize } (N_1(u, v, \lambda, q_1), \ldots, N_k(u, v, \lambda, q_k)) \\
&\text{subject to } \\
&\sum_{i=1}^k \lambda_i (\nabla_u f_i(u, v) + \nabla_{xx} f_i(u, v) q_i) \geq 0 \\
&\lambda > 0, \sum_{i=1}^k \lambda_i = 1
\end{align*}\)

where

\(M_i(x, y, \lambda, p_i) = f_i(x, y) - y^T \sum_{i=1}^b \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) p_i) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i\)

\(N_i(u, v, \lambda, q_i) = f_i(u, v) - u^T \sum_{i=1}^k \lambda_i (\nabla_u f_i(u, v) + \nabla_{xx} f_i(u, v) q_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i\).

\(f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}, (1 \leq i \leq k)\) are thrice differentiable functions;
\( q_i \) \((1 \leq i \leq k)\) are vectors in \( R^n \) and \( p_i (1 \leq i \leq k) \) are vectors in \( R^m \),
\( p = (p_1, p_2, \ldots , p_k), q = (q_1, q_2, \ldots , q_k) \) and \( \lambda_i \in R, (1 \leq i \leq k), \lambda = (\lambda_1, \lambda_2, \ldots , \lambda_k)^T \).

**Theorem 1.2.17 (Weak Duality).** Let \((x, y, \lambda, p)\) be feasible for \((WMP)\) and \((u, v, \lambda, q)\) be feasible for \((WMD)\). Suppose that the following conditions hold:

(a) for \( i = 1, 2, \ldots , k \), \( f_i(., v) \) is second order \((F, \rho_i)\) convex in the first variable at \( u \) and \( f_i(x, .) \) is second order \((G, \sigma_i)\) concave in the second variable at \( y \).

(b) \( \sum_{i=1}^{k} \lambda_i (\rho_i d^2_1(x, u) + \sigma_i d^2_2(v, y)) \geq 0 \)

(c) \( F(x, u; \alpha) + u^T \alpha \geq 0, \) for all \( \alpha \in R^n \)

(d) \( G(v, y; b) + y^T b \geq 0, \) for all \( b \in R^m \).

Then

\[
(M_1(x, y, \lambda, p_1), M_2(x, y, \lambda, p_2), \ldots , M_k(x, y, \lambda, p_k))
\leq (N_1(u, v, \lambda, q_1), N_2(u, v, \lambda, q_2), \ldots , N_k(u, v, \lambda, q_k))
\]

**Theorem 1.2.18 (Strong Duality).** Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) be an efficient solution of \((WMP)\), fix \( \lambda = \bar{\lambda} \) in \((WMD)\) and suppose that

(i) \( \nabla_{xy} f_i(\bar{x}, \bar{y}) \) is positive definite for all \( i = 1, 2, \ldots , k \) and

\[ \sum_{i=1}^{k} \bar{\lambda}_i \bar{P}_i^T [\nabla_{xy} f_i(\bar{x}, \bar{y})] \geq 0 \]

or,
\( \nabla_{y_j} f_i(\bar{x}, \bar{y}) \) is negative definite for all \( i = 1, 2, \ldots, k \) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i^T \bar{p}_i^T [\nabla_{y_j} f_i(\bar{x}, \bar{y})] \leq 0.
\]

(ii) The vectors \( \{\nabla_{y_j} f_1(\bar{x}, \bar{y}), \nabla_{y_j} f_2(\bar{x}, \bar{y}), \ldots, \nabla_{y_j} f_k(\bar{x}, \bar{y})\} \) are linearly independent.

Then \( (\bar{x}, \bar{y}, \bar{\lambda}, \bar{\rho} = 0) \) is feasible for (WMD) and the corresponding values of the objective functions of (WMP) and (WMD) are equal. Further if the hypotheses of Theorem 1.2.17 are satisfied for all feasible solution of (WMP) and (WMD) then \( (\bar{x}, \bar{y}, \bar{\lambda}, \bar{\rho} = 0) \) is a properly efficient solution of (WMD).

**Theorem 1.2.19 (Converse Duality).** Let \( (\bar{\mu}, \bar{\nu}, \bar{\lambda}, \bar{\rho}) \) be an efficient solution of (WMD). Fix \( \lambda = \bar{\lambda} \) in (WMP) and suppose that

(i) \( \nabla_{xx} f_i(\bar{\mu}, \bar{\nu}) \) is positive definite for all \( i = 1, 2, \ldots, k \) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T [\nabla_{x_x} f_i(\bar{\mu}, \bar{\nu})] \geq 0, \text{ or } \nabla_{xx} f_i(\bar{\mu}, \bar{\nu}) \) is negative definite for all \( i = 1, 2, \ldots, k \) and
\[
\sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T [\nabla_{x_x} f_i(\bar{\mu}, \bar{\nu})] \leq 0.
\]

(ii) the vectors \( \{\nabla_{x_x} f_1(\bar{\mu}, \bar{\nu}), \nabla_{x_x} f_2(\bar{\mu}, \bar{\nu}), \ldots, \nabla_{x_x} f_k(\bar{\mu}, \bar{\nu})\} \) are linearly independent.

Then \( (\bar{\mu}, \bar{\nu}, \bar{\lambda}, \bar{\rho} = 0) \) is a feasible solution of (WMP) and the corresponding values of objective functions of (WMP) and (WMD) are equal.

Moreover, if the hypotheses of Weak Duality Theorem 1.2.17 are satisfied for all feasible solutions of (WMP) and (WMD), then \( (\bar{\mu}, \bar{\nu}, \bar{\lambda}, \bar{\rho} = 0) \) is a properly efficient solution of (WMP).
We next consider the following pair of Mond-Weir type second order multiobjective programming problems.

\begin{align*}
\text{(MSP)} \quad & \text{minimize} \quad (f_i(x, y) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i, \ldots, f_k(x, y) - \frac{1}{2} p_k^T \nabla_{yy} f_k(x, y) p_k) \\
& \text{subject to} \\
& \quad \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) p_i) \leq 0 \\
& \quad y^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) p_i) \geq 0 \\
& \quad \lambda > 0
\end{align*}

\begin{align*}
\text{(MSD)} \quad & \text{maximize} \quad (f_i(u, v) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i, \ldots, f_k(u, v) - \frac{1}{2} q_k^T \nabla_{xx} f_k(u, v) q_k) \\
& \text{subject to} \\
& \quad \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v) q_i) \geq 0 \\
& \quad u^T \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v) q_i) \leq 0 \\
& \quad \lambda > 0
\end{align*}

**Theorem 1.2.20 (Weak Duality).** Let \((x, y, \lambda, p)\) be feasible for (MSP) and \((u, v, \lambda, q)\) for (MSD). Suppose that the following conditions hold:

(i) \[ \sum_{i=1}^k \lambda_i f_i(, y) \text{ is second order (} F, \rho \text{) pseudoconvex at } u \text{ and } \sum_{i=1}^k \lambda_i f_i(x, .) \text{ is second order (} G, \sigma \text{) pseudoconcave at } v. \]

(ii) \[ F(x, u; a) + u^T a + \rho d_1^2(x, u) \geq 0, \text{ for all } a \in \mathbb{R}^n \]

(iii) \[ G(v, y; b) + y^T b + \sigma d_2^2(v, y) \geq 0, \text{ for all } b \in \mathbb{R}^m \]

Then
Theorem 1.2.21 (Strong Duality). Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})\) be a weak efficient solution of (MSP). Fix \(\lambda = \bar{\lambda}\) in (MSD) and suppose that

(i) \(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is positive definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \lambda_i \bar{p}_i^T [\nabla_{xx} f_i(\bar{x}, \bar{y})] \geq 0\); or \(\nabla_{yy} f_i(\bar{x}, \bar{y})\) is negative definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \lambda_i \bar{p}_i^T [\nabla_{xx} f_i(\bar{x}, \bar{y})] \leq 0\).

(ii) The set \(\{\nabla_{xx} f_1(\bar{x}, \bar{y}) + \nabla_{yy} f_1(\bar{x}, \bar{y}) \bar{p}_1, \nabla_{xx} f_2(\bar{x}, \bar{y}) + \nabla_{yy} f_2(\bar{x}, \bar{y}) \bar{p}_2, ..., \nabla_{xx} f_k(\bar{x}, \bar{y}) + \nabla_{yy} f_k(\bar{x}, \bar{y}) \bar{p}_k\}\) is linearly independent.

Then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is feasible for (MSD) and the corresponding values of the objective functions are equal. Moreover, if the hypotheses of Weak Duality Theorem 1.2.20 hold for all feasible solution of (MSP) and (MSD) then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)\) is a properly efficient solution of (MSD).

Theorem 1.2.22 (Converse Duality). Let \((\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})\) be efficient for (MSD), fix \(\lambda = \bar{\lambda}\) in (MSP) and suppose that

(i) \(\nabla_{xx} f_i(\bar{u}, \bar{v})\) is positive definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \lambda_i \bar{q}_i^T [\nabla_{xx} f_i(\bar{u}, \bar{v})] \geq 0\); or \(\nabla_{xx} f_i(\bar{u}, \bar{v})\) is negative definite for all \(i = 1, 2, ..., k\) and \(\sum_{i=1}^{k} \lambda_i \bar{q}_i^T [\nabla_{xx} f_i(\bar{u}, \bar{v})] \leq 0\).
The set \( \{ \nabla_x f_1(\overline{u}, \overline{v}) + \nabla_x f_1(\overline{u}, \overline{v})\overline{q}_1, \ldots, \nabla_x f_k(\overline{u}, \overline{v}) + \nabla_x f_k(\overline{u}, \overline{v})\overline{q}_k \} \) is linearly independent.

Then \((\overline{u}, \overline{v}, \overline{\lambda}, \overline{p} = 0)\) is feasible for (MSP) and the corresponding values of the objective functions are equal.

Moreover, if the hypotheses of Weak Duality Theorem 1.2.20 are satisfied for all feasible solutions of (MSP) and (MSD), then \((\overline{u}, \overline{v}, \overline{\lambda}, \overline{p} = 0)\) is a properly efficient solution for (MSP).

In Section 3.2 we extend the definitions of second order cone convex, second order cone pseudoconvex and second order strongly cone pseudoconvex functions defined by Suneja et al [131] to two variables and study their interrelations by virtue of examples. A pair of Mond-Weir type second order symmetric dual programs over arbitrary cones is formulated and corresponding duality results are established for these programs using second order cone pseudoconvex and second order strongly cone pseudoconvex functions.

The following pair of second order Mond-Weir type symmetric dual nonlinear programming problems are considered and duality results are proved:

**(SP)**  
\[
\text{K-minimize } f(x, y) - \frac{1}{2} r^T \nabla_{yy} f(x, y) r
\]

subject to

\[
-\sum_{i=1}^{p} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) r_i) \in C^*_2
\]

\[
y^T \sum_{i=1}^{p} \lambda_i (\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y) r_i) \geq 0
\]

\[
x \in C_1, \ \lambda \in \text{ int } K^+, \ \lambda_i \neq 0, \ r_i \in R^m, \ i = 1, 2, \ldots, p.
\]
(SD) \[ K\text{-maximize } f(u,v) - \frac{1}{2} q^T \nabla_{xx} f(u,v) q \]

subject to

\[ \sum_{i=1}^{p} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \in C_1^+ \]

\[ u^T \sum_{i=1}^{p} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \leq 0 \]

\[ v \in C_2, \lambda \in \text{int } K^+, \lambda_i \neq 0, q_i \in R^n, i = 1, 2, ..., p \]

where \( f_i: R^n \times R^m \to R \), \( i = 1, 2, ..., p \) are thrice continuously differentiable functions. \( C_1 \) and \( C_2 \) are closed convex cones with nonempty interiors in \( R^n \) and \( R^m \) respectively, \( C_1^+ \) and \( C_2^+ \) are positive polar cones of \( C_1 \) and \( C_2 \) respectively, \( K \) is a closed convex pointed cone in \( R^p \) with nonempty interior and \( K^+ \) is its positive polar cone.

**Theorem 1.2.23 (Weak Duality).** Let \((x, y, \lambda, r)\) and \((u, v, \lambda, q)\) be feasible solutions of (SP) and (SD) respectively. Assume that \( f(\cdot, y) \) is second order strongly \( K \)-pseudoconvex in the first variable at \( u \) for fixed \( v \) with respect to \( q = (q_1, q_2, ..., q_p) \) where \( q_i \in R^n \) and \(-f(x, \cdot)\) is second order \( K \)-pseudoconvex in the second variable at \( y \) for fixed \( x \) with respect to \( r = (r_1, r_2, ..., r_p) \), \( r_i \in R^m \). Then

\[ \left[ f(u,v) - \frac{1}{2} q^T \nabla_{xx} f(u,v) q \right] - \left[ f(x,y) - \frac{1}{2} r^T \nabla_{yy} f(x,y) r \right] \notin \text{int } K \]

**Theorem 1.2.24 (Strong Duality).** Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})\) be a weak minimum of (SP). Fix \( \lambda = \bar{\lambda} \) in (SD) and suppose that
(a) either the Hessian matrix \( \nabla_{yy} f_i \) is positive definite for all 
\[ i = 1, 2, \ldots, p \] and \( \sum_{i=1}^{p} \kappa_{ii} \nabla_{yy} f_i \geq 0 \); or the Hessian matrix \( \nabla_{yy} f_i \) is negative definite for all \( i = 1, 2, \ldots, p \) and \( \sum_{i=1}^{p} \kappa_{ii} \nabla_{yy} f_i \leq 0 \).

(b) the set \( \{ \nabla_{yy} f_1, \ldots, \nabla_{yy} f_p \} \) is linearly independent,

where \( f = (f_1, f_2, \ldots, f_p) \), \( f_i = f_i(\bar{x}, \bar{y}) \), \( i = 1, \ldots, p \). Then \( (\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0) \) is a feasible solution for (SD) and the corresponding values of the objective functions are equal. Moreover if the hypotheses of Weak Duality Theorem 1.2.23 are satisfied for all feasible solutions of (SP) and (SD), then \( (\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0) \) is a weak maximum of (SD).

In Chapter 4 we study higher order optimality and duality results for optimization problems. This chapter is divided into two sections.

In Section 4.1 we introduce new class of higher order \( (F, \rho_i, \sigma_j) \)-type-I functions for a multiobjective programming problem. In this section we consider the following multiobjective nondifferentiable fractional program

\[
\text{(FP) minimize } \left[ \frac{f_i(x) + s(x \mid C_i)}{g_i(x) - s(x \mid D_i)}, \ldots, \frac{f_k(x) + s(x \mid C_k)}{g_k(x) - s(x \mid D_k)} \right]
\]

subject to

\[ h_j(x) + s(x \mid E_j) \leq 0, \quad j = 1, 2, \ldots, m \]

where \( x \in S \subseteq \mathbb{R}^n \), \( f_i, g_i : S \to \mathbb{R} \) \( (i = 1, 2, \ldots, k) \) and \( h_j : S \to \mathbb{R} \) \( (j = 1, 2, \ldots, m) \) are continuously differentiable functions,
$f_i(x) + s(x | C_i) \geq 0$ and $g_i(x) - s(x | D_i) > 0$ for all $x \in S$; $C_i$, $D_i$ and $E_j$ are compact convex sets in $\mathbb{R}^n$ and $s(\cdot | A)$ denotes the support function of a compact convex set $A$.

We formulate a higher order Mond-Weir type dual for (FP) and establish weak and strong duality theorems by considering the functions involved to be higher order $(F, \rho_i, \sigma_j)$-type I.

(MFD) maximize
\[
\begin{bmatrix}
    f_1(u) + u^T z_1 \\
    \vdots \\
    f_k(u) + u^T z_k
\end{bmatrix}
\]
subject to
\[
\sum_{i=1}^{k} \lambda_i \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^{m} y_j \left( h_j(u) + u^T w_j \right)
\]
\[
+ \sum_{i=1}^{k} \lambda_i \nabla_p L_i(u,p) + \sum_{j=1}^{m} y_j \nabla_q H_j(u,q) = 0
\]
\[
\sum_{j=1}^{m} y_j (h_j(u) + u^T w_j) + H_j(u,q) - q^T \nabla_q H_j(u,q) \geq 0
\]
\[
\sum_{i=1}^{k} \lambda_i \left( L_i(u,p) - p^T \nabla_p L_i(u,p) \right) \geq 0
\]
\[
z_i \in C_i, \quad v_i \in D_i, \quad i = 1, 2, \ldots, k, \quad w_j \in E_j, \quad j = 1, 2, \ldots, m
\]
\[
y_j \geq 0, \quad j = 1, 2, \ldots, m
\]
\[
\lambda_i \geq 0, \quad i = 1, 2, \ldots, k, \quad \sum_{i=1}^{k} \lambda_i = 1.
\]

Theorem 1.2.25 (Weak Duality Theorem). Let $x$ be a feasible solution for (FP) and $(u, z, v, y, \lambda, w, p, q)$ be feasible for (MFD). Suppose that:

(i) \[
\begin{bmatrix}
    f_i(\cdot) + (\cdot)^T z_i \\
    \vdots \\
    f_k(\cdot) + (\cdot)^T z_k
\end{bmatrix}, \quad h_j(\cdot) + (\cdot)^T w_j
\]
is higher order $(F, \rho_i, \sigma_j)$-type-I with respect to $L_i$ and $H_j$, at $u$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$,

(ii) \[
\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geq 0, \quad \lambda_i > 0,
\]
then the following cannot hold:
\[
\frac{f_j(x) + s(x | C_j)}{g_j(x) - s(x | D_j)} \leq \frac{f_j(u) + u^T z_i}{g_j(u) - u^T v_i}, \quad i = 1, 2, \ldots, k
\]

and

\[
\frac{f_r(x) + s(x | C_r)}{g_r(x) - s(x | D_r)} < \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r}, \text{ for some } r = 1, 2, \ldots, k.
\]

**Theorem 1.2.26 (Strong Duality Theorem).** If \( \bar{x} \) is an efficient solution of (FP), \( L_i(\bar{x}, 0) = 0, \nabla_p L_i(\bar{x}, 0) = 0, i = 1, 2, \ldots, k, H_j(\bar{x}, 0) = 0, \nabla_q H_j(\bar{x}, 0) = 0, j = 1, 2, \ldots, m \) and a constraint qualification is satisfied for (FP, \( \bar{x} \)) for at least one \( r = 1, 2, \ldots, k \) then, there exist \( \bar{\lambda} \in R^k, \bar{y} \in R^m, \bar{z}_i \in R^n, \bar{v}_j \in R^n \) and \( \bar{\alpha}_j \in R^n, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m \) such that \((\bar{x}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\alpha}_j, \bar{\omega}, \bar{\rho} = 0, \bar{q} = 0)\) is a feasible solution of (MFD) and the corresponding values of the objective functions of (FP) and (MFD) are equal. Further if the conditions of Weak Duality Theorem 1.2.25 are satisfied for all feasible solutions of primal problem (FP) and all feasible solution of the dual (MFD) then \((\bar{x}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\alpha}_j, \bar{\omega}, \bar{\rho} = 0, \bar{q} = 0)\) is an efficient solution of (MFD).

We also formulate the following higher order Schaible type dual for the problem (FP) and study weak and strong duality results for the pair.

**SFD**

maximize \((\alpha_1, \alpha_2, \ldots, \alpha_k)\)

subject to

\[
\nabla \left\{ \sum_{i=1}^{k} \lambda_i \left[ (f_i(u) + u^T z_i) - \alpha_i (g_i(u) - u^T v_i) \right] \\
+ \sum_{j=1}^{m} \gamma_j (h_j(u) + u^T w_j) \right\} + \sum_{i=1}^{k} \lambda_i \nabla_p \left( K_i(u, p) - \alpha_i G_i(u, p) \right)
\]
\[ + \sum_{j=1}^{m} y_j \nabla_q H_j(u, q) = 0 \]

\[ \sum_{i=1}^{k} \lambda_i \left\{ [(f_i(u) + u^T z_i) - \alpha_i(g_i(u) - u^T v_i)] + (K_i(u, p) - \alpha_i G_i(u, p)) \right\} \]

\[ - p^T \nabla_p (K_i(u, p) - \alpha_i G_i(u, p)) \geq 0. \]

\[ \sum_{j=1}^{m} y_j \left\{ (h_j(u) + u^T w_j) + H_j(u, q) - q^T \nabla_q H_j(u, q) \right\} \geq 0. \]

\[ z_i \in C_i, \quad v_i \in D_i, \quad i = 1, 2, ..., k, \quad w_j \in E_j, \quad j = 1, 2, ..., m \]

\[ y_j \geq 0, \quad j = 1, 2, ..., m \]

\[ \lambda_i \geq 0, \quad i = 1, 2, ..., k, \quad \sum_{i=1}^{k} \lambda_i = 1 \]

\[ \alpha_i \geq 0, \quad i = 1, 2, ..., k \]

**Theorem 1.2.27 (Weak Duality Theorem).** Let \( x \) be feasible for (FP) and \((u, \alpha, z, v, w, y, \lambda, p, q)\) be feasible for (SFD). Suppose that \((f_i(\cdot) + (\cdot)^T z_i, h_j(\cdot) + (\cdot)^T w_j)\) are higher order \((F, p_i, \sigma_i)\)-type I with respect to \(K_i\) and \(H_j\) and \([- (g_i(\cdot) - (\cdot)^T v_i), h_j(\cdot) + (\cdot)^T w_i]\) are higher order \((F, p_i, \sigma_i)\)-type I with respect to \(- G_i\) and \(H_j\) and \(\sum_{i=1}^{k} \lambda_i^i + \sum_{j=1}^{m} \sigma_j \geq 0, \lambda_i > 0\), where \(\lambda_i^i = \rho_i(1 + \alpha_i)\). Then the following cannot hold

\[ \frac{f_i(x) + s(x \mid C_i)}{g_i(x) - s(x \mid D_i)} \leq \alpha_i, \quad i = 1, 2, ..., k \]

and

\[ \frac{f_r(x) + s(x \mid C_r)}{g_r(x) - s(x \mid D_r)} < \alpha_r, \quad \text{for some } r = 1, 2, ..., k. \]

**Theorem 1.2.28 (Strong Duality Theorem).** Let \( \bar{x} \) be an efficient solution of (FP) and \( K_i(\bar{x}, 0) = 0, \nabla_p K_i(\bar{x}, 0) = 0, G_i(\bar{x}, 0) = 0, \)
Assume that a constraint qualification is satisfied for (FP) for at least one \( r = 1, 2, ..., k \). Then there exist \( \lambda \in \mathbb{R}^k \), \( \bar{y} \in \mathbb{R}^m \), \( \bar{a} \in \mathbb{R}^k \), \( \bar{z}_i \in \mathbb{R}^n \), \( \bar{v}_i \in \mathbb{R}^n \) and \( \bar{w}_j \in \mathbb{R}^n \) such that \((\bar{x}, \bar{a}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0)\) is feasible, for (SFD) and the corresponding values of the objective functions of (FP) and (SFD) are equal. Further if the conditions of Weak Duality Theorem 1.2.27 are satisfied for all feasible solutions of the primal problem (FP) and all feasible solutions of the dual (SFD) then \((\bar{x}, \bar{a}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0)\) is an efficient solution of (SFD).

In section 4.2 we introduce higher order cone convex, higher order cone pseudoconvex, higher order strongly cone pseudoconvex and higher order cone quasiconvex functions. Their interrelations and relations with some classes of functions available in the literature are studied by virtue of examples over arbitrary cones. Higher order sufficient optimality conditions are proved for the vector optimization problem (VP) in the form of following theorems.

**Theorem 1.2.29.** Let \( f \) be higher order \( K \)-convex at \( \bar{x} \in X_0 \) with respect to \( H : S \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( g \) be higher order \( Q \)-convex at \( \bar{x} \in X_0 \) with respect to \( G : S \times \mathbb{R}^n \rightarrow \mathbb{R}^p \), where \( G \) and \( H \) are differentiable vector valued functions. Suppose that there exist \( 0 \neq \bar{\lambda} \in K^+ \) and \( \bar{\mu} \in Q^+ \), such that

\[
(x - \bar{x})^T [\nabla_x (\bar{\lambda}^T f)(\bar{x}) + \nabla_p (\bar{\lambda}^T H)(\bar{x}, p) + \nabla_x (\bar{\mu}^T g)(\bar{x}) + \nabla_p (\bar{\mu}^T H)(\bar{x}, p)] \geq 0 \quad \forall x \in X_0 \tag{1.10}
\]

\[
(\bar{\mu}^T g)(\bar{x}) + (\bar{\mu}^T H)(\bar{x}, p) - p^T \nabla_p (\bar{\mu}^T G)(\bar{x}, p) = 0 \tag{1.11}
\]
\[(\bar{\lambda}^TH)(\bar{x},p) - p^T \nabla_p (\bar{\lambda}^TH)(\bar{x},p) = 0 \quad (1.12)\]

Then \(\bar{x}\) is a weak minimum of (VP).

**Theorem 1.2.30.** Let \(f\) be higher order \(K\)-convex at \(\bar{x} \in X_0\), with respect to \(H\) and \(g\) be higher order \(Q\)-convex at \(\bar{x}\) with respect to \(G\). Suppose that there exist \(\bar{\lambda} \in K^{S^+}\) and \(\bar{\mu} \in Q^+\), such that (1.10), (1.11) and (1.12) hold. Then \(\bar{x}\) is a minimum of (VP).

**Theorem 1.2.31.** Let \(f\) be higher order \(K\)-convex at \(\bar{x} \in X_0\), with respect to \(H\) and \(g\) be higher order \(Q\)-convex at \(\bar{x}\), with respect to \(G\). Suppose that there exists \(\bar{\mu} \in Q^+\), such that (1.10), (1.11) and (1.12) hold with \(\bar{\lambda}\) replaced by \(\lambda\) for every \(\lambda \in K^{S^+}\). Then \(\bar{x}\) is a strong minimum of (VP).

We further associate the following scalar optimization problem with (VP) to study higher order optimality conditions for a Benson proper minimum

\[\text{(VP}_\lambda) \quad \text{minimize } (\lambda^T f)(x)\]

subject to \(-g(x) \in Q\)

where \(\lambda \in K^{S^+}\)

The following higher order dual is associated with (VP) and duality results are proved.

\[\text{(HD)} \quad K\text{-maximize } f(u) + H(u, p) - p^T \nabla_p H(u, p)\]

subject to

\[(x - u)^T \{\nabla_x [(\lambda^T f)(u)] + (\mu^T g)(u)\} + \nabla_p [(\lambda^T H)(u, p) + (\mu^T G)(u, p)] \geq 0 \quad \forall x \in X_0\]

\[(\mu^T g)(u) + (\mu^T G)(u, p) - p^T \nabla_p (\mu^T G)(u, p) \geq 0\]

where \(0 \neq \lambda \in K^+, \mu \in Q^+, u \in S\).
Theorem 1.2.32 (Weak Duality). Let \( x \) be feasible for (VP) and \((u, \lambda, \mu, p)\) for (HD). Suppose that \( f \) is higher order \( K \)-convex at \( u \) with respect to \( H \) and \( g \) is higher order \( Q \)-convex at \( u \), with respect to \( G \), where \( G \) and \( H \) are differentiable vector valued functions, then

\[
f(u) + H(u, p) - p^T \nabla_p H(u, p) - f(x) \notin \text{int } K
\]

Theorem 1.2.33 (Strong Duality). Let \( \bar{x} \) be a weak minimum for (VP) at which Slater type constraint qualification is satisfied and \( H(\bar{x}, 0) = 0, G(\bar{x}, 0) = 0, \nabla_p G(\bar{x}, 0) = 0 \) and \( \nabla_p H(\bar{x}, 0) = 0 \). Then there exist \( 0 \neq \bar{\lambda} \in K^+, \bar{\mu} \in Q^+ \) such that \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{p} = 0)\) is feasible for (HD) and the corresponding values of the objective functions of (VP) and (HD) are equal. Moreover if the conditions of Weak Duality Theorem 1.2.32 are satisfied for all feasible solutions of (VP) and (HD) then \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{p} = 0)\) is a weak maximum for (HD).