CHAPTER 4
THEORETICAL ESTIMATION

In this chapter, we will discuss in details the procedure for the theoretical estimation of the antenna model considering rectangular configuration. We will also discuss the procedure for theoretical estimation of the transmission line model, cavity model and CAD model of RMSA to analyze and study the accuracy of the models in characterizing key antenna parameters.

4.1 Theoretical estimation for the rectangular microstrip antenna model

4.1.1 Transmission line model

The equivalent representation of RMSA in transmission line model is shown in Fig.5. Due to finite dimensions of the patch, the fields at the edge of patch along the length and width undergo fringing. As a result electrical length becomes slightly more than its physical length [1]. The fringing extension also applies for width. However, the effect of length extension can substantially influence resonant frequency and wave propagation velocity [1]. An effective dielectric constant $\varepsilon_{re}$ has values in the range of $1 \leq \varepsilon_{re} \leq \varepsilon_r$ [1].

4.1.1.1 Estimation for resonant frequency

The resonant frequency of transmission line proposed by Sengupta et al. can be expressed as [43]:

$$f_r = f_{r0} \left[ 1 - \frac{2h}{\varepsilon_{re}(W)lna} \right] \left/ \left( 1 + \frac{2h}{\varepsilon_{re}(W)lna} ln \left( \frac{2L\sqrt{\varepsilon_{re}(W)\frac{1}{\rho h}}} {\varepsilon_{re}(W)ln(\frac{2L}{\varepsilon_{re}(W)lna})} \right) \right) \right]$$

(4.1)
Where \( f_{r0} = \frac{c}{2L\sqrt{\varepsilon_{re}(W)}} \), \( \alpha = 1 + 1.393 \frac{h}{W} + 0.667 \frac{h}{W} \ln(\frac{h}{W} + 1.44) \) and \( \gamma = 1.78107 \)

The transmission line model resonant frequency proposed by Hammersted can be expressed as [45]:

\[
f_{rh} = \frac{c}{2(L+\Delta L)\sqrt{\varepsilon_{re}}}
\]  \hspace{1cm} (4.1a)

Where \( \varepsilon_{re} \) and \( \Delta L \) are effective dielectric constant and fringing length extension.

### 4.1.1.2 Estimation for effective dielectric constant

The effective dielectric constant depends upon substrate thickness and frequency. At lower frequency, the relative permittivity is almost constant. The low frequency approximation of \( \varepsilon_{re} \) for thin substrate can be expressed as [1]:

\[
\varepsilon_{re} = \frac{(\varepsilon_r+1)}{2} + \frac{(\varepsilon_r-1)}{2} \left[ 1 + 12 \frac{h}{W} \right]^{-\frac{1}{2}}
\]  \hspace{1cm} (4.2)

Where \( \varepsilon_r \) is the relative permittivity of substrate materials, \( h \) is thickness of the substrate and \( W \) is microstrip line width.

The expression for \( \varepsilon_{re} \) given by Ramesh Garg et. al. [6] can be expressed as:

\[
\varepsilon_{re} = \frac{(\varepsilon_r+1)}{2} + \frac{(\varepsilon_r-1)}{2} \left[ 1 + \frac{10}{u} \right]^{-ab}
\]  \hspace{1cm} (4.2a)

Where \( u=W/h \),

\[
a = 1 + \frac{1}{49} \ln \left[ \frac{u^4 + \left( \frac{u}{0.57} \right)^2}{u^4 + 0.432} \right] + \frac{1}{18.7} \ln \left[ 1 + \ln \left( 1 + \frac{u}{18.1} \right)^3 \right]
\]

and \( b = 0.564 \left( \frac{\varepsilon_r-0.9}{\varepsilon_r+0.3} \right)^{0.053} \)

The effect of metal strip thickness \( t \) on effective dielectric constant can be included by replacing \( u \) by \( u'_r \) defined as [6]:


\[
\begin{align*}
\varepsilon_{ref} &= \frac{(\varepsilon_r+1)}{2} + \frac{(\varepsilon_r-1)}{2} \left[ 1 + \frac{10}{u''_r} \right]^{-a'_h} \\
\text{Where } a'_h &= 1 + \frac{1}{49} \ln \left[ \frac{u''_r + (\frac{u''_r}{18.1})^2}{u''_r^4 + 0.432} \right] + \frac{1}{18.1} \ln \left[ 1 + \ln \left( 1 + (u''_r/18.1)^3 \right) \right] \\
\varepsilon_{re}(f) &= \varepsilon_r - \frac{(\varepsilon_r-\varepsilon_{ref})}{(1+F(f))} \\
\text{Where } F(f) &= F_1F_2[(0.1844 + F_3F_4)f_n]^{1.5763} \\
F_1 &= 2.7488 + u(0.6315 + 0.525/(1 + 0.0157f_n)^{20}) - 0.065683 \exp(-8.7513u) \\
F_2 &= 0.33622(1 - \exp(-0.03442\varepsilon_r)) \\
F_3 &= 0.03636 \exp(-4.6u)(1 - \exp[-(f_n/38.7)^4.97]) \\
F_4 &= 1 + 2.751(1 - \exp[-(\varepsilon_r/15.916)^8]) \\
\varepsilon_r &= (W/h) \text{, } f_n = f \times h \text{ (GHz mm) = 47.713} \times k_0 \times h \\
\text{To account for finite thickness of patch metallization, } u \text{ in above equations is replaced by } u', \text{ given by } [7]: \\
u' &= [W + (W' - W)/\varepsilon_r]/h
\end{align*}
\]
Where \( W' = W + (t/\pi) \left[ 1 + \ln \left( 4/\sqrt{(t/h)^2 + (1/\pi)^2 / \left( \frac{W}{t} + 1 \right)^2} \right) \right] \)

Where \( t \) is patch metallization thickness.

The expression for Dynamic permittivity is given by [49]:

\[
\varepsilon_{\text{dyn}} = \sqrt{\varepsilon_{re}(L)\varepsilon_{re}(W)}
\]

(4.2d)

Where \( \varepsilon_{re}(L) = \frac{(\varepsilon_r+1)}{2} + \frac{(\varepsilon_r-1)}{2} \left[ 1 + 10 \frac{h}{L} \right]^{-\frac{1}{2}} \)

\[
\varepsilon_{re}(W) = \frac{(\varepsilon_r+1)}{2} + \frac{(\varepsilon_r-1)}{2} \left[ 1 + 10 \frac{h}{W} \right]^{-\frac{1}{2}}
\]

The more accurate expression of \( \varepsilon_{re} \) can be expressed as [7]:

\[
\varepsilon_{re} = 0.5(\varepsilon_r + 1) + 0.5(\varepsilon_r - 1)/G
\]

(4.2e)

Where \( G = \left[ 1 + \frac{10}{u} \right] \frac{a}{b} - \frac{ln4}{\pi} \frac{t}{\sqrt{Wh}} \), \( u=W/h \),

\[
a=1+\frac{1}{49} \ln\left(\left(\frac{u}{52}\right)^2 / (u^4 + 0.432)\right) + \ln\left(1 + \ln\left(1 + (u/18.1)^3\right)\right) / 18.7,
\]

\( t \)=thickness of patch metallization and

\[
b=0.564* \exp[-0.2/(\varepsilon_r + 0.3)]
\]

4.1.1.3 Estimation for fringing length extension

The dimensions of the patch antenna along its length extended on each end by a distance \( \Delta L \), which is given empirically by [1]:

\[
\Delta L = 0.412 \frac{h}{W} \left( \frac{\varepsilon_{re}+0.3}{W/h+0.264} \right) \left( \frac{\varepsilon_{re}-0.258}{W/h+0.8} \right)
\]

(4.3)

4.1.1.4 Estimation for effective width, length and ground plane

The expression for effective width for RMSA can be expressed as [7]:

\[
W_e = 2\pi h / \ln \left[ h F / W' + \sqrt{1 + (2h/W')^2} \right]
\]

(4.4)
Where
\[
F = 6 + (2\pi - 6)e x p \left[ (-4\pi^2/3)(h/W)^{3/4} \right]
\]

The expression for frequency dependent effective width of the RMSA can be expressed as [7]:
\[
W_e(f) = \frac{W}{3} + (C_w + A_w)^\frac{1}{3} - (C_w - A_w)^\frac{1}{3}
\]
(4.4a)

Where
\[
A_w = (W/3)^3 + (D_w/2)(W_e - W/3)
\]

\[
B_w = (D_w/3) - (W/3)^2
\]

\[
C_w = (A_w^2 + B_w)^\frac{1}{2}
\]

\[
D_w = c^2/[4f^2(\varepsilon_{re}(f) - 1)]
\]

The actual length, effective length and effective width can be expressed as [1]:
\[
L = \frac{c}{2f_{re}\sqrt{\varepsilon_{re}}} - 2\Delta L
\]
(4.5)

\[
L_{ef} = L + 2\Delta L
\]
(4.5a)

\[
W_e = c/(2f_{e}\sqrt{(\varepsilon_e + 1)/2}) + 2\Delta L
\]
(4.6)

The ground plane dimensions are given as [1]:
\[
L_g = L + 5.8h
\]
(4.7)

\[
W_g = W + 5.8h
\]
(4.8)

4.1.1.5 Estimation for input impedance

The RMSA of length L, width W, and excited from the back by a coaxial probe of radius \(r_0\) is shown in Fig.2. The exact location of feed line is usually governed from
impedance and polarization considerations. It is assumed that propagation takes place
along the ±x directions.

The characteristic admittance and propagation constant of transmission line are given
by [46]:

\[ Y_c = \frac{1}{Z_c} = \frac{W \alpha \sqrt{\varepsilon_r}}{120 \pi \varepsilon_r h} \] (4.9)

\[ \beta = \frac{\omega}{c} \sqrt{\varepsilon_{re}} \] (4.10)

with

\[ \varepsilon_{re} = 0.5 * (\varepsilon_r + 1) + 0.5 * (\varepsilon_r - 1) * [1 + 10h/W]^{-0.5} \]

\[ \alpha = 1 + 1.393 \left( \frac{h}{W} \right) + 0.667 \left( \frac{h}{W} \right) \log \left( \frac{h}{W} + 1.444 \right) \]

And \( \varepsilon_r, \omega = 2\pi f, c \) are the relative permittivity of the substrate, radian frequency and
speed of light in free space, respectively.

The approximate expression for conductance G and susceptance B with the
assumption that the patch to be a semi-infinite parallel plate waveguide, radiating into
free space, using image principle and aperture admittance of open-end of a parallel plate
waveguide radiating into free space can be written as [46]:

\[ G = \frac{\beta h}{2\alpha \varepsilon_r} \] (4.11)

\[ B = \frac{\beta h}{\pi \alpha \varepsilon_r} \log \left( \sqrt{\varepsilon_{re}} \frac{2\pi e}{\gamma \beta h} \right) \] (4.12)

Where \( \gamma = 1.78107, e = 2.71828 \)

The normalized inductive reactance of the probe is obtained by considering the
probe excitation of a parallel plate waveguide propagating only in its dominant mode
and can be expressed as [46]:

\[ X_L = \frac{\beta W a}{2\pi} \log \left( \frac{2}{\gamma \beta r_0} \right) \] (4.12a)
Where $r_0$ is probe radius.

At dominant mode the propagation constant is connected to $G$ and $B$ by a transcendental equation and can be expressed as [46]:

$$\tan(\beta L) = \frac{2B}{B^2 + G^2 + 1}$$  \hspace{2cm} (4.13)

At resonance the antenna fed at $x_f$ from the radiating edge conductance can be expressed as [46]:

$$g_r = \frac{2G}{\cos^2(\beta x_f) + (B^2 + G^2)\sin^2(\beta x_f) - B\sin(2\beta x_f)}$$  \hspace{2cm} (4.13a)

The resonant resistance $R_r$ of an antenna excited from one edge ($x_f = 0$) can be expressed as [46]:

$$R_r = Z_c / 2G$$  \hspace{2cm} (4.13b)

The propagation constant $\beta$ and phase shift $\delta$ at resonance is given by [46]:

$$\beta_r = \frac{\pi(1-\delta/\pi)}{L}$$  \hspace{2cm} (4.13c)

With

$$\delta = \frac{2h}{\varepsilon_r\varepsilon_0\tan\theta} \log\left[\frac{\varepsilon_r 2e_L}{\sqrt{\varepsilon_r 2L}}/\gamma h\right]$$  \hspace{2cm} (4.14)

The input resistance of the antenna at resonance by neglecting the effect of probe, dielectric and conduction losses can be expressed as [46]:

$$R_r = \frac{120 \sqrt{\varepsilon_r}}{(1-\delta/\pi)\left(\frac{L}{W}\right)}$$  \hspace{2cm} (4.15)

The approximate transmission line model in which each radiating slot is represented by a parallel equivalent admittance $Y$ (with conductance $G$ and susceptance $B$) the input impedance can be expressed as [1]:

$$Z_{in} = 1/Y_1$$  \hspace{2cm} (4.16)
Where $Y_1$ is slot admittance and can be expressed as:

$$Y_1 = G_1 + jB_1$$

(4.17)

$G_1$ & $B_1$ are slot conductance and susceptance respectively.

For thin substrate with $h<0.1 \lambda_0$, slot conductance, susceptance and radiation resistance are given as [1]:

$$G_1 = \frac{W}{120 \lambda_0} (1 - (k_0 h)^2/24)$$

(4.17a)

$$B_1 = \frac{W}{120 \lambda_0} (1 - 0.636 \ln(k_0 h))$$

(4.17b)

$$R_r = \frac{1}{G_r}$$

(4.17c)

With

$$G_r = 2G_1$$

(4.17d)

The resonant input resistance at resonance, taking into account the mutual conductance $G_{12}$ between the slots and feeding position is given as [1]:

$$R_{in} = \frac{1}{2} (G_1 \pm G_{12}) \cos^2 \left( \frac{\pi}{L} y_0 \right)$$

(4.18)

$$G_{12} = \frac{1}{120 \pi^2} \int_0^\pi \left[ \sin(k_0 W/2 \cos \theta)/\cos \theta \right]^2 J_0(k_0 L \sin \theta) \sin^3 \theta d\theta$$

(4.18a)

where $G_1$ & $G_{12}$ are self and mutual conductance in two slot model, $y_0$ is feed location and $J_0$ is Bessel function of first kind of order zero. The plus (+) sign is used for modes with odd (anti-symmetric) resonant voltage distribution below the patch and between the slots, while the minus (−) sign is used for even (symmetric) modes resonant voltage distribution.

Another more simplified expressions of slot conductance ($G_s$) and susceptance ($B_s$) valid for low permittivity thin substrate satisfying $0.05 \leq k_0 h \leq 0.6$ and $2.45 < \varepsilon_r < 2.65$ are given by [6]:
\[ G_s = W \frac{7.75 + 2.2k_0h + 4.8(k_0h)^2}{1000A_0} \left( 1 + \frac{(\varepsilon_r - 245)(k_0h)^3}{1.3} \right) \] (4.18b)

\[ B_s = 0.01668 \frac{\Delta l W \varepsilon}{k_0 \varepsilon_r} \] (4.18c)

With

\[ \frac{\Delta l}{h} = \frac{0.95}{1 + 0.85k_0h} - \frac{0.075(\varepsilon_r - 245)}{1 + 10k_0h} \]

The approximate radiation resistance can be expressed as:

\[ R_r = \frac{1}{G_r} \] (4.18d)

With

\[ G_r = 2G_s \] (4.18e)

4.1.1.6 Estimation for Q and BW

The BW of the RMSA is inversely related to its quality factor Q. In deriving radiation \( Q_r \) of the edge-fed antenna, the effect of probe and losses associated with dielectric and patch conductor is ignored and lump circuit parameters for RMSA can be developed near resonance [46].

Considering a parallel combination of R, L, C having resonant frequency \( \omega_r = \frac{1}{\sqrt{LC}} \), the equivalent R & C of the patch antenna near resonance can be written as [46]:

\[ R = \frac{\alpha \varepsilon_r}{\beta \varepsilon} \] (4.19)

\[ C = \frac{\varepsilon_0 \varepsilon_r \alpha W L}{2h} + \frac{\varepsilon_0 W}{\pi} \log \left( \frac{\sqrt{\varepsilon_r} \varepsilon}{\gamma h} \right) \] (4.20)

The first term is half the equivalent static capacitance of the patch and the second term is the dynamic contribution to the capacitance due to the two radiating slot. Using the basic definition, \( Q_r \) can be written as [46]:

\[ Q_r = \omega_r CR \] (4.21)
The expression for the radiation quality factor $Q_r$ of the patch antenna can be written as [46]:

$$Q_r = \frac{\beta_r l^2}{4g} \left| \frac{\epsilon_{re} a L}{2h} + \frac{1}{\pi} \log \left( \frac{\sqrt{\epsilon_{re} 2L}}{g\beta_r h} \right) \right|$$

(4.21a)

$$Q_r = \frac{\epsilon_{re} a L}{2h} + \frac{1}{\pi} \log \left( \frac{\sqrt{\epsilon_{re} 2L}}{g h} \right)$$

(4.21b)

The theoretical VSWR $= 2:1$ and fractional BW are:

$$BW = \frac{1}{\sqrt{2Q_r}}$$

(4.22)

An approximate transmission line BW can be expressed as [50]:

$$BW = 3.77(\epsilon_r - 1)/\epsilon_r^2 (W/L)(h/\lambda_0)$$

(4.23)

4.1.2 Cavity model

4.1.2.1 Estimation for resonant frequency

The cavity model resonant frequency based on dynamic permittivity model proposed by Schneider can be calculated as [49]:

$$f_{mn} = \frac{c}{2\sqrt{\epsilon_{dyn}}} \sqrt{\left( m/W_{eff} \right)^2 + \left( n/L_{eff} \right)^2}$$

(4.24)

Where

$$W_{eff} = W + \left( \frac{W_{eq}-W}{2} \right) \frac{\epsilon_{re}(L)+0.3}{\epsilon_{re}(L)-0.258} \quad \text{and} \quad L_{eff} = L + \left( \frac{W_{eq}-L}{2} \right) \frac{\epsilon_{re}(W)+0.3}{\epsilon_{re}(W)-0.258}$$

$$L_{eq} = \frac{120nh}{Z_c(L)\sqrt{\epsilon_{re}(L)}} \quad \text{and} \quad W_{eq} = \frac{120nh}{Z_c(W)\sqrt{\epsilon_{re}(W)}}$$

With

$$Z_c(U) = 60\pi/\sqrt{\epsilon_r[U/2h + X + Y(1.451 + \ln(U/2h + 0.94))]^{-1}}$$

$$X = 0.082*(\epsilon_r - 1)/\epsilon_r, \quad Y = (\epsilon_r - 1)/2\pi\epsilon_r$$
Where $U$ could be either $W$ or $L$ in the above expression.

The resonance frequency in $TM_{010}$ or $TM_{100}$ mode can be calculated as [49]:

$$f_{010} = \frac{c}{2L_{eff} \sqrt{\varepsilon_{dyn}}} \quad (4.24a)$$

$$f_{100} = \frac{c}{2W_{eff} \sqrt{\varepsilon_{dyn}}} \quad (4.24b)$$

4.1.2.2 Estimation for fringing length extension

The improved expression for length extension can be expressed as [7]:

$$\Delta L = hE_1E_3E_5/E_4 \quad (4.25)$$

Where

$$E_1 = 0.434907 e^{0.81 + 0.26 (W/h)^{0.8544 + 0.236}}$$

$$E_2 = 1 + (W/h)^{0.371} / (2.358 \times \varepsilon_r + 1)$$

$$E_3 = 1 + 0.5274 \times \arctan[0.084(W/h)^{1.9413/E_2}] / e^{0.9236}$$

$$E_4 = 1 + 0.0377\times \arctan[0.067(W/h)^{1.457}] \times (6 - 5 \times \exp[0.036(1 - \varepsilon_r)])$$

$$E_5 = 1 - 0.218 \times \exp(-7.5W/h)$$

4.1.2.3 Estimation for radiation conductance and resistance

Cavity model slot conductance and radiation conductance can be expressed as [1]:

$$G_s = I_1/120\pi^2 \quad (4.26)$$

$$R_r = 1/G_r \quad (4.26a)$$
\[ G_r = 2G_s \]  

(4.26b)

With \( I_1 = -2 + \cos(k_0W) + k_0WS_i(k_0W) + \sin(k_0W)/(k_0W) \) &

\[ S_i(x) = \int_0^{x\sin x} \frac{dx}{x} \]

The expression for slot conductance is given by [6]:

\[ G_s = \frac{1}{m\eta_0} \left\{ \left( wS_i(w) + \frac{\sin w}{w} + \cos w - 2 \right) \left( 1 - \frac{s^2}{24} \right) \right\} + \frac{s^2}{12} \left( \frac{1}{3} + \frac{\cos w}{w^2} - \frac{\sin w}{w^3} \right) \]  

(4.26c)

Where \( w=k_0W \), \( s=k_0h \) and \( S_i(x) \) is sine integral defined earlier.

The mutual conductance can be expressed as [6]:

\[ G_m = G_sM_g \]  

(4.27)

The factor \( M_g \) accounts for mutual coupling between the slots and can be expressed as [6]:

\[ M_g = g_m / g_s \]  

(4.27a)

Where \( g_m \) and \( g_s \) are defined as:

\[ g_m = \frac{k_0}{2\eta_0} \left\{ \left( 1 - \frac{(kh)^2}{24} \right) J_0(k_0L_e) + \frac{(kh)^2}{24} J_2(k_0L_e) \right\} \]  

(4.27b)

\[ g_s = \frac{k_0}{2\eta_0} \left( 1 - \frac{(kh)^2}{24} \right) \]  

(4.27c)

Further \( M_g \) can be simplified as:

\[ M_g = J_0(k_0L_e) + \frac{(kh)^2}{24(k_0h)^2} J_2(k_0L_e) \]  

(4.27d)

Where \( J_0 \) and \( J_2 \) are Bessel function of first kind of order 0 and 2 respectively.

The two slot model radiation conductance and radiation resistance accounting mutual coupling for odd (+) and even (−) mode respectively can be expressed as [6]:
\[ G_r = 2(G_x + G_m) \quad (4.27e) \]
\[ R_r = \frac{1}{G_r} \quad (4.28) \]

The expression for mutual conductance and mutual susceptance, are applicable for any pair of radiating slots belonging to same patch or different patches.

### 4.1.2.4 Estimation for input impedance, Q and BW

The cavity model simplifies the analysis of patch antenna with simple equations which can be used to model the patch antenna. In this model the patch cavity is modeled as a parallel RLC circuit, while the probe inductance is modeled as a series inductor. The input impedance of this circuit is approximately described by [8]:

\[ Z_{in} = jX_f + \frac{R}{1+2Q(f_0-1)} \quad (4.29) \]

Where \( f_0 \) is the resonance frequency, \( R \) is the input resistance at the resonance of the RLC circuit (where the input resistance of the patch is maximum), \( X_f = \omega L_p \) is the (feed) reactance of the coaxial probe and \( Q \) is the total quality factor of the patch cavity.

The input resistance in \( TM_{01} \) and \( TM_{10} \) cavity mode in terms of Q is given as [8]:

\[ R_{01} = 120 \lambda_{01} \frac{Q h}{\varepsilon_r w e e} c o s^2 \left( \frac{\pi x_0}{L_e} \right) \quad 0 \leq y_0 \leq \frac{L}{2} \quad (4.29a) \]

\[ R_{10} = 120 \lambda_{10} \frac{Q h}{\varepsilon_r w e e} c o s^2 \left( \frac{\pi x_0}{W_e} \right) f_0^2 \left( \frac{nw}{2w_e} \right) \quad 0 \leq x_0 \leq \frac{W}{2} \quad (4.29b) \]

Where \( \lambda_{01} = 2L_e \sqrt{\varepsilon_{ef}} \), \( \lambda_{10} = 2W_e \sqrt{\varepsilon_{ef}} \), \( f_0 = \frac{\sin \left( \frac{nw}{2w_e} \right)}{\frac{nw}{2w_e}} \) and \( w \) = feed width, approximately five times the probe diameter \( (w=5 \times (2 \times r_0)) \), \( r_0 \) = coaxial probe radius.

The location of feed points \( (x_0, y_0) \) are adjusted to obtain the required matching point to the feed line.
The correct feeding location \( x_f \) for desired resistance of matching circuit for transmission line or co-axial cable of 50 ohms characteristic impedance can be expressed as [6]:

\[
x_f = \frac{L_{ef}}{\pi} \cos^{-1} \left[ \sqrt{\frac{50}{R_{in}}} \cos \left( \frac{\pi x_0}{L_{ef}} \right) \right]
\]  
(4.30)

Total quality factor of the patch cavity is given by [9]:

\[
\delta_{eff} = \frac{1}{Q} = \frac{1}{Q_c} + \frac{1}{Q_d} + \frac{1}{Q_r} + \frac{1}{Q_{sw}}
\]  
(4.31)

Where \( Q_c \) is due to the conductor loss, \( Q_d \) is dielectric loss of the substrate, \( Q_r \) is radiation loss and \( Q_{sw} \) is the loss associated due to surface wave, each term can be defined as [9]:

\[
Q_c = \frac{h}{\delta_c}
\]  
(4.31a)

with

\[
\delta_c = \frac{1}{\pi} \sqrt{\frac{2}{120 \sigma_c}}
\]

\[
Q_d = \frac{1}{\tan \delta}
\]  
(4.31b)

The radiation quality factor \( Q_r \) of cavity can be expressed as [9]:

\[
Q_r = \frac{\varepsilon_r L W}{60 \hbar \varepsilon_{0m} \varepsilon_{o0} G_r}
\]  
(4.31c)

Where \( \varepsilon_{0m} = \varepsilon_{0n} = 1 \) for \( m=n=0 \) and \( \varepsilon_{0m} = \varepsilon_{0n} = 2 \) when \( m, n \neq 0 \)

\[
Q_{sw} = \frac{A_z}{1 - A_z} Q_r
\]  
(4.31d)

Where

\[
A_z = 1 - 3.4 K_{ef} \text{ for } \frac{h}{\lambda} < 0.06
\]

\[
A_z = 1 - 3.4 K_{ef} + \frac{1600}{\varepsilon_r^3} (K_{ef}^3 - 100 K_{ef}^{5.6}) \text{ for } 0.06 < \frac{h}{\lambda} < 0.16
\]
\[ Q_r = \left[ \frac{120 k_0 h G_r}{\varepsilon_r L W (1 - 3.4 K_{ef})} + \frac{1}{\pi h} \sqrt{\frac{\lambda_0}{120 \sigma_c}} + \tan \delta \right]^{-1} \] (4.32)

Where \( K_{ef} = \frac{h}{\lambda} \sqrt{\varepsilon_r - 1} \), \( \sigma_c \) is the conductivity of patch metallization, \( G_r \) is the radiation conductance.

The cavity model BW is given by:

\[ BW = \frac{1}{\sqrt{2 Q_r}} \] (4.33)

4.1.2.5 Estimation for radiation efficiency

The propagation of surface wave in addition to dielectric and conductor loss for thin substrate approximation has profound effect on efficiency of patch antenna. The radiation efficiency, without accounting into dielectric and conduction losses can be expressed as [8]:

\[ \varepsilon_r^0 = \frac{P_r}{P_r + P_{sw}} \] (4.34)

Where \( P_r = 40 k_0^2 (k_0 h)^2 c_1 \)

\[ P_{sw} = 30 \pi k_0^2 \varepsilon_r X / (\varepsilon_r - \frac{1}{\sqrt{X}}) + (k_0 h) \left[ 1 + \varepsilon_r^2 \right] \]

\[ c_1 = 1 - \frac{1}{n_1^2} + \frac{2}{n_1^2} n_1 = \sqrt{\varepsilon_r H_r} \]

\[ X = x_0^2 - 1 \quad Y = \varepsilon_r - x_0^2 \quad \text{and} \quad x_0 = 1 + 0.5 (\varepsilon_r - 1) * k * h / \varepsilon_r \]

4.1.2.6 Estimation for directivity and gain

Directivity for two slot model is [1]:

\[ D_2 = \frac{(2 \pi W / \lambda_0)^2 \pi}{l_2} \] (4.35)

Where

\[ l_2 = \int_0^\pi \int_0^\pi \left[ \sin(k_0 W / 2 \cos \theta) / \cos \theta \right]^2 \sin^3 \theta \cos^2(k_0 L_e / 2 \sin \theta \cos \varphi) d \theta d \varphi \]
And simplified form of directivity can be written as [6]:

\[ D_0 = \frac{4(k_0 W)^2}{12\rho^2} \]  \hspace{1cm} (4.35a)

The gain of the antenna is related to efficiency and directivity and can be expressed as [1]:

\[ G = \varepsilon_r D_0 \]  \hspace{1cm} (4.36)

### 4.1.2.7 Estimation for radiation power pattern

The radiation patterns of the \( TM_{10} \) mode are calculated by modeling the antenna as a combination of two parallel slots of length \( W \) and width \( h \) and separated by a distance \( L \) apart. It is assumed that the radiation from the patch is linearly polarized with the electric field directed along the patch length. If voltage across either radiating slot is taken as \( V_0 \), the radiation fields are obtained by multiplying the radiation pattern of one slot with the array factor \( 2 \cos(k_0 L \sin \theta) \cos(\Phi)/2 \).

The power pattern is expressed in terms of radiation fields in the principal planes of a RUMSA, operating in \( TM_{10} \) mode including the effect of the ground plane and substrate of the antenna [6]:

For the \( \Phi=0 \) plane or E-plane power pattern are given as [6]:

\[ |E_\theta(\theta)|^2 = \varepsilon_r \left[ 1 + \varepsilon_r \cot^2(k_0 h \sqrt{\varepsilon_r}) \right] T_1(\theta) T_2(\theta) \]  \hspace{1cm} (4.37)

Where

\[ T_1(\theta) = \frac{\cos^2(k_0 L \sin \theta)}{(\varepsilon_r - \sin^2 \theta)^2} \] \hspace{1cm} and \hspace{1cm} \[ T_2(\theta) = \frac{(\varepsilon_r - \sin^2 \theta) \cos^2 \theta}{(\varepsilon_r - \sin^2 \theta) + \varepsilon_r \cot^2(k_0 h \sqrt{\varepsilon_r} \sin^2 \theta)} \]

\[ E_\phi(\theta) = 0 \]

For the \( \theta =0 \) plane or H-plane power pattern are given as [6]:

\[ |E_\phi(\theta)|^2 = \left[ 1 + \varepsilon_r \cot^2(k_0 h \sqrt{\varepsilon_r}) \right] P_1(\theta) P_2(\theta) \]  \hspace{1cm} (4.37a)
Where \( P_1(\theta) = \frac{\cos^2 \theta}{(\varepsilon_r - \sin^2 \theta) \cot^2 (k_0 h / \sqrt{\varepsilon_r - \sin^2 \theta}) + \cos^2 \theta} \) and \( P_2(\theta) = \frac{\sin (k_0 W \sin \theta / 2)}{(k_0 W \sin \theta / 2)} \)

\[ E_\theta (\theta) = 0 \]

4.1.3 CAD model

4.1.3.1 Estimation for resonant frequency

The CAD model resonant frequency based on magnetic current proposed by James et al can be expressed as [44]:

\[ f_r = \frac{f_{ro} \varepsilon_r}{\sqrt{\varepsilon_r (\varepsilon'_{ro}) (1 + \delta)}} \]  \hspace{1cm} (4.38)

Where \( \delta = \left( \frac{h}{W} \right) \left[ 0.882 + \frac{0.164 (\varepsilon_r - 1)}{\varepsilon_t} + \frac{(\varepsilon_r + 1)}{\varepsilon_r} \left[ 0.758 + \log \left( \frac{W}{h} + 1.88 \right) \right] \right] \)

4.1.3.2 Estimation for input resistance and feed reactance

A CAD formula for the input resistance \( R \) is given by [8]:

\[ R_{in} = \left( \frac{4}{\pi} \right) (\mu_r \eta_0) \left( \frac{1}{\delta_{ef}} \right) \left( \frac{W_c}{L_c} \right) \left( \frac{h}{\lambda_0} \right) \cos^2 \left( \frac{\pi x_0}{L_c} \right) \hspace{1cm} 0 \leq x_0 \leq \frac{L}{2} \]  \hspace{1cm} (4.39)

And effective loss tangent \( \delta_{ef} \) is defined as:

\[ \delta_{ef} = \left[ \tan \delta + \left( \frac{R_s}{\pi \eta_0 \mu_r} \right) \left( \frac{1}{\pi \lambda_0} \right) + \left( \frac{16}{3} \right) \left( \frac{p c_1}{\pi} \right) \left( \frac{h}{\lambda_0} \right) \left( \frac{W_c}{L_c} \right) \left( \frac{1}{e_r \tau} \right) \right] \]

If dielectric and conductor losses are neglected then simplified expression for input resistance reduces to:

\[ R_{in} = \left( \frac{3}{4 \pi} \right) \left( \frac{16}{3} \right) \left( \frac{\varepsilon_r \mu_r \eta_0}{pc_1} \right) \left( e_r^{\text{sw}} \right) \left( \frac{l_c}{W_c} \right) \cos^2 \left( \frac{\pi x_0}{L_c} \right) \]  \hspace{1cm} (4.39a)

A CAD formula for the feed reactance due to the probe is given by [8]:

\[ X_f = \frac{\eta_0}{2 \pi} \mu_r (k_0 h) \left[ -\gamma + \ln \left( \frac{2}{(k_0 h \sqrt{\varepsilon_r \mu_r})} \right) \right] \]  \hspace{1cm} (4.40)
Where \( \gamma = 0.577216 \) is Euler’s constant, \( r_p = \) probe radius

### 4.1.3.3 Estimation for radiation efficiency

The radiation efficiency for a horizontal electric dipole model can be expressed in terms of \( P_{HED}^{sp} \), power radiated into space by a unit strength horizontal electric dipole on the lossless substrate and \( P_{sw}^{HED} \), the power radiated into surface wave by the dipole as [6]:

\[
e_{r}^{HED} = \frac{P_{HED}^{sp}}{P_{HED}^{sp} + P_{sw}^{HED}} \tag{4.41}
\]

Where

\[
P_{HED}^{sp} = \frac{1}{\lambda_0}(k_0h)^2(80\pi^2\mu_r^2 c'_{1}),
\]

\[
c'_{1} = 1 - \frac{1}{n_1^2} + \frac{2/\varepsilon}{n_1} \quad \text{and} \quad n_1 = \sqrt{\varepsilon_r \mu_r},
\]

\[
P_{sw}^{HED} = \frac{1}{\lambda_0}(k_0h)^360\pi^3\mu_r^3 \left(1 - \frac{1}{n_1^2}\right)^3
\]

The improved expression for \( P_{sw}^{HED} \) with higher order terms in the approximation for nonmagnetic substrate (\( \mu_r = 1 \)) can be expressed as [8]:

\[
P_{sw}^{HED} = \frac{\eta\varepsilon_0\omega^2}{8} \frac{\varepsilon_r [x_0^2 - 1]}{\sqrt{\varepsilon_r [x_1 + 1] + (k_0h) [\varepsilon_r x_0 + 1] |x_0^2 - 1|}} \tag{4.41a}
\]

where \( x_1 = \frac{x_0^2 - 1}{\varepsilon_r - x_0^2} \), \( x_0 = 1 + \frac{-\varepsilon_r^2 + \alpha_0 \alpha_1 + \varepsilon_r \sqrt{\varepsilon_r^2 - 2\alpha_0 \alpha_1 + \alpha_1^2}}{\varepsilon_r^2 - \alpha_1^2} \)

\[
\alpha_0 = s \tan(k_0hs), \quad s = \sqrt{\varepsilon_r - 1} \quad \text{and} \quad \alpha_1 = -\frac{1}{s} \left[\tan(k_0hs) + \frac{k_0hs}{\cos^2(k_0hs)}\right]
\]

The CAD formula of radiation efficiency for horizontal electric dipole model of MSA can be expressed as [8]:

\[
e_{r}^{CAD} = e_{r}^{HED} / (1 + e_{r}^{HED} \left[ \delta_{ef} + \left( \frac{R_e}{\eta_0 \mu_r} \right) \left( \frac{1}{\lambda_0} \right) \right] * Q_{sp} ) \tag{4.42}
\]
The resistance due to skin effect on surface of the patch and ground plane metallization at frequency \( \omega=2\pi f \), is \( R_s = \sqrt{\omega \mu_0 / 2\sigma_e} \), where \( \sigma_e \) is the conductivity of the patch metallization and \( \delta_{ef} \) is the effective loss tangent defined earlier.

The space-wave \( Q \) factor formula in terms of CAD is given by [8]:

\[
Q_{sp} = \frac{3}{16} \left( \frac{\epsilon_r}{\rho_c} \right) \left( \frac{L_e}{W_e} \right) \left( \frac{1}{h/\lambda_0} \right)
\]  

(4.42a)

Where \( p \)-factor is defined as:

\[
p = 1 + \frac{a_2}{10} (k_0 W_e)^2 + (a_2^2 + 2a_4) \left( \frac{3}{560} \right) (k_0 W_e)^4 + c_2 \frac{1}{5} (k_0 L_e)^2 + a_2 c_2 \left( \frac{1}{70} \right) (k_0 W_e)^2 (k_0 L_e)^2
\]

The surface-wave \( Q \) factor is related to the space-wave \( Q \) factor as [8]:

\[
Q_{sw} = Q_{sp} \frac{e_{pw}}{1 - e_{pw}^{sw}}
\]

(4.43)

where \( e_{pw}^{sw} \) is the radiation efficiency accounting only for surface-wave loss.

The \( Q \) for radiated power is:

\[
\frac{1}{Q_r} = \frac{1}{Q_{sw}} + \frac{1}{Q_{sp}}
\]

(4.44)

Efficiency in terms of space wave \( Q_{sw} \) and radiation \( Q_r \) is:

\[
e_{rsw} = Q_r / Q_{sw}
\]

(4.45)

4.1.3.4 Estimation for reflection coefficient and VSWR

Reflection coefficient \( \Gamma \) can be expressed as [9]:

\[
|\Gamma| = \frac{Z_{in} - R_{mn}}{Z_{in} + R_{mn}}
\]

(4.46)

With \( R_{mn} = \frac{1}{G_{mn}} = \frac{60 \sigma_0 \epsilon_{mn} Q \lambda_{mn}}{\epsilon_r LW} \psi_{mn}^2 (x_0, y_0) f_0^2 \left( \frac{mnw}{2W} \right) \)
And the Eigen functions \( \psi_{mn}(x_0, y_0) \), \( f_0 \) are defined as:

\[
\psi_{mn}(x_0, y_0) = \cos\left(\frac{m\pi x_0}{W}\right) \cos\left(\frac{n\pi y_0}{L}\right) \quad \text{and} \quad f_0 = \frac{\sin\left(\frac{m\pi W}{2W}\right)}{\left(\frac{2W}{2W}\right)}
\]

The reflection coefficient \( \Gamma \) and VSWR can be expressed in terms of \( Q \) as:

\[
|\Gamma| = \left[1 + \frac{4}{Q^2}\right]^{-0.5}
\]

Where \( S = f/f_{mn} - f_{mn}/f \)

\[
\text{VSWR} = (1 + |\Gamma|)/(1 - |\Gamma|)
\]

4.1.3.5 Estimation for BW, directivity and gain

The BW is defined with respect to a minimum value of VSWR. The BW equation in terms of CAD formulation with VSWR less than equal to two can be expressed as [8]:

\[
\text{BW} = \frac{1}{\sqrt{2}} \left[ \tan\delta + \left(\frac{R_e}{\pi \eta_0 \mu_f}\right) \left(\frac{1}{h/\lambda_0}\right) + \left(\frac{16}{3}\right) \left(\frac{p c_1}{\varepsilon_f}\right) \left(\frac{h}{\lambda_0}\right) \left(\frac{W_c}{L_c}\right) \left(\frac{1}{\varepsilon_f^{ref}}\right) \right]
\]

(4.47)

If dielectric and conductor losses are neglected the BW formula simplifies to:

\[
\text{BW}^0 = \frac{1}{\sqrt{2}} \left[ \left(\frac{16}{3}\right) \left(\frac{p c_1}{\varepsilon_f}\right) \left(\frac{h}{\lambda_0}\right) \left(\frac{W_c}{L_c}\right) \left(\frac{1}{\varepsilon_f^{ref}}\right) \right]
\]

(4.47a)

The CAD formula for directivity is given as [8]:

\[
D_0 = \left(\frac{\eta_0}{40\pi}\right) \left(\frac{1}{p c_1}\right) \left[ \frac{\tan^2(k_0 h n_1)}{(k_0 h n_1)^2} \right] \left[ \frac{1+(\frac{p c_1}{\varepsilon_f})\tan^2(k_0 h n_1)}{1+(\frac{p c_1}{\varepsilon_f})\tan^2(k_0 h n_1)} \right]
\]

(4.48)

CAD formula for gain is can be expressed as [8]:

\[
G = e^{h ED} D_0
\]

(4.49)
4.1.4 Estimation for effect of finite size ground plane on resonant frequency

The finite ground plane gives rise to diffraction of radiation from the edges of the ground plane resulting in change in radiation pattern, radiation conductance and resonance frequency. If patch size metallization of an antenna is equal to the dimension of the ground plane, the resonant frequency is higher compared to that of an infinite sized ground plane antenna. Denoting ground plane extension by \( d \), the fractional change in resonant frequency is given by [6]:

\[
\frac{\delta f}{f_r} = \frac{1}{\pi} \frac{\varepsilon_r + \frac{1}{h}}{\varepsilon_r \lambda_0} \ln \left( \frac{W}{h} \right), \quad k_0d \to 0 \text{ for } \frac{W}{h} \gg 1
\]  

(4.50)

For other value of \( d \), not satisfying the limit of the expression can be written as [6]:

\[
\frac{\delta f}{f_r} = -\frac{240}{\sqrt{\varepsilon_r \mu_0}} \frac{h}{G_d} 
\]  

(4.50a)

With \( G_d = \frac{k_0W}{\omega_0} \left[ J_0^2(k_0d) - Y_0^2(k_0d) \right] \)

Where \( J_0 \) and \( Y_0 \) are zero order Bessel function of first and second kind and \( \eta_0 \) is characteristics impedance of free space.

4.1.5 Estimation for effect of probe reactance on resonant frequency

The relative upward shift in resonant frequency caused due to probe inductance can be expressed as [46]:

\[
\frac{\delta f_c}{f_r} = \frac{\pi}{2} \left( \frac{h}{L} \right)^2 \left( \frac{W}{L} \right) \frac{1}{\varepsilon_r e^2} \log \left( \frac{2L}{\gamma \pi \eta_0} \right)
\]  

(4.51)

4.1.6 Tolerance analysis of RMSA

The deviation in the patch dimensions is mainly due to the variation of effective electrical length from the designed values which can lead to disagreement between the design and actual resonant frequencies. These deviations include the irregularities in substrate thickness, variations in relative permittivity of the substrate or small change in antenna length and width. The fabrication error occurs due to inaccuracy in etching.
The manufacturing tolerance in inaccurate patch dimension, change in relative permittivity and substrate thickness alter the resonant frequency, which in turn change input impedance and BW.

The effect of tolerance in patch parameters namely $\Delta h$, $\Delta \varepsilon_r$ and $\Delta L'$ on fractional change in resonant frequency can be expressed as [6]:

$$\frac{|\Delta f_r|}{f_r} = \left[ \left(\frac{\Delta L'}{L}\right)^2 + \left(\frac{0.5}{\varepsilon_{re}}\right)^2 \left(\frac{\partial \varepsilon_{re}}{\partial h} \Delta h + \left(\frac{\partial \varepsilon_{re} \Delta \varepsilon_r}{\partial \varepsilon_r}\right)^2\right)\right]$$

(4.52)

Using eqn.(4.1a) and eqn.(4.2a) for the change in effective dielectric constant with respect to relative permittivity and substrate thickness can be expressed as [6]:

$$\frac{\partial \varepsilon_{re}}{\partial \varepsilon_r} = 0.5 \left[ 1 + \left( 1 + \frac{10h}{W} \right)^{-ab} \right]$$

(4.52a)

$$\frac{\partial \varepsilon_{re}}{\partial h} = -\varepsilon_r \left( 1 + \frac{5ab}{W} \left( 1 + \frac{10h}{W} \right)^{-ab-1} \right)$$

(4.52b)

In the next chapter we will discuss in details the simulations procedure of transmission line, cavity and CAD models of the RMSA.