Chapter 3

Angular power spectrum estimation

3.1 Introduction

The angular power spectrum of cosmic microwave background (CMB) temperature fluctuations is a measurable physical quantity that depends sensitively on the physics of the early universe. In particular, the shape of the angular power spectrum and the locations and heights of its peaks relate directly to parameters in the underlying cosmological models [43]. As such, it has been used extensively as an acid test of the relative merit of competing cosmological models, and as a rich source of information about cosmological parameters.

Traditionally, and almost exclusively, cosmologists have resorted to model-based parametric statistical methods for estimating the CMB angular power spectrum from data. Parametric regression methods require the functional form of the unknown regression function \( f \) to be pre-specified. The adjustable parameters in \( f \), finite in number that is independent of the data size, are usually estimated by maximizing an appropriate likelihood function or a posterior distribution. In the cosmological context, it is conventional to employ parametric models that
attempt to capture the essential physics of the problem via the pre-specified functional form, and any pre-existing knowledge about parameters is incorporated in the estimation process via appropriate prior distributions. Nonparametric function estimation methods, on the other hand, assume no specific functional form for \( f \), except for mild regularity conditions such as smoothness assumptions, membership to a function space, etc. In this approach, the number of parameters that define the unknown regression function \( f \) is either infinite or grows proportionally with the data size, and the estimate \( \hat{f}_N \) of \( f \) is obtained by balancing bias and variance of \( \hat{f}_N \) via optimal smoothing. Nonparametric methods are therefore model-independent, and are based on fewer and less restrictive assumptions about \( f \). This, in turn, implies that any inferences about \( f \) made from nonparametric regression analyses tend to be more data-driven as opposed to being primarily model-driven. In other words, to a greater extent nonparametric function estimation methods tend to infer what is rather than what should be. As such, nonparametric regression methods can be meaningfully employed as sanity-enforcing mechanisms on parametric analyses, thereby making the conclusions drawn more conservative. For example, a feature seen in a parametric fit that survives in a nonparametric analysis is likely to be a real and robust feature of the data itself, and not merely an artifact that is seen because a parametric model expects it to be there.

Alternative methodologies, such as the nonparametric methodology used in this thesis (Chapter 2) may allow posing inferential questions that are difficult to address using conventional methods. For example, this particular nonparametric methodology allows validating model-based, parametric fits against the confidence set for the nonparametric fit to the same data, possibly to rule them out as candidates for the true but unknown regression function. This methodology also has the formal advantage of being able to provide realistic uncertainties on any number of features of the angular power spectrum that are simultaneously valid at the same level of confidence. Such desirable features are arguably lacking in most mainstream methodologies, including Bayesian, that are commonly used in cosmology. An incisive, insightful, and discerning discussion about the relative merits of this
methodology over statistical methods conventionally employed in cosmology can be found in [1, 60], and is best read in the original.

The four CMB angular power spectrum data sets [10, 76–78] released by the Wilkinson Microwave Anisotropy Probe (WMAP) [79] mission, representing cumulative observations at the end of 1, 3, 5, and 7 years of operation, present a unique opportunity for statistical analysis. For example, till date these are claimed to represent the most precise and extensive full-sky CMB measurements ever made (only to be superseded by the recently released 9-year data set [27], and by the Planck mission [30]). These four data sets represent the evolution of data over a period of about a decade, thereby making it possible to assess progressive and possibly systematic resolution of features of the spectrum. From a statistical perspective, each of these moderately large data sets (minimum of 899 data points for the WMAP 1-year release) is not only heteroskedastic, but also has substantial correlations that arise in typical data pipelines [80–84].

In this chapter, we present a comparative nonparametric analysis of the WMAP 1-, 3-, 5-, and 7-year data sets for the angular power spectrum. Specifically, for each data realization, we address the following three key questions: (a) How well is the angular power spectrum determined by the data alone? (b) How well is the ΛCDM model supported by a model-independent, nonparametric, data-driven analysis? (c) What are the realistic uncertainties on peak/dip locations and heights? (d) Is there any evidence of harmonicity of peaks? Our analysis is based on a nonparametric function estimation methodology [1, 60], which is discussed in Chapter 2 together with our extensions; i.e., a prescription for obtaining nonparametric fits that are closer to cosmological expectations on smoothness, and a method for sampling cosmologically meaningful power spectrum variations from the confidence set of a nonparametric fit. Results are presented in Sec. 3.3, and conclusion in Sec. 3.4.
3.2 Methods

The four WMAP angular power spectrum data sets used in this work, $\Lambda$CDM parametric fits for the CMB angular power spectrum, and likelihood codes that produce their respective Fisher (inverse covariance) matrices are obtained from the WMAP data archive http://lambda.gsfc.nasa.gov/. For all four data realizations, it turned out that (a) the weight matrix $W$ (see Appendix F) is numerically positive definite, and (b) the confidence set for the full-freedom monotone fit is completely nested inside that for the smoother restricted-freedom monotone fit. It is worth noting that our nonparametric fits and confidence sets are not too sensitive to the details of the covariance matrix (this was also pointed out in [60]). Most computations were done using the R statistical computing environment [85]. We used the QL codes [72] for the monotone risk minimization problem (Eq. 2.38). Our nonparametric fits, obtained using the method outlined in Sec. 2.2.4, are shown in Fig. 3.1 and 3.2 for the WMAP 1-, 3-, 5-, and 7-year data sets.

We are given CMB angular power spectrum data of the form

$$Y_l = C_l + \epsilon_l,$$  \hfill (3.1)

consisting of $N$ data points observed over multipole index range $l_{\text{min}} \leq l \leq l_{\text{max}}$ (data points in Figures 3.1 and 3.2). Here, $C_l$ stands for the value of the true but unknown power spectrum at multipole index $l$. The noise variables $\epsilon_i$ are assumed to have a mean-0 normal distribution with known covariance matrix $\Sigma$.

If we compare the above equation with Eq. 2.9 we realize that $C_l \equiv f(x)$. Therefore, the nonparametric methodology which is described in Chapter 2 is applicable here on CMB angular power spectrum data. So, by assuming the $f(x)$ belongs to an appropriate $L_2$ function space, we expand it in a complete orthonormal basis $\{\phi_j(x)\}$, as

$$f(x) = \sum_{j=0}^{\infty} \beta_j \phi_j(x).$$
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**Figure 3.1:** Nonparametric fits for WMAP 1- (top) and 3-year (bottom) data sets. $x$- and $y$-ranges are identical across plots. Green: full-freedom monotone fit ($\text{EDoF} \approx 80.2, 76.5$ respectively); blue: NSS fit ($\text{EDoF} = 12, 10$ respectively); red: restricted-freedom monotone fit ($\text{EDoF} \approx 9.4, 9.5$ respectively); solid gray: best $\Lambda$CDM-based parametric fit; dashed gray: power spectrum for an H$\Lambda$CDM model. See Table 3.1 for details of model-based power spectra.
Figure 3.2: Nonparametric fits for WMAP 5- (top) and 7-year (bottom) data sets. $x$- and $y$-ranges are identical across plots. Green: full-freedom monotone fit ($\text{EDoF} \approx 60.4, 102.9$ respectively); blue: NSS fit ($\text{EDoF} = 13, 20$ respectively); red: restricted-freedom monotone fit ($\text{EDoF} \approx 14.4, 14.1$ respectively); solid gray: best $\Lambda$CDM-based parametric fit; dashed gray: power spectrum for an H$\Lambda$CDM model. See Table 3.1 for details of model-based power spectra.
A basis that has proven useful in the CMB angular power spectrum context is the cosine basis which is already described in the previous chapter (Eq. 2.10). In the CMB context, any contiguous range of $N$ integer multipole indices $l_{\text{min}} \leq l \leq l_{\text{max}}$ can be formally mapped onto this equispaced grid, hence we will not make any categorical distinction between data index $i$ and the corresponding multipole index $l$.

The true angular power spectrum $C_l \equiv f(x_l)$ is estimated as $\hat{C}_l \equiv \hat{f}_N(x_l)$ via coefficient estimates $\hat{\beta}_j$, which are estimated as $\hat{\beta}_j = \lambda_j Z_j$. The shrinkage parameters $\lambda_j$ are obtained by minimizing the risk estimator (Eq. 2.35) under Monotone and NSS constraints (Eq(s). 2.21, 2.22). The minimization procedure is explained in Sections 2.2.3 and 2.2.4.

### 3.2.1 Obtaining a smoother monotone fit that is closer to cosmological expectations

To motivate the discussion in this section, consider the full-freedom monotone fit to the WMAP data sets (green curves in Fig. 3.1 and 3.2). By **full-freedom monotone fit**, we mean the fit that minimizes risk over the entire monotone-admissible region (Fig. 2.1) without any restriction on the EDoF of the fit. This fit turns out to be quite wiggly especially at the high-$l$ end because of the high noise variance here. Such wiggleness implies presence of high-frequency components in the fit, which in turn implies a large number of EDoF in the fit.

Without cosmological pre-conditioning (i.e., from a completely agnostic and data-driven perspective) and when viewed in relation to the data, it is clear that this fit is not at all unreasonable, given the high noise levels at the high-$l$ end. However, all cosmological models suggest far smoother shapes for the angular power spectrum. In the context of the present methodology, one candidate for a smoother fit is the NSS fit (i.e., one that has the minimal risk over the set of $(N+1)$ NSS fits). Indeed, this possibility has been exploited, e.g., in [1]. However, the NSS fit (see the blue curve in Fig. 3.1 and 3.2) may also turn out to be somewhat unsatisfactory with
respect to cosmological expectations (and, some times, also with respect to trends reflected in the full-freedom monotone fit). This is primarily because of the limited freedom available in the NSS class. Notice again that the NSS fit is not entirely unreasonable from an agnostic viewpoint.

The monotone set, on the other hand, offers the possibility of harnessing local minima in the risk function that are constrained to lie in appropriate “smoother” subsets of the full monotone set. This may be achieved in two distinct ways that may be combined for greater effect:

1. By imposing one of the additional constraints (Eq. 2.39) on the EDoF of the fit. Examples of such restricted-freedom monotone fits are the red curves in Fig. 3.1 and 3.2.

2. By truncating the expansion (Eq. 2.12) to $p$ number of coefficients ($p < N$) and then performing monotone risk minimization over this subset of the full monotone set.

In practice, such smoother restricted-freedom monotone fit can be obtained by gradually reducing the value of $q$ (Eq. 2.39) starting from the EDoF of the NSS fit until all low-amplitude, high-frequency wiggles in the fit disappear. Generally, the resulting fit has a lower risk than the NSS fit with EDoF = $q$, and is manifestly consistent with trends captured by the full-freedom monotone fit. We find it useful to present (or consider) all three fits (NSS, full-freedom monotone, and restricted-freedom smoother monotone) together. This helps build a realistic picture about estimated trends in the data, and thereby about the shape of the underlying true spectrum. Like the NSS fit, the smoother restricted-freedom monotone fit will generally be more biased than the full-freedom monotone fit. This greater bias, however, is partially compensated for by a larger risk which results in a larger confidence radius value (Eq. 2.44), a larger confidence set, and therefore more conservative inferences.
3.2.2 Probing the confidence set for uncertainties on features of the fit

In this work, we need to probe the confidence set for a fit for two purposes: (a) for validating cosmological models against a nonparametric fit (see Sec. 3.3.3). (b) for finding the uncertainties on specific features of the fit such as peak heights and locations, and below we describe our method to scan and sample the confidence set for the later. Our particular method for probing the confidence set for determining uncertainties on features of the fit is based on the following observations:

1. The confidence set $D_{N,\alpha}$ on the vector of coefficients $(\beta_0, \ldots, \beta_{N-1})$, by construction, is centered at the vector of estimates $(\hat{\beta}_0, \ldots, \hat{\beta}_{N-1})$.

2. The confidence interval defined in Eq. 2.46 requires locating extreme variations in any functional $T$ of the power spectrum $f$; e.g., location or height of a peak. The largest possible variations in $T$ will be located as far away from the center of the confidence set as possible, i.e., on its surface.

3. Cosmologically meaningful and sufficiently smooth variations in the fitted spectrum $\hat{C}_l$ are most likely to be located in the projection of the full confidence set onto the lowest $M$ dimensions.

We therefore generate a uniform sample from the projection of the full confidence set surface onto the lowest $d$ dimensions, where $2 \leq d \leq M$, with $M \lesssim 23$. For convenience, we use the smoother restricted-freedom monotone fit for this purpose, with the justification that the confidence set corresponding to the full-freedom monotone fit happens to be nested inside that for this fit, for all four data realizations. The Cholesky factorization $W = u^T u$ is used to transform the original confidence ellipsoid $D_{N,\alpha}$ (whose principal axes may not be aligned with coordinate directions in the $\beta$-space) into a sphere of the form $\{ \psi : \|\psi - \hat{\psi}\|^2 \leq r^2_\alpha \}$, where $\psi = u\beta$ and $\hat{\psi} = u\hat{\beta}$. The surface of this $\psi$-sphere can be efficiently sampled with uniform density using a standard algorithm (Sec. 3.4.1.E.6 on p.130 of [86]), and then transformed back into the $\beta$-space, preserving uniformity of density because
of the linearity of the transformation. From a sufficiently large sample of such variations of the power spectrum, we further selected those functions for which successive peaks and dips are separated by at least 50 multipole moments $l$. This cosmologically-motivated selection criterion ensures that (a) the sampled functions are sufficiently smooth, and (b) high-frequency wiggles are not counted as peaks or dips when estimating uncertainties on locations and heights of peaks and dips (see Sec. 3.3.4). Based on cosmological considerations, we restricted the search to functions with 3 peaks (WMAP 1-, 3-, and 5-year data) or 4 peaks (7-year data). The set of functions thus sampled is used to estimate uncertainties on specific features of the fit. As an aside, we note that the confidence set construct and the formal guarantees related to confidence intervals do not necessarily imply a uniform density over the confidence set; uniform sampling is used here as a convenient computational device for scanning the confidence set surface in an unbiased fashion.

3.3 Results and discussion

3.3.1 Likelihood versus nonparametric risk

Fig. 3.3 illustrates the contrasting behavior of the nonparametric risk (red curve) and the WMAP 1-year likelihood function (green curve) [9] for the WMAP 1-year data [76], as a function of the EDoF of all NSS fits. This figure is motivated by the fact that cosmologists, by and large, are better-acquainted with parametric likelihood-based methods. Each integer value on the horizontal axis represents one NSS fit, from the zero function at EDoF = 0 to the fit that simply interpolates through the data (EDoF = $N$). Optimal smoothing for NSS shrinkage occurs at EDoF = 12 where the nonparametric risk function attains its minimum over the NSS set of fits. Likelihood function, on the other hand, keeps on improving with the EDoF indefinitely.
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3.3.2 How well is the power spectrum determined by data alone?

Considering that the noise in all four data sets is highly heteroskedastic and noise levels are especially high for large $l$, it would be useful to make an assessment of how such noise in the data affects local uncertainties in the fitted spectrum, and to quantify how well the angular power spectrum value at each $l$ is determined by the data.

To this end, we compute, for each data set, the approximate 95% confidence interval on each $\hat{C}_l$ using 5000 function variations from the confidence set as outlined in
Sec. 3.2.2. The length of this vertical confidence interval at given $l$, divided by the absolute value of the fit, $|\hat{C}_l|$, provides an approximate indication of how well each $\hat{C}_l$ is determined via the following interpretation: a value $\ll 1$ indicates that the fit is well determined by the data, whereas values $\gg 1$ imply that the data contain little or no information about the height of the power spectrum. This approach, which is inspired by the boxcar probe approach of [1], has the practical advantage of not having adjustable parameters (i.e., the boxcar width) in the procedure.

![Figure 3.4](image_url)

**Figure 3.4:** The results of a probe of the confidence sets for the WMAP 1- (black), 3- (blue), 5- (red), and 7-year (green) nonparametric restricted-freedom monotone fits (Fig. 3.1 and 3.2) to determine how well the angular power spectrum is determined by the data alone. The quantity plotted for each data realization is the total vertical variation at each $l$ within the respective 95% ($2\sigma$) confidence set, divided by the absolute value of the fit. This quantity is an approximate measure of how well the angular power spectrum is determined by the data: Values $\ll 1$ indicate that the fit is tightly determined by the data, whereas values $\gg 1$ indicate that the data contain little or no information about the height of the angular power spectrum for that $l$. Disregarding the low-$l$ region, the color-coded vertical lines indicate the approximate $l$-value at which each curve rises above 1.

In Fig. 3.4, we plot this height, scaled by the value of the fit, as a function of the multipole index $l$, for all four data realizations. We see that the range of
$l$-values over which the fit is well-determined expands consistently between 1-, 3- and 5-year data realizations, from $l \approx 546$ (1-year), to $l \approx 667$ (3-year), to $l \approx 804$ (5-year). On the other hand, while the $l$-range of the data expanded substantially between WMAP 5 and 7, the information contained in the data does not appear to have grown proportionately beyond $l \approx 842$ for the 7-year data.

### 3.3.3 How well is the $\Lambda$CDM model supported by a model-independent, nonparametric, data-driven analysis?

In each of the four plots in Fig. 3.1 and 3.2, we have also included parametric fits based on the $\Lambda$CDM [10, 76–78] and HCDM [87, 88] models (see figure caption for details). The parametric $\Lambda$CDM-based fits turn out to be quite close to the respective nonparametric fits wherever the data are precise. This is remarkable considering that our nonparametric method does not rely on any cosmologically-motivated prior information whatsoever. Moreover, the parametric $\Lambda$CDM-based fit appears to get closer to the respective nonparametric fit across the four WMAP data releases. The closeness of a parametric fit ($C_{l_{\text{min}}}$, ..., $C_{l_{\text{max}}}$) to the corresponding nonparametric fit ($\hat{C}_{l_{\text{min}}}$, ..., $\hat{C}_{l_{\text{max}}}$) can be measured through the distance

$$d(C, \hat{C}) \approx \sqrt{\frac{1}{N} \sum_{l=l_{\text{min}}}^{l_{\text{max}}} \left( \frac{C_l - \hat{C}_l}{\sigma_l} \right)^2}.$$  

Using Eq. 2.44, this distance can be further expressed as the smallest confidence level $\alpha$ beyond which the parametric fit is rejected as a candidate for the true spectrum.

Table 3.1 lists distances of parametric fits based on the two cosmological models ($\Lambda$CDM and HCDM) from the nonparametric full-freedom monotone fit for the corresponding data realization (see caption for specific details and description). The progression of distance values between a parametric $\Lambda$CDM fit and the corresponding nonparametric fit clearly shows that the two are getting closer as the data become precise. In contrast to this, the angular power spectrum generated by
Table 3.1: Distances of two model-based power spectra from our non-parametric fits

<table>
<thead>
<tr>
<th>Data</th>
<th>ΛCDM</th>
<th>HΛCDM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-year</td>
<td>0.1540</td>
<td>0.1493</td>
</tr>
<tr>
<td>$r_α = 0.3679$</td>
<td>16.70%</td>
<td>15.87%</td>
</tr>
<tr>
<td>3-year</td>
<td>0.1462</td>
<td>0.2057</td>
</tr>
<tr>
<td>$r_α = 0.3653$</td>
<td>14.52%</td>
<td>29.03%</td>
</tr>
<tr>
<td>5-year</td>
<td>0.1419</td>
<td>0.3552</td>
</tr>
<tr>
<td>$r_α = 0.3563$</td>
<td>19.66%</td>
<td>94.82%</td>
</tr>
<tr>
<td>7-year</td>
<td>0.1238</td>
<td>0.3550</td>
</tr>
<tr>
<td>$r_α = 0.3551$</td>
<td>9.08%</td>
<td>94.98%</td>
</tr>
</tbody>
</table>

Note: These are the distances of two model-based (ΛCDM and HΛCDM) power spectra from our nonparametric (full-freedom monotone) fits. The HΛCDM model \[87, 88\] considered here for illustrative purposes is defined by a small neutrino fraction ($Ω_ν h^2 = 0.00275$) with corresponding adjustment to the dark energy content ($Ω_Λ = 0.729756$), and the rest of the parameters (including zero curvature) being identical to that of the best ΛCDM model [10] for the 7-year data (power spectrum generated using the CAMB software [47]). ΛCDM-based fits used are the best parametric fits for the corresponding data realization [10, 76–78]. $r_α$ is the confidence radius at $α = 0.05$ (i.e., 95% confidence level $≈ 2σ$). Percentages reported are the confidence levels corresponding to these distances; these can be interpreted as asymptotic probabilities with which the corresponding parametric fit is ruled out as a candidate for the true but unknown spectrum. Notice the dramatic progression, as the data become precise, of (a) how this HΛCDM model is pushed to the boundary of the confidence set, and (b) how the ΛCDM model gets closer to the non-parametric fit.
the particular HACDM model considered, which is almost as strong a contender for the true power spectrum as the ΛCDM fit with respect to the 1-year confidence set, is progressively pushed away to the boundary of the 95% confidence set for the 7-year confidence set. Visually, this trend can be understood on the basis of the differences between the HACDM power spectrum and the nonparametric fit (e.g., differences in the heights of the first peak; see Fig. 3.1 and 3.2) that result into pushing this particular HACDM model out of the confidence set. Given the formal guarantees of this methodology (Sec. 2.2.5), the WMAP 7-year data thus rules out, at ≈ 95% confidence level, the particular HACDM model considered.

3.3.4 Uncertainties on locations and heights of peaks and dips

We now consider the problem of determining uncertainties on the locations and heights of peaks and dips in the nonparametrically fitted spectrum. The motivation for this exercise comes from the fact that peak locations and heights contain valuable information about cosmological models and parameters [89, 90].

Following the prescription outlined in Sec. 3.2.2, we sampled a set of 5000 function variations from the confidence set for each data realization. Peak and dip locations and heights were recorded for each peak and dip over this set of functions. This results into an empirical scatterplot that is indicative of the joint distribution of location and height for each peak or dip, under the assumption of uniform surface density on the confidence set $D_{N,\alpha}$.

Fig. 3.5 shows the results of probing the 95% confidence sets for uncertainties on peaks and dips, as outlined above, with 5000 acceptable function variations for each data realization. The box around a peak or a dip represents the largest horizontal and vertical variations in the scatter. In accordance with the confidence interval defined in Eq. 2.46, these form the 95% confidence intervals on the location and height of a peak/dip. Table 3.2 lists these confidence intervals together with 95% confidence intervals on peak height ratios.
Figure 3.5: Nonparametric uncertainties on peak and dip locations and heights for the WMAP 1- (top left), 3- (top right), 5- (bottom left), and 7-year (bottom right) data sets. Nonparametric fit displayed for reference is the restricted-freedom monotone fit in Fig. 3.1 and 3.2. The number of acceptable function variations sampled from the confidence set for each data realization is 5000.
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The following features of these results are worth pointing out. As is well-known, the first peak was very clearly resolved in the 1-year data itself. Our results are manifestly consistent with this observation, in the sense that its box does not overlap with any other box. However, the uncertainty on the first peak does not shrink appreciably across the four data realizations. Further, our results clearly indicate that the second peak is resolved cleanly only in the 5-year data, whereas the third and fourth peaks are not resolved completely even in the 7-year data.

3.3.5 Uncertainties on the acoustic scale ($l_A$) and peak shift ($\phi_m$) parameters

Consider the following relationship [43, 89] between the location $l_m$ of the $m$th peak, the acoustic scale $l_A$, and the shift parameter $\phi_m$:

$$l_m = l_A(m - \phi_m).$$

Substituting the end-points of the 95% confidence interval for the $m$th peak location, this relationship results into a hyperbolic band of allowed values in the $l_A - \phi_m$ plane. Such bands, derived from 95% confidence intervals on the first three peaks (Table 3.2), are shown in Fig. 3.6. Additional information from other sources is required to constrain these bands to physically meaningful regions in the $l_A - \phi_m$ plane. For example, if we assume $l_A = 300$ [44] then, based on the 7-year data, the 95% confidence intervals for $\phi_m$ will be $\phi_1 : (0.1600, 0.3767), \phi_2 : (0.0367, 0.3600), \phi_3 : (-0.2167, 0.7300)$. Conversely, additional constraints on $\phi_m$ could be used to generate a confidence interval on $l_A$. From a model-independent point of view, we note that the $(l_A, \phi_m)$ bands for different peaks $m$ appear to overlap around $\phi_m \approx 0$ and $200 \lesssim l_A \lesssim 400$. We interpret this as a nonparametric revelation of the nearly harmonic structure of peaks in the CMB power spectrum.
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3.3.6 The low-\(l\) up-turn from a nonparametric viewpoint

Another interesting feature in Fig. 3.5 is the tiny but clearly observable scatter for the very first dip at the low-\(l\) end. This scatter corresponds to extreme power spectrum variations that reside on the surface of the 95% confidence set and have an up-turn at low \(l\) values. In the \(\Lambda\)CDM cosmology, such up-turn at the low-\(l\) end is primarily the result of the integrated Sachs-Wolfe (ISW) effect, and is seen in all parametric \(\Lambda\)CDM fits in Fig. 3.1 and 3.2. It would therefore be interesting to see what could be said about the low-\(l\) up-turn (and thereby about the ISW effect) based on the nonparametric confidence set.
Notice that our nonparametric fits, which are at the center of their respective confidence sets, do not show a low-$l$ up-turn. However, the 7-year parametric $\Lambda$CDM fit, e.g., does show a clear up-turn at the low-$l$ end. This parametric fit is at a distance corresponding to confidence level of about 10% ($\approx 0.12\sigma$; see Table 3.1) from our 7-year nonparametric full-freedom monotone fit. This means that the confidence set for the 7-year nonparametric fit contains spectra with a low-$l$ up-turn at most as far away as the 7-year parametric fit. We therefore conclude conservatively that the low-$l$ up-turn as a feature of the CMB angular power spectrum cannot be ruled out at any confidence level in excess of about 10%. Actually, there are indications in our results (not shown) that such up-turned variations of the spectrum may be much closer to the center of the confidence set for the 7-year full-freedom monotone fit; this needs further investigation.

### 3.3.7 Nonparametric indication of the harmonicity of peaks

In the early universe, before the last scattering the baryon-photon fluid underwent an acoustic oscillation. The imprint of this oscillation is supposed to be a series of harmonic peaks in angular power spectrum. We have illustrated these harmonic features in Fig. 3.7 for the WMAP 7-year data. In this figure, we depict a 2-dimensional color-coded histogram for the $(l_i, l_j)$ scatter, where $l_i$ is the position of the $i$th peak, for a sample of 68000 4-peaked spectra from the 95% (2$\sigma$) confidence set. We see that the arrangement of the six peaks in this histogram over a regular rectangular grid, as a nonparametric evidence for the basic physics of harmonicity of acoustic oscillations of the baryon-photon fluid. The peak-peak locations for the $\Lambda$CDM parametric fit [10] (crosses), and those for nonparametric fit (circles) are also plotted. We see the close agreement between nonparametric and parametric inferences. This model-independent representation of harmonicity is in approval with parametric achievements.
Figure 3.7: Nonparametric indication of the harmonicity of peaks in the CMB angular power spectrum as inferred from the WMAP 7-year data. This is a 2-dimensional color-coded histogram for the \((l_i, l_j)\) scatter, where \(l_i\) is the position of the \(i\)th peak, for a sample of 68000 4-peaked spectra from the 95% \((2\sigma)\) confidence set. Color indicates \(\log(\text{count})\) over an \(l\)-grid with unit spacing in both directions. We interpret the arrangement of the six peaks in this histogram over a regular rectangular grid as nonparametric evidence for the basic physics of harmonicity of acoustic oscillations of the baryon-photon fluid that gave rise to CMB anisotropies. Also indicated are the peak-peak locations for the \(\Lambda\)CDM parametric fit [10] (crosses), and those for our nonparametric fit (circles), connected by dotted and dashed lines respectively, as a guide to the eye.
3.4 Conclusion

In this work, we have presented a comparative nonparametric analysis of the WMAP 1-, 3-, 5-, and 7-year data releases for the CMB angular power spectrum, using a nonparametric function estimation methodology [1, 60]. In the context of this methodology, we have also presented our own numerical formulation for minimization of the inverse-noise-weighted risk function subject to monotonicity constraints, and a prescription for obtaining monotone nonparametric fits that are closer to cosmological expectations on smoothness. For all data realizations, we have presented results pertaining to the following questions: (a) how well is the angular power spectrum determined by the data alone, (b) how well is the ΛCDM model supported by a model-independent, nonparametric, data-driven analysis, and (c) what are the realistic uncertainties on peak/dip locations and heights.

The motivation for the analysis presented here was to explore what could be inferred about the CMB angular power spectrum in a model-independent, data-driven manner. On the other hand, the basic physics of the CMB is quite well established. It would therefore be useful to connect a nonparametric/model-independent analysis such as ours with the known physics of the CMB angular power spectrum. This is reserved for the future.

To conclude, we have demonstrated the threefold utility of the nonparametric methodology used here for cosmological function estimation problems: as a method with sound formal guarantees, as a sanity-enforcing mechanism on parametric model-based analyses, and as a method that allows interesting inferential questions to be addressed and answered in a data-driven manner.
### Table 3.2: 95% confidence intervals on several features of the angular power spectrum

<table>
<thead>
<tr>
<th>Data</th>
<th>Peak Location</th>
<th>Peak Height</th>
<th>Dip Location</th>
<th>Dip Height</th>
<th>Peak Location Ratio</th>
<th>Peak Height Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-year</td>
<td>$l_1 : (186,252)$</td>
<td>$h_1 : (4968,6133)$</td>
<td>$l_{1+\frac{1}{2}} : (378,507)$</td>
<td>$h_{1+\frac{1}{2}} : (1152,2029)$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$l_2 : (440,680)$</td>
<td>$h_2 : (1879,3933)$</td>
<td>$l_{2+\frac{1}{2}} : (494,772)$</td>
<td>$h_{2+\frac{1}{2}} : (-2929,2868)$</td>
<td>$l_2/l_1 : (1.953,3.217)$</td>
<td>$h_2/h_1 : (0.329,0.724)$</td>
</tr>
<tr>
<td></td>
<td>$l_3 : (559,897)$</td>
<td>$h_3 : (41,10733)$</td>
<td>$l_{3+\frac{1}{2}} : (639,900)$</td>
<td>$h_{3+\frac{1}{2}} : (-11376,9962)$</td>
<td>$l_3/l_1 : (2.495,4.385)$</td>
<td>$h_3/h_1 : (0.0075,1.906)$</td>
</tr>
<tr>
<td>3-year</td>
<td>$l_1 : (184,254)$</td>
<td>$h_1 : (5130,6199)$</td>
<td>$l_{1+\frac{1}{2}} : (384,456)$</td>
<td>$h_{1+\frac{1}{2}} : (1402,1917)$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$l_2 : (479,606)$</td>
<td>$h_2 : (2172,2997)$</td>
<td>$l_{2+\frac{1}{2}} : (593,856)$</td>
<td>$h_{2+\frac{1}{2}} : (80,2145)$</td>
<td>$l_2/l_1 : (2.028,3.000)$</td>
<td>$h_2/h_1 : (0.380,0.555)$</td>
</tr>
<tr>
<td></td>
<td>$l_3 : (646,977)$</td>
<td>$h_3 : (1506,6045)$</td>
<td>$l_{3+\frac{1}{2}} : (729,1000)$</td>
<td>$h_{3+\frac{1}{2}} : (-4501,5113)$</td>
<td>$l_3/l_1 : (2.761,4.801)$</td>
<td>$h_3/h_1 : (0.264,1.082)$</td>
</tr>
<tr>
<td>5-year</td>
<td>$l_1 : (187,251)$</td>
<td>$h_1 : (5249,6301)$</td>
<td>$l_{1+\frac{1}{2}} : (387,445)$</td>
<td>$h_{1+\frac{1}{2}} : (1489,1934)$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$l_2 : (485,596)$</td>
<td>$h_2 : (2306,2955)$</td>
<td>$l_{2+\frac{1}{2}} : (608,793)$</td>
<td>$h_{2+\frac{1}{2}} : (971,2095)$</td>
<td>$l_2/l_1 : (2.040,2.963)$</td>
<td>$h_2/h_1 : (0.381,0.525)$</td>
</tr>
<tr>
<td></td>
<td>$l_3 : (663,982)$</td>
<td>$h_3 : (1863,4635)$</td>
<td>$l_{3+\frac{1}{2}} : (731,1000)$</td>
<td>$h_{3+\frac{1}{2}} : (-2978,3157)$</td>
<td>$l_3/l_1 : (2.883,4.672)$</td>
<td>$h_3/h_1 : (0.321,0.833)$</td>
</tr>
<tr>
<td>7-year</td>
<td>$l_1 : (187,252)$</td>
<td>$h_1 : (5177,6377)$</td>
<td>$l_{1+\frac{1}{2}} : (390,442)$</td>
<td>$h_{1+\frac{1}{2}} : (1512,1931)$</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$l_2 : (492,589)$</td>
<td>$h_2 : (2328,3015)$</td>
<td>$l_{2+\frac{1}{2}} : (623,803)$</td>
<td>$h_{2+\frac{1}{2}} : (1074,2063)$</td>
<td>$l_2/l_1 : (2.060,2.887)$</td>
<td>$h_2/h_1 : (0.386,0.534)$</td>
</tr>
<tr>
<td></td>
<td>$l_3 : (681,965)$</td>
<td>$h_3 : (1871,4119)$</td>
<td>$l_{3+\frac{1}{2}} : (733,1104)$</td>
<td>$h_{3+\frac{1}{2}} : (-3111,2709)$</td>
<td>$l_3/l_1 : (3.000,4.553)$</td>
<td>$h_3/h_1 : (0.323,0.732)$</td>
</tr>
<tr>
<td></td>
<td>$l_4 : (815,1193)$</td>
<td>$h_4 : (-1391,7726)$</td>
<td>$l_{4+\frac{1}{2}} : (951,1200)$</td>
<td>$h_{4+\frac{1}{2}} : (-11102,6364)$</td>
<td>$l_4/l_1 : (3.606,6.047)$</td>
<td>$h_4/h_1 : (-0.244,1.340)$</td>
</tr>
</tbody>
</table>

Note: Negative values for heights and height ratios is a reflection of the high noise at the high-$l$ end. Here, $l_m$ ($h_m$) stands for the location (height) of the $m$th peak, and $l_{m+\frac{1}{2}}$ ($h_{m+\frac{1}{2}}$) denotes the location (depth) of the $m$th dip.