6.1 INTRODUCTION

A Cosmological model is a mathematical description of the universe, which tries to explain the reasons of its current aspect, and to describe its evolution during time. As mentioned in chapter-1, cosmological analysis regarding mystery of universe must account for the observations, and be able to make predictions that later observations may be able to check. Einstein has written the equations, which govern a universe filled with matter, but he thought that the universe had to be static. So, he introduced a term, called the Cosmological Constant, into his equations, in order to obtain this result. Afterward, in view of the results of Hubble [17], he returned from this idea and admitted that universe can effectively be expanding. Soon after him, the Dutch De Sitter, the Russian Friedmann and the Belgian Lemaître introduce non-static universes as solutions for the Einstein equations of relativity. Based upon a method proposed by Rahman [136] & Berman [132], Einstein field equation is expressed as:

\[ R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij} \]  

(6.1)

The Friedmann-Lemaître-Robertson-Walker metric (FLRW) [152] which rules the evolution of the universe in this model, is expressed in the form

\[ ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]  

(6.2)

Where \((r, \theta, \phi)\) are the polar coordinates, \(R(t)\) the scale factor (positive), and \(k\) is +1, 0 or -1 depending upon the geometry of the universe. This describes the space-time geometry of the universe coordinate system \((r, \theta, \phi)\). The time ‘\(t\)’ is the proper time for each particle, \(k\) is the curvature index such that \(k = 1\) for a close model, \(k = 0\) for a flat model and \(k = -1\) for an open model, \(R(t)\) is the scale factor.
6.2 COSMOLOGICAL MODELS WITH $8\pi G \rho = C_1 R^{-2n}$ & $\Lambda c^2 = -C_2 R^{-2n}$

In section 4.7 of Chapter-4, we have seen a Cosmological Model constructed under the assumptions $8\pi G \rho = C_1 t^{-2n}$ & $\Lambda c^2 = -C_2 t^{-2n}$, (6.3) and we have presented all the possibilities regarding suitability and validation of Cosmological Model under the above assumption. Here in this section we shall construct a Cosmological Model under the assumption

$$8\pi G \rho = C_1 R^{-2n} \quad & \quad \Lambda c^2 = -C_2 R^{-2n}$$

(6.4)

The fundamental equations governing the homogeneous & isometric Cosmological Model are as follows:

The acceleration equation is

$$3 \frac{\dot{R}}{R} = -4\pi G \left[ \frac{3p}{c^2} + \rho - \frac{\Lambda c^2}{4\pi G} \right]$$

(6.5)

and the velocity equation is

$$\frac{3\dot{R}^2}{R^2} + \frac{3k c^2}{R^2} = 8\pi G \rho + \Lambda c^2$$

(6.6)

The pressure-density relation is

$$p = \beta \rho c^2$$

(6.7)

The variation of density $\rho$ is derived from the conservation equation

$$\dot{\rho} = -3 (1 + \beta ) \rho \frac{\dot{R}}{R}$$

(6.8)

which on integration under the assumptions $R = R_0$, $\rho = \rho_0$

gives

$$\rho = \rho_0 \left( \frac{R}{R_0} \right)^{-3(1+\beta)}$$

(6.9)

The variation of $G$ & $\Lambda$ is governed by the equation

$$8\pi G \dot{\rho} + \dot{\Lambda} c^2 = 0$$

(6.10)

Also for a realistic cosmological model of the universe we must have
Numerous cosmological consequences with gauge ……

\[
G > 0 \text{ & } \dot{G} < 0 \\
\Lambda < 0 \text{ & } \dot{\Lambda} > 0
\] (6.11)
during the expansion of the cosmological model. As a consequence of this result the equation (6.5) implies that

\[
\ddot{R} < 0
\]

It means the Cosmological Model is either ever expanding or oscillatory with the above assumptions given in equation (6.4) where \( C_1 > 0, C_2 > 0 \) & \( n > 0 \)

After differentiation of equations in (6.4) we have

\[
8\pi\dot{G}\rho + 8\piG\dot{\rho} = -2nC_1R^{-2n-1}\ddot{R}
\] (6.12)

and

\[
\dot{\Lambda}c^2 = 2nC_2R^{-2n-1}\ddot{R}
\] (6.13)

Adding equations (6.12) & (6.13), we have by using equation (6.10)

\[
8\piG\dot{\rho} = 2n(C_2 - C_1)R^{-2n-1}\ddot{R}
\] (6.14)

Using first equation in (6.4) & equation (6.14), we get

\[
\frac{\dot{\rho}}{\rho} = \frac{2n(C_2 - C_1)}{C_1}\frac{\ddot{R}}{R}
\] (6.15)

Comparing equation (6.15) with equation (6.8), we get

\[
n = \frac{3(1 + \beta)C_1}{2(C_1 - C_2)}
\] (6.16)

Dividing equation (6.12) by first equation in (6.4), we get

\[
\frac{\dot{G}}{G} + \frac{\dot{\rho}}{\rho} = -2n\frac{\ddot{R}}{R}
\] (6.17)

By using equation (6.15) in (6.17), we get

\[
\frac{\dot{G}}{G} = -2n\frac{C_2}{C_1}\frac{\ddot{R}}{R}
\] (6.18)

Dividing equation (6.13) by 2\textsuperscript{nd} equation in (6.4), we have

\[
\frac{\dot{\Lambda}}{\Lambda} = -2n\frac{\ddot{R}}{R}
\] (6.19)

Substituting the value of \( n \) from (6.16) in (6.18) & (6.19), we get
Numerous cosmological consequences with gauge ……

\[
\frac{\dot{G}}{G} = -\frac{3(1 + \beta) C_1 \dot{R}}{C_1 - C_2} \quad \text{(6.20)}
\]

\[
\frac{\dot{\Lambda}}{\Lambda} = -\frac{3(1 + \beta) C_1 \dot{R}}{C_1 - C_2} \quad \text{(6.21)}
\]

Under the realistic consideration (6.11), we get

\[C_1 - C_2 > 0 \quad \text{(6.22)}\]

Integrating the equations (6.20) & (6.21) under the initial conditions

\[G = G_0, \quad \Lambda = \Lambda_0 \quad \text{&} \quad R = R_0 \text{ at } t = t_0 \quad \text{(6.23)}\]

we get

\[
\frac{G}{G_0} = \left( \frac{R}{R_0} \right)^{-\frac{3(1 + \beta) C_2}{C_1 - C_2}} \quad \text{(6.24)}
\]

\[
\frac{\Lambda}{\Lambda_0} = \left( \frac{R}{R_0} \right)^{-\frac{3(1 + \beta) C_1}{C_1 - C_2}} \quad \text{(6.25)}
\]

with the help of equation (6.16) we have

\[3\beta C_1 = (2n - 3)C_1 - 2nC_2 \quad \text{(6.26)}\]

For \( \beta \geq 0, C_1 > 0 \quad \text{&} \quad C_2 > 0 \), we may conclude that our assumption predict that

\[n > \frac{3}{2} \quad \text{(6.27)}\]

By using equations (6.4) in equation (6.6), we get

\[
R^{n-1} \dot{R} = \sqrt{\left( \frac{C_1 - C_2}{3} \right) - kc^2 R^{2(n-1)}} \quad \text{(6.28)}
\]

For any suitable value of \( n > \frac{3}{2} \), equation (6.28) may be integrated and provide us the variation of \( R \) on \( t \). Now we will discuss the above equation (6.28) under following cases:
6.3 FLAT COSMOLOGICAL MODEL

In this case we have  \( k = 0 \),

\[
R^{n-1} \ddot{R} = \sqrt{\frac{C_1 - C_2}{3}}
\]

\[
R^{n-1} dR = \left( \sqrt{\frac{C_1 - C_2}{3}} \right) dt
\]  (6.29)

On integration, we get

\[
\frac{R^n}{n} = \left( \sqrt{\frac{C_1 - C_2}{3}} \right) t + A
\]  (6.30)

where A is the constant of integration

Using initial condition \( R(0) = 0 \), which is the singular state (the Big-bang) of the cosmological model, we have

\[
R^n = n \left( \sqrt{\frac{C_1 - C_2}{3}} \right) t, \quad n > 3/2
\]  (6.31)

If we take \( n = 2 \),

\[
R^2 = 2 \left( \sqrt{\frac{C_1 - C_2}{3}} \right) t
\]

\[
R = \sqrt{2} \left( \frac{C_1 - C_2}{3} \right)^{1/4} t^{1/2},
\]

As \( C_1 > C_2 \), let \( C_1 = C_2 + b \), then \( b > 0 \), and

\[
R = \sqrt{2} \left( \frac{b}{3} \right)^{1/4} t^{1/2}
\]  (6.32)

\[
\dot{R} = \left( \frac{1}{\sqrt{2}} \right) \left( \frac{b}{3} \right)^{1/4} t^{-1/2}
\]  (6.33)

Therefore

\[
H = \frac{\dot{R}}{R} = \frac{1}{2t}
\]  (6.34)
Numerous cosmological consequences with gauge ……

\[ \dddot{R} = -\frac{1}{2\sqrt{2}} \left( \frac{b}{3} \right)^{1/4} t^{-3/2} \]  

(6.35)

The deceleration parameter is \( q \) is defined as

\[ q = -\frac{\dddot{R}}{R \dot{H}} = 1 \]  

(6.36)

since \( q > 0 \), there is deceleration of scaling factor.

Similary for \( n > 2 \),

\[ R \propto t^{1/n}, \quad n = 3, 4, 5, \ldots \ldots \]  

(6.37)

As we have already discussed in the previous chapters that when \( 0 < \beta < \frac{1}{3} \), the universe is at the stage when the matter energy and radiation energy are comparable to each other. In similar way the above equations are valid at the stage when the matter energy and radiation energy are comparable.

6.4 OPEN COSMOLOGICAL MODEL

In this case we have \( k = -1 \), By equation (6.28)

\[ \frac{R^{n-1}}{\sqrt{\left( \frac{C_1 - C_2}{3} + c^2 R^{2(n-1)} \right)}} dR = dt, \]  

(6.38)

\[ \frac{R^{n-1}}{\sqrt{\left( \frac{b}{3} + c^2 R^{2(n-1)} \right)}} dR = dt, \]  

(6.39)

where \( C_1 > C_2 \), taking \( C_1 = C_2 + b \), \( b > 0 \)

If \( \frac{b}{3c^2} = b_1 \), we have

\[ \frac{R^{n-1}}{\sqrt{b_1 + R^{2(n-1)}}} dR = cd t \]  

(6.40)
For $n = 2$, we get
\[ \frac{R}{\sqrt{b_1 + R^2}} \, dR = c \, dt \] (6.41)

On integration, we get
\[ \sqrt{b_1 + R^2} = ct + A \] (6.42)

where $A$ is the constant of integration.

Now using $R(0) = 0$, we have
\[ \sqrt{b_1 + R^2} - \sqrt{b_1} = ct \] (6.43)

Substituting the value of $b_1$ in (6.43), we get the variation of $R$ on $t$ as:
\[ \sqrt{\frac{C_1 - C_2}{3c^2} + R^2} - \sqrt{\frac{C_1 - C_2}{3c^2}} = ct \] (6.44)

For $C_1 = C_2$, we have
\[ R \propto t^{1/2} \] (6.45)

For $0 < \beta < \frac{1}{3}$, the above equations hold when the matter energy and the radiation energy are comparable in the cosmological model. The evolution of the cosmological model in the radiation dominated era and in the matter dominated era are obtained from the above equations by considering $\beta = \frac{1}{3}$ and $\beta = 0$ respectively.

For $n = 2$, we have
\[ 3\beta C_1 = C_1 - 4C_2 \] (6.46)

For $\beta = 0$, $C_1 = 4C_2$; model is physible for matter dominated era.

For $\beta = \frac{1}{3}$, $C_2 = 0 \Rightarrow \Lambda = 0$ by equ (6.4); model may not be physible for radiation dominated era.
### 6.5 CLOSED COSMOLOGICAL MODEL

Here, \( k = 1 \). By equation (6.28) we get

\[
\frac{R^{n-1}}{\sqrt[3]{C_1 - C_2 - c^2 R^{2(n-1)}}} \, dR = dt \tag{6.47}
\]

where \( C_1 > C_2 \), let \( C_1 = C_2 + b \), \( b > 0 \)

we get

\[
\frac{R^{n-1}}{\sqrt[3]{\left(\frac{b}{3} - c^2 R^{2(n-1)}\right)}} \, dR = dt \tag{6.48}
\]

\[
\frac{R^{n-1}}{\sqrt[3]{(b_1 - R^{2(n-1)})}} \, dR = cdt \tag{6.49}
\]

where \( \frac{b}{3c^2} = b_1 \)

For \( n = 2 \), we have

\[
\frac{R}{\sqrt[3]{b_1 - R^2}} \, dR = cdt \tag{6.50}
\]

On integration, we get

\[-\sqrt[3]{b_1 - R^2} = ct + A \tag{6.51}\]

Now using \( R(0) = 0 \), we have

\[\sqrt[3]{b_1} - \sqrt[3]{b_1 - R^2} = ct \tag{6.52}\]

Substituting the value of \( b_1 \) in (6.52), we get the variation of \( R \) on \( t \) as:

\[\sqrt[3]{\frac{C_1 - C_2}{3c^2}} - \sqrt[3]{\frac{C_1 - C_2}{3c^2} - R^2} = ct \tag{6.53}\]

So here we have discussed all the three possible cosmological models (flat, open & close) separately. Now we will discuss the Cosmological Model under this assumption for both radiation dominated universe and matter dominated universe.
6.6 RADIATION DOMINATED UNIVERSE

Here we have \( \beta = \frac{1}{3} \), then \( p = \frac{1}{3} \rho c^2 \)

\[
3p = \rho c^2
\]  
(6.54)

From equation (6.9), we have

\[
\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-4}
\]  
(6.55)

From equation (6.24), we have

\[
\frac{G}{G_0} = \left(\frac{R}{R_0}\right)^{-\frac{4c_1}{c_1-c_2}}
\]  
(6.56)

From equation (6.25), we have

\[
\frac{\Lambda}{\Lambda_0} = \left(\frac{R}{R_0}\right)^{-\frac{4c_1}{c_1-c_2}}
\]  
(6.57)

From equation (6.16), we have

\[
n = \frac{2C_1}{C_1-C_2}, \quad n > 3/2
\]  
(6.58)

6.7 MATTER DOMINATED UNIVERSE

Here we have \( \beta = 0; \rho = 0 \)

From equation (6.9), we have

\[
\frac{\rho}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3}
\]  
(6.60)

From equation (6.24), we have
Numerous cosmological consequences with gauge ……

\[ \frac{G}{G_0} = \left( \frac{R}{R_0} \right)^{\frac{3C_1}{c_1-c_2}} \]  \hspace{1cm} (6.61)

From equation (6.25), we have

\[ \frac{\Lambda}{\Lambda_0} = \left( \frac{R}{R_0} \right)^{\frac{3C_1}{c_1-c_2}} \]  \hspace{1cm} (6.62)

From equation (6.16), we have

\[ n = \frac{3C_1}{2(C_1 - C_2)} \]  \hspace{1cm} (6.63)

### 6.8 VALIDATION

From f-gravity theory we may assume:

\[ G = G_i = 6.7 \times 10^{31} \text{ c.g.s \ units} \]  \hspace{1cm} (6.64)

\[ R = R_i = 10^{13} \text{ cm} \]  \hspace{1cm} (6.65)

\[ \rho = \rho_i = 10^{17} \text{ g.cm}^{-3} \]  \hspace{1cm} (6.66)

Now from equation (6.4) and with use of present day observational data [154, 155].

\[ G = G_0 = 6.7 \times 10^{-8} \text{ c.g.s \ units} \]  \hspace{1cm} (6.67)

\[ R = R_0 = 10^{28} \text{ cm} \]  \hspace{1cm} (6.68)

\[ \rho = \rho_0 = 3 \times 10^{31} \text{ g.cm}^{-3} \]  \hspace{1cm} (6.69)

we have from equation (6.4)

\[ C_1 = 8\pi G_i \rho_i R_i^{2n} \]  \hspace{1cm} (6.70)

& \hspace{1cm} \[ C_1 = 8\pi G_0 \rho_0 R_0^{2n} \]  \hspace{1cm} (6.71)

From equations (6.70) & (6.71), we have

\[ \left( \frac{R_0}{R_i} \right)^{2n} = \frac{G_i \rho_i}{G_0 \rho_0} \]  \hspace{1cm} (6.72)

On simplification with above values,
Numerous cosmological consequences with gauge ……

we get the approximate value of  \( n = 2.9 \)  \( \text{(6.73)} \)

For this value of \( n \), equation (6.70) gives

\[ C_1 = 1.7 \times 10^{125} \text{ c.g.s. units (approximately)} \]  \( \text{(6.74)} \)

Substituting the value of \( n \) from equation (6.73) & value of \( C_1 \) in equation (6.58), we get for radiation dominated era

\[ C_2 = 5.3 \times 10^{124} \text{ c.g.s. units (approximately)} \]  \( \text{(6.75)} \)

Similarly on substituting the value of \( n \) from (6.73) & value of \( C_1 \) in equation (6.63), we get for matter dominated era

\[ C_2 = 8.2 \times 10^{124} \text{ c.g.s. units (approximately)} \]  \( \text{(6.76)} \)

Here we have two values of \( C_2 \) because of two stages of evolution. For the purpose of further estimation we may take the average of the two numerical values of \( C_2 \), i.e.

\[ C_2 = 6.8 \times 10^{124} \text{ c.g.s. units (approximately)} \]  \( \text{(6.77)} \)

From the above equation it is clear that we may not set a value for \( n \) where \( C_1 \) & \( C_2 \) are known. Also neither the present day observational data nor the estimated data obtained from f-gravity theory are sufficient to determine the constants \( C_1 \) & \( C_2 \). However, if we consider both data at a time, it would be possible to determine the constants \( n, C_1 \) & \( C_2 \).

From equation (6.4), we have

\[ \Lambda = -C_2 e^{-2} R^{-2n} \]  \( \text{(6.78)} \)

\[ \Rightarrow \Lambda_i = -C_2 e^{-2} R_i^{-2n} \]  \( \text{(6.79)} \)

Substituting the value of \( R_i, n \) & \( C_2 \) from equations (6.65), (6.73) & (6.77) in equation (6.79), we get

\[ \Lambda_i = -7.6 \times 10^{29} \text{ cm}^{-2} \text{ (approx.)} \]

From equation (7.78), we have

\[ \Lambda_0 = -C_2 e^{-2} R_0^{-2n} \]  \( \text{(6.80)} \)

Substituting the value of \( R_0, n \) & \( C_2 \) from equations (6.28), (6.73) & (6.77) in equation (6.80), we get
\[ \Lambda_0 = -7.6 \times 10^{-58} \text{ cm}^{-2} \quad \text{(approx.)} \]

6.9 CONCLUDING REMARKS

On the basis of discussion in this chapter, the following conclusions may be made:

- As per calculated values of \( C_1 \) & \( C_2 \) shown in equations (6.74 & 6.77), we may conclude that with these values we have:
  \[ R < 10^{27} \text{ cm} \quad \text{(approx.)} \]

- In this chapter, under the following assumptions:
  \[ 8\pi G \rho = C_1 R^{-2n} \quad \& \quad \Lambda c^2 = -C_2 R^{-2n} \]

The variation of \( G \) & \( \Lambda \) has been studied and it is found that the flat model may be constructed for any positive value of \( n \). The construction of open and closed model is possible for \( n = 2 \) only otherwise one may propose for approximate calculations which have been discussed in chapter 10.