CHAPTER-5
MATHEMATICAL PROPERTIES OF COSMOLOGICAL MODELS WITH VARIABLE POSITIVE COSMOLOGICAL CONSTANT

5.1 INTRODUCTION

As discussed in section (1.5), in Einstein’s equations for static universe, the cosmological constant behaves as a force under certain conditions; it grows with distance and is independent of mass. W. de Sitter formulated what he considered a non-physical empty universe which predicted a red shift, increasing for object at greater distances. In 1922, Friedmann presented an interpretation of gravitational field theory which showed that two non-static universes were possible and that the universe must be contracting or expanding, not static [96].

At the time of Einstein, the Cosmological Constant $\Lambda$ was having no physical meaning but recent Cosmological observations indicate that the expansion of the universe is accelerating and this has led to a great deal of theoretical activities. Non-static cosmological expansion raises a variety of interesting mathematical questions with kinds of non-empty universes. Here in this chapter we have investigated the mathematical properties which have been obtained and discussed with many other challenges, which remain look to solve by peer researchers. We have also presented the kind of non-empty universes as per our published paper [146], with according to contemporary classification of cosmological models as proposed by Berman [132], Rahmann [136] under suitable assumptions given by Mishra [7].

Non-static expansion plays a role in cosmology in two different regimes. The first in the very early universe while the second in the period between the decoupling of the microwave background radiation and the present epoch. Non-static expansion in the early universe is associated with the name inflation which was introduced by Guth [20]. One of the interesting aspects of the Inflation is that it claimed to solve certain queries in cosmology but this could not satisfy the queries related with:

- Homogeneous & Isotropic nature of universe
- Flatness Problem
Horizon Problem

Recently, a very strong observational evidence came into picture that the velocity of recession of distinct galaxies is accelerating. This is associated with Dark Energy introduced by Caldwell, Dave & Steinhardt [147]. After decoupling we have number of different evidences for cosmic acceleration as:

5.1.1 SUPERNOVAE OF TYPE Ia

A type Ia supernova is a sub-category of cataclysmic variable stars that results from the violent explosion of a white dwarf star. A white dwarf is the remnant of a star that has completed its normal life cycle and has ceased nuclear fusion. However, white dwarfs of the common carbon-oxygen variety are capable of further fusion reaction that release a great deal of energy if their temperatures rise high enough. Physically, white dwarfs with a low rate of rotation are limited to masses that are below the Chandrasekhar limit of about 1.38 solar masses. If a white dwarf gradually accretes mass from a binary companion, its core is believed to reach the ignition temperature for carbon fusion as it approaches the limit. Within a few seconds of initiation of nuclear fusion, a substantial fraction of matter in the white dwarf undergoes a runaway reaction, releasing enough energy \((1 - 2 \times 10^{44} \text{ joules})\) to unbind the star in a supernova explosion. This category of supernovae produces consistent peak luminosity because of the uniform mass of white dwarf that explode via the accretion mechanism. The stability of this value allows these explosions to be used as standard candles to measure the distance to their host galaxies because the visual magnitude of the supernovae depends primarily on the distance [148]. In 1998, published observations of Type Ia supernovae by the High-z Supernova search team [23] followed in 1999 by the Supernova Cosmology Project [149] and Tonry et al. [150] suggested that the expansion of the universe is accelerating.
5.1.2 MICROWAVE BACKGROUND FLUCTUATION

The cosmic microwave background is the afterglow radiation left over from the hot Big Bang. It is now generally agreed among both astronomers and physicists alike that the universe was created some 10 to 20 billion years ago in a leviathan explosion dubbed the "Big Bang". But till date the after effect of big-bang is to be verified by research. The exact nature of the initial event is still cause for much speculation, and it's fair to say that we know very little about the first instant of creation. Nevertheless we do know that the Universe used to be incredibly hotter and more dense than it is today. Expansion and cooling after this cataclysm of the Big Bang resulted in the production of all of the physical contents of the Universe which we see today. Namely: light in the form of "photons"; matter in the form of "leptons" (electrons, positrons, muons) and "baryons" (protons, antiprotons, neutrons, antineutrons); more esoteric particles like "neutrinos" and perhaps some exotic "dark matter" particles; and the subsequent formulation of the Universe's first chemical elements.

Perhaps the most conclusive (and certainly among the most carefully examined) piece of evidence for the Big Bang is the existence of an isotropic radiation bath that permeates the entire Universe known as the "cosmic microwave background" (CMB). The word "isotropic" means the same in all directions; the degree of anisotropy of the CMB is about one part in a thousand. In 1965, two young radio astronomers, Arno Penzias and Robert Wilson, almost accidentally discovered the CMB using a small, well-calibrated horn antenna. It was soon determined that the radiation was diffuse, emanated uniformly from all directions in the sky, and had a temperature of approximately 2.7 Kelvin (ie 2.7 degrees above absolute zero).

If the universe was once very hot and dense, the photons and baryons would have formed plasma, i.e. a gas of ionized matter coupled to the radiation through the constant scattering of photons off ions and electrons. As the universe expanded and cooled there came a point when the radiation (photons) decoupled from the matter - this happened about a few hundred thousand years after the Big Bang. That radiation cooled and is now at 2.7 Kelvin. The fact that the spectrum (of the radiation is almost
exactly that of a "black body" (a physicist's way of describing a perfect radiator) implies that it could not have had its origin through any prosaic means. This has led to the death of the steady state theory for example. In fact the CMB spectrum is a black body to better than 1% accuracy over more than a factor of 1000 in wavelength. This is a much more accurate black body than any we can make in the laboratory!

5.1.3 GALAXY CLUSTERING

Galaxies are preferentially found in groups or larger agglomerations called clusters. The Local Group consists of our own galaxy, the larger spiral galaxy Andromeda (M31) and several smaller satellites, including the Large and Small Magellanic Clouds. Regular clusters have a concentrated central core and a well-defined spherical structure. These are subdivided according to their richness, that is, the number of galaxies within 1.5 Mpc of the center (known as the Abell radius). Typically, they have a size in the range 1-10 Mpc and a mass $M \sim 10^{15}$ solar masses (one followed by 15 zeros, that is, a million billion suns).

5.2 KINDS OF NON-EMPTY UNIVERSES

On the basis of our study and discussion of cosmological models under various suitable assumptions in foregoing chapters, we wish to discuss the kinds of non-empty universes.

5.2.1 EINSTEIN'S MODEL OR STATIC MODELS.

Einstein field equations are

$$\frac{\dot{R}}{R} = -\frac{Kc^2}{2}\left(\frac{\rho}{3} + \frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

(5.1)
\[
\frac{3\dot{R}^2}{R^2} + \frac{3kc^2}{R^2} = K\rho c^2 + \Lambda c^2
\]  
(5.2)

where \( K = \frac{8\pi G}{c^2} \)

In this case \( R \) remains unchanged and the above equation (5.1) gives

\[
\Lambda = \frac{K}{2} \left( \rho + 3 \frac{p}{c^2} \right)
\]  
(5.3)

\[
\frac{3k}{R^2} = K\rho + \Lambda
\]  
(5.4)

Consequence of which are that \( \rho \) & \( p \) remain unchanged as well and that \( \Lambda > 0, k = +1 \). This model is unstable because change in one or two of the quantities \( R, \rho, \) and \( p, \) the models compresses themselves up to the special state of the infinite density. Eddington [151] was the first, who pointed out the instability of the Einstein model.

### 5.2.2 MODELS OF THE FIRST KIND

Here we can discuss asymptotic, monotone and oscillating models.

#### 5.2.2.1 Asymptotic models

Asymptotic models of the first kind change their volumes monotonically during their expansion or contraction between the special state of infinite density, when \( t = t_0 \) & a static state, when \( t \to +\infty \) or \( -\infty \).

This is possible under \( \Lambda > 0 \) & \( k = +1 \).

#### 5.2.2.2 Monotone models

Monotone models of first kind change their volume monotonically during their expansion between the special state of the infinite density when \( t = t_0 \) & infinite cleanliness when \( t \to +\infty \) or \( -\infty \).

This is under \( \Lambda > 0 \), \( k = 0 \) & \( k = -1 \).
5.2.2.3 Oscillating models

Oscillating models of the first kind expands their volume from the special state, when \( t = t_1 \) continue expansion up to \( t = t_0 \).

This is possible with \( \Lambda \geq 0 \) under \( k = +1 \) or with \( \Lambda < 0 \) under any numerical value of \( k \).

5.2.3 MODELS OF SECOND KIND

Again it is required to discuss the Asymptotic & Monotone models of second kind.

5.2.3.1 Asymptotic model of second kind

Asymptotic models of the second kind change their volume during their monotone expansion or compression between a static state, corresponding to the Einstein model, when \( t \to +\infty \) or \( \infty \) under \( \Lambda > 0 \) & \( k = +1 \).

5.2.3.2 Monotone Models of second kind

Monotone models of second kind compress their volumes from the state of infinite of infinite cleanliness when \( t = -\infty \), up to a state of their maximal volume when \( t = t_0 \) under \( \Lambda > 0 \) & \( k = +1 \). As discussed in the above section, we may conclude that in case of \( p > 0 \) or \( p = 0 \), the possible kinds of the models may be Shown in the following table:
Table-5.1: Different kinds of Cosmological Models

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$k = -1$</th>
<th>$k = 0$</th>
<th>$k = +1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda &gt; 0$</td>
<td>Monotone models of first kind</td>
<td>Monotone models of first kind</td>
<td>i. Asymp. Models of I &amp; II kind ii. Monotone modes of I &amp; II kind iii. Oscillating models of I kind</td>
</tr>
<tr>
<td>$\Lambda = 0$</td>
<td>Oscillating models of first kind</td>
<td>Monotone models of first kind</td>
<td>Monotone models of first kind</td>
</tr>
<tr>
<td>$\Lambda &lt; 0$</td>
<td>Oscillating models of first kind</td>
<td>Oscillating models of first kind</td>
<td>Oscillating models of first kind</td>
</tr>
</tbody>
</table>

5.3 RADIATION DOMINATED UNIVERSE

Considering again the equation (4.37) for $\beta = \frac{1}{3}$, we have

$$\rho \propto R^{-4}$$

and for small $t$, we have from Friedmann equations

$$R \propto t^{1/2}$$

So that from (5.5) & (5.6) we can write

$$\rho \propto t^{-2}$$

In this case the curvature term is negligible for smaller value of $t$ in respect of Friedmann’s equation

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi}{3} G \rho$$

So $k = 0, \pm 1$ do not change the physical results. As we know that the radiation in this epoch is modeled by a black body. The density of the black body radiation is related to its temperature by

$$\rho_r = \alpha r T_r^4$$
where \( \alpha_1 = 8.42 \times 10^{-36} \text{ g cm}^{-3} \text{ K}^{-4} \)

As per equation (5.5), the temperature of radiation is inversely proportional to the scale factor of the universe i.e. \( T_r \propto R^{-1} \), where \( R \) is very small, in case of early universe. So \( T_r \) may be very high with the help of above discussed equations we can write:

\[
\rho_0 R_0^3 = \rho_m R^3 \tag{5.10}
\]

By equations (5.10) & (5.5), we will get

\[
\frac{\rho_r}{\rho_m} = \left( \frac{\rho_r}{\rho_m} \right)_0 \left( \frac{R_0}{R} \right) \tag{5.11}
\]

This formula shows that the ratio of the radiation and matter densities is not invariant, rather, it decreases as the universe expands.

Although \( \left( \frac{\rho_r}{\rho_m} \right)_0 \) is very small, being only about \( 10^{-3} \) in the early universe i.e. when

\[
\left( \frac{R_0}{R} \right) \geq 1, \quad \text{we had} \quad \frac{\rho_r}{\rho_m} \geq 1. \tag{5.12}
\]

It means radiation was dominant component of universe and its temperature was

\[
T_r = (T_r)_0 \left( \frac{R_0}{R} \right) = (T_r)_0 (1 + z) \tag{5.13}
\]

For \( (T_r)_0 = 2.7 K \) and \( z \sim 10^3 \), this implies

\[
T_r > 2.7 \times 10^3 \approx 3000 K \tag{5.14}
\]

This phase of universe is called the radiation dominated universe.

### 5.4 Metric and Properties of Matter

The space is a constant curvature space having homologous expansions and compressions with
Chapter 5: Mathematical Properties of Cosmological Models & Relation of $\Lambda$ with Dark Energy

\[ C = \frac{3k}{R^2}; \quad k = 0, \pm 1 \quad \& \quad R = R(t) \]  \hspace{1cm} (5.15)

Euclidean spatial coordinates has the metric

\[ ds^2 = c^2 dt^2 - R^2 \left( \frac{dx^2 + dy^2 + dz^2}{1 + \frac{k}{4} \left( x^2 + y^2 + z^2 \right)^2} \right) \]  \hspace{1cm} (5.16)

For the above equation (5.16), the metric is not Euclidean, the detail is well known from geometry that a space of $k = 1$ is locally spherical, a space of $k = 0$ is locally Euclidean and a space for $k = -1$ is locally hyperbolic. So taking spherical symmetry w. r. t. any point, here we may conclude that the space for $k = 1$ is elliptic (actually, doubly-connected) or spherical (actually, simply connected) and spaces for $k = 0$ and $k = -1$ are Euclidean and hyperbolic respectively (both spaces are infinite and simply connected).

Gauss invented a method for the mathematical treatment of continua in general, in which ‘size relations’ are defined. To every point of the continuum are assigned as many numbers as the continuum has dimensions. This is done in such a way, that only one meaning can be attached to the assignment, and the numbers which differ by an indefinitely small amount are assigned to adjacent points. The Gaussian coordinate system is a logical generalization of the Cartesian coordinate system. It is also applicable to non-Euclidean continua, but only when, with respect to the defined ‘size or ‘distance’.

Let us consider $g_{ij}$ and putting it into the Einstein equations of gravitations we have:

\[ G^{ij} - \frac{1}{2} g^{ij} G = -KT^{ij} - \Lambda g^{ij} \]  \hspace{1cm} (5.17)

We obtain that

\[ \rho = \rho(t) \quad \& \quad p = p(t) \]  \hspace{1cm} (5.18)

exists, where $T^{ij}$ can be expressed by the formulas

\[ T^{00} = \frac{1}{g_{00}} \left( \rho + \frac{p}{c^2} \right) - \frac{p}{c^2} g^{00} \]  \hspace{1cm} (5.19)
\[ T^{0j} = -\frac{\rho}{c^2} g^{0j} \]  \hspace{1cm} (5.20)

\[ T^{ik} = -\frac{p}{c^2} g^{ik} \]  \hspace{1cm} (5.21)

In other words

\[ T^{ij} = \left( \rho + \frac{p}{c^2} \right) \frac{dx^i}{ds} \frac{dx^j}{ds} - \frac{p}{c^2} g^{ij} \]  \hspace{1cm} (5.22)

where \( \frac{dx^i}{ds} \) is a four-dimensional velocity, which characterizes the mean motion of matter in the neighborhood of every point. The pioneer work was done in relativistic cosmology by Einstein and Friedmann. They did not take the pressure into account, Friedmann was the first who considered the static models, so the case of \( \rho > 0, \ p = 0 \) is known as Friedmann’s case of an inhomogeneous universe. The first, who introduce \( p > 0 \) was Lemaitre.

In an open universe, the sum of the angles of a triangle is lower than 180°. In a closed universe (like the surface of Earth), this sum is always greater than 180° as shown in Figure 5.1, [145]:

**Figure-5.1: Universe seems like**
5.5 CONCLUDING REMARKS

This chapter provides the mathematical analysis of cosmological models with non-static (accelerated) expansion. The exposition with a review of results in the case of positive cosmological constant has been discussed in brief and it is highlighted that the cosmological constant $\Lambda$ may be replaced by a non-linear scalar field. It has been also focused that there are close relations between models with scalar field & models with perfect fluids whose equation of state is more or less exotic. As discussed in section (5.3), the radiation was dominant in the universe and its temperature was more than $2.7 \times 10^3 K$. We can assume this as 3000K. On the basis of the result published by other cosmologists we may say that an efficient check with suitable assumptions is required in suitable framework, which might be the effective behavior common to a large class of high energy theories. By considering the equations presented in section (5.4) we may conclude that

- In an open universe, the sum of the angles of a triangle is lower than $180^\circ$.
- In a closed universe (like the surface of the Earth), this sum is always greater than $180^\circ$ as shown in figure (5.1).