CHAPTER 4
COSMOLOGICAL MODELS WITH VARIABLE G AND Λ

4.1 INTRODUCTION

A cosmological model is a mathematical description of the universe which tries to explain the reason of its current aspects and also describes its evolution during time. It must account for the observations and be able to make predictions that later observations will be able to check. The cosmological models discussed in this thesis are based on Einstein General theory of Relativity because this theory provides us the best agreement for large scale behavior.

By taking the following field equations as discussed in Chapter-1:

\[ R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij} \]  
\( (i, j = 1, 2, 3, 4) \)

but with a slight modification as variable gravitational constant ‘G’ & variable cosmological constant ‘Λ’. Here the conservation law is same as in General Relativity as discussed in section (1.9), we have

\[ T^{ij}_{\,\,;i} = 0 \]  
\( (i, j = 1, 2, 3, 4) \)

To avoid the creation or vanishing of the energy in the universe.

The equation (4.2) when applied to the field equation will give

\[ \frac{8\pi G}{c^4} T^{ij} + \Lambda_{;i} g^{ij} = 0 \]  
\( (i, j = 1, 2, 3, 4) \)

which governs the variation of G & Λ. The field equations (4.1), (4.2) & (4.3) are not derived from the Lagrangian formulation. If one attempts to derive the field equations via Lagrangian formulation, one has to add the term of G and Λ in the Lagrangian density and the resulting field equation contains undetermined constants.
An important conclusion of the above equation (4.3) is that the cosmological constant ‘Λ’ depends upon ‘G’ as well as energy momentum contained in the universe. Contracting the equation (4.3) we will obtain

$$Λ_{,i} = -\frac{8\pi G}{c^4} T_{i}^{,i}$$

(4.4)

(i, l = 1, 2, 3)

Putting the value of $T_{i}^{,i}$ from (4.2), we get

$$Λ_{,i} = -\frac{8\pi G}{c^4} G_{,i} R_{i}^{,i} + \frac{4\pi}{c^4} G_{,i} R - \frac{8\pi}{c^4} G_{,i} \Lambda$$

(4.5)

(i, j, l = 1, 2, 3, 4)

This equation shows that how $G$, $Λ$, curvature tensor $R_{i}^{,i}$ and curvature scalar $R$ are related to each other.

Taking the trace of equation (4.1) (i.e. contracting the equation (4.1) with $g^{ij}$) and using $T = g^{ij} T_{ij}$, $R = g^{ij} R_{ij}$ & $g^{ij} g_{ij} = 4$, we get

$$R = \frac{8\pi G}{c^4} T + 4 \Lambda$$

(4.5a)

(here $T$ may be called energy-momentum scalar)

Putting the value of $R$ from (4.5a) in (4.1) we get

$$R_{ij} = -\frac{8\pi G}{c^4} \left[ T_{ij} - \frac{1}{2} T g_{ij} \right] + Λ g_{ij}$$

(4.6)

(i, j = 1, 2, 3, 4)

This is another form of Einstein field equation (4.1).

### 4.2 EQUATIONS GOVERNING COSMOLOGICAL MODEL

For a homogeneous and isotropic universe the energy momentum tensor by perfect fluid distribution as:

$$T^{ij} = \left( \frac{P}{c^2} + ρ \right) u^i u^j - \frac{P}{c^2} g^{ij}$$

(4.7)

(i, j = 1, 2, 3, 4)
where \( p \) is the proper pressure, \( \rho \) is the proper density of the fluid particle, \( u_i \) and \( u^j \) are defined as:

\[
\begin{align*}
  u_i &= g_{ij} u^j \\
  u^j &= \frac{dx^j}{ds}
\end{align*}
\]

\( (i, j = 1, 2, 3, 4) \) \( (4.8) \)

here \( u^j \) is four-velocity vector of the fluid particle and since the fluid particles are at rest w. r. t. the commoving coordinate system \((r, \theta, \phi)\), therefore we have

\[
  u^i = (0, 0, 0, c)
\]

\( (i, j = 1, 2, 3, 4) \) \( (4.9) \)

Putting the value from (1.6), (4.7) & (4.9) in (1.25), we get after some simplifications

\[
- \frac{2 \dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k c^2}{R^2} = 8\pi G \rho - \Lambda c^2
\]

\( (4.10) \)

and

\[
\frac{3 \dot{R}^2}{R^2} + \frac{3k c^2}{R^2} = 8\pi G \rho + \Lambda c^2
\]

\( (4.11) \)

where dot on a symbol denote the derivative w. r. t. ‘t’. Adding equations (4.10) & (4.11), we get

\[
- \frac{2 \dot{R}}{R} + \frac{2 \dot{R}^2}{R^2} + \frac{2k c^2}{R^2} = 8\pi G \left( \frac{p}{c^2} + \rho \right)
\]

\( (4.12) \)

Differentiating the equation (4.11) with respect to ‘t’, we get

\[
8\pi (\dot{G} \rho + G \dot{\rho}) = -\frac{3 \ddot{R}}{R} \left( - \frac{2 \dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{2k c^2}{R^2} \right) - \dot{\Lambda} c^2
\]

\( (4.13) \)

which is simplified with the help of equation (4.12) as

\[
(\dot{G} \rho + G \dot{\rho}) + \frac{3 \ddot{R}}{R} G \left( \frac{p}{c^2} + \rho \right) + \frac{\dot{\Lambda} c^2}{8\pi} = 0
\]

\( (4.14) \)

Multiplying equation (4.14) by \( R^3 \), we get

\[
c^2 R^3 \left( \dot{G} \rho + \frac{\dot{\Lambda} c^2}{8\pi} \right) + G \left[ c^2 \frac{d}{dt} (\rho R^3) + 3 \rho R^2 \dot{R} \right] = 0
\]

\( (4.15) \)

The conservation equation (4.2) when simplified with the help of the above equations we get the conservation law as
\[ c^2 \frac{d}{dt}(\rho R^3) + 3 pR^2 \dot{R} = 0 \quad (4.16) \]

and
\[ 8\pi G \rho + \Lambda c^2 = 0 \quad (4.17) \]

using the above equations we get
\[ 3\ddot{R} = -4\pi GR \left( \frac{3p}{c^2} + \rho - \frac{\Lambda c^2}{4\pi G} \right) \quad (4.18) \]

The equation (4.18) is known as acceleration equation. The equation (4.11) is known as velocity equation and the equation (4.17) is known as conservation equation.

### 4.3 Age of the Universe with Pressure–Density Relation

For all the calculation purpose in this thesis, we have considered the dependence of pressure \( p \) on density \( \rho \) in a homogeneous & isotropic cosmological model as:
\[ p = \beta \rho c^2 \quad (4.19) \]

where \( \beta \) is a constant and here two types of cosmological models may be discussed under two possible values of \( \beta \).

(i) For \( \beta = 0 \), by equation (4.19) we get \( p = 0 \) and then the obtained model is known as pressure less cosmological model. Such cosmological models are used to study the universe at present epoch.

(ii) For \( \beta = \frac{1}{3} \), by equation (4.19) we get \( p = \frac{1}{3} \rho c^2 \), and then the cosmological model is filled with radiation. Such cosmological models are used to study the universe in the past when it was radiation dominated. In this way, it is interesting to notice that in a cosmological model where we are considering the variation of \( G \) & \( \Lambda \) as the universe expands, it would be significant to know how the pressure and density vary during the expansion of the universe starting from the Big–Bang. Therefore, here we consider neither \( \beta = \frac{1}{3} \) nor \( \beta = 0 \) as considered in
the usual cosmological models, but the value of $\beta$ from other consideration as suggested by Mishra & Pande [7]. However keeping into consideration of the matter dominated era and the radiation dominated era we may accept value of $\beta$ lies between $0$ & $1/3$.

i.e. $0 < \beta < \frac{1}{3}$

Putting the value of $p$ from equation (4.19) in the conservation equation (4.16), we get for $c = 1$:

$$\dot{\rho} + 3 \frac{\dot{R}}{R} (p + \rho) = 0$$

(4.20)

the above equation may be integrated by taking $R = R_0$ & $\rho = \rho_0$, where $R_0$ is the radiation & $\rho_0$ is the density of the universe at present epoch. By simplifications, we can write the equations of density & pressure as:

$$\rho = \rho_0 \left( \frac{R}{R_0} \right)^{-3(1+\beta)}$$

(4.21)

$$p = \beta \rho_0 \left( \frac{R}{R_0} \right)^{-3(1+\beta)} c^2$$

(4.22)

The detail of the values of $\beta$ is presented below:

(i) For $\beta = 0$

This is the case of regular Newtonian matter. An example would be cold dark matter (CDM) (discussed in chapter-8).

Because the matter is pressure less, we have $p = 0$. Since $\rho \neq 0$, equation (4.19) gives $\beta = 0$.

Hence by equation (4.21), we can say that

$$\rho \propto R^{-3}$$

(4.22a)

This result makes the sense based on mass conservation, as density is proportional to the inverse of volume.
(ii) For $\beta = \frac{1}{3}$

This is extreme opposite to Newtonian matter and an example may be a relativistic gas composed of photons or very light neutrinos.

From Kinetic theory, it may be shown that $\beta = \frac{1}{3}$ for a highly-relativistic substance.

Thus by equation (4.22), we have

$$\rho \propto R^{-4}$$

(4.22b)

The energy density goes down by another factor of $R$ relative to the Newtonian matter density because not only does the density of matter decrease as the inverse of the volume, but the energy of the particles is also decreased due to redshifting with the energy loss proportional to $R^{-1}$.

(iii) For $\beta = -1$

The hypothesized matter with the present equation of state is sometimes referred to as ‘vacuum energy’.

Taking $\beta = -1$, $p = -\rho c^2$ and $\dot{\rho} = 0$. This scenario represents a universe that is continually being filled with matter as it expands.

(iv) For $\beta \leq -\frac{1}{3}$

This is the case of dark energy (discussed in detail in chapters-8 & 9) refers to a more general form of negative pressure. As long as $\beta \leq -\frac{1}{3}$, then $\ddot{R}$ is positive and the universe is accelerating.

4.4 VARYING $\Lambda$-MODEL

As discussed in sections (1.3, 1.8, 2.10 & 4.1) the present estimates of the values of $\Lambda$ are very small. One approach to resolve this fact with observations is to assume that $\Lambda$ is not a pure constant but rather decreases continuously with cosmic expansion. Therefore we can say that $\Lambda$ is extremely small because the universe is old. Many
cosmologists have established this fact regarding decay of this term [132, 133]. Waga et al [134] has given their cosmological model with varying cosmological term at lower lensing rate by considering the statistics of gravitational lensing, which is a powerful tool in constraining model of the universe. This is due to the fact that in a varying \( \Lambda \)-cosmology, the distance to an object with red-shift '\( z \)' is smaller than the distance to the same object in a constant \( \Lambda \)-model. So the probability that light coming from the object is affected by a fore-ground galaxy is reduced in a decaying \( \Lambda \)-cosmology [135]. To incorporate a variable cosmological constant \( \Lambda \), a very small way has been proposed by Berman [132] & Rahman [136]. According to this approach, the field equations are same as in general relativity, mentioned as (4.1) where \( G \) & \( \Lambda \) are variable. The conservation law is same as in General theory of Relativity as mentioned in (4.2).

Therefore the conservation law equation (4.2) when applied in equation (4.1), we will get equation (4.3), which governs the variation of \( G \) & \( \Lambda \). Since the experiment test of General theory of Relativity is based upon the Schwarzschild solution of the field equation which is valid around a spherically symmetric gravitating body. The space around the gravitating body is empty, means

\[
T^\ ij = 0 \quad \text{ (4.23)}
\]

(\( i, j = 1, 2, 3, 4 \))

The equation (4.3) implies that \( \Lambda \) is a constant and the conservation equation (4.2) is trivial. The equation (4.1) reduces to

\[
R^\ ij = \Lambda g^\ ij \quad \text{ (4.24)}
\]

(\( i, j = 1, 2, 3, 4 \))

which is same as the field equation of general relativity in empty space. Therefore the empty space solution of general relativity & the empty space solution of the field equation (4.1) are same and the effect of the variation of \( G \) & \( \Lambda \) may not be seen in the experimental test like the gravitational redshift of the spectral lines, the advance of perihelion and gravitational deflection of light rays. In cosmological models effect of the variation of \( G \) & \( \Lambda \) may be seen only in those cosmological models which are non empty.
One of the objective of the present study is to discuss the nature of non empty homogeneous and isotropic cosmological models constructed with the help of the field equations as stated (4.1), (4.2) & (4.3).

4.5 VARYING G - MODEL

Theories regarding variable $G$ – model was first proposed by Milne, Jordan [13] & Dirac [11], later Brans & Dicke [14] put forwarded more elaborating theories of varying $G$ – model. A variation of $G$ with time has a considerable effect on the evaluation of Earth & Sun and on the orbit of the moon & other planets. As the Gravitational constant ‘$G$’ reduced the Earth expanded to its present size. The star like Sun has been brighter in the past if $G$ decreases with time [137]. The effect of this on life on Earth could be enhanced by the fact that the Earth must be moving away from the Sun if $G$ is decreasing. A varying $G$ leads to the variation in the Moon’s distance & the period. The orbits of the planets are also modified, and this could show up in radar time-decay experiments [138]. Some theories in which $G$ varies, also predicts other changes which can mask the above discussed effects. In the case of big-bang, one stronger limit on $G$ may be imposed as:

$$\left| \frac{\dot{G}}{G} \right| \leq 10^{-12} \text{ yr}^{-1} ; \quad [139-141]$$

In the construction of non singular cosmological models by incorporating f-gravity effect at the beginning of the expansion of the universe Sinha & Sivaram [16] proposed that

$${t = 0}, \quad \Lambda = \Lambda_f = 10^{28} \text{ cm}^2 ; \quad G = -G_f = -6.7 \times 10^{-30} \text{ c.g.s. units.} \quad (4.27)$$

We would like to propose here that the author(s) have considered $G < 0$ & $\Lambda > 0$, the same situation may also be possible when we consider $G > 0$ & $\Lambda < 0$. In the present work we shall see that our approach as suggested by [142] is more suitable than the first approach under the field equations.
proposed by Berman [132] & Rahman [136]. It has been also noticed that the Brans &Dicke theory may have slightly different prediction from General theory of Relativity for the detection of light by the Sun & for the advance of planets [14]. But if Helium is synthesized in the big-bang, there is even a stronger limit as suggested by Row-Robertson [138]. If G has had greater values in the past then it has now expect a small mass density to have the same effect as a large mass density has later. An increasing ‘G’ would lead to a contraction of the earth. It means the mystery of big-bang plays the important role in construction of cosmological model.

4.6 NEED OF MODIFICATION IN GENERAL THEORY OF RELATIVITY

Recently [135, 143], a modification has been proposed treating G & Λ non constant coupling scalars as in the Einstein field equation

\[ R_{ij} - \frac{1}{2} R g_{ij} = -8\pi G T_{ij} - \Lambda g_{ij} \]  

(4.28)

where G & Λ are coupling scalars. This approach could be a limit of viable such as covariant 5-dimension theory [144, 145]. Different possibilities for variable G may be investigated and the related problem of the standard model may be solved as in the inflationary scenario because G couples geometry to matter and in the expanding universe scenario G as \( G = G(t) \).

With the above discussed assumptions we may conclude that there are many models in which G increases with time and other models in which it decreases with time. To incorporate a variable ‘G’ & variable ‘Λ’ a very simple way has been proposed by Berman [132] & Rahman [136]. According to this approach the field equations are similar as equation (4.1) of General theory of Relativity:

\[ R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij} \]  

(4.29)

\[ (i, j = 1, 2, 3, 4) \]  

63
here \( G \) & \( \Lambda \) are not constants, but variables.

The conservation law is same as in General theory of Relativity

\[
T^{ij} ;_{ij} = 0 \quad ; \quad (i, j = 1, 2, 3, 4) \quad (4.30)
\]

therefore the conservation law (4.30) when applied in the equation (4.29), we get

\[
\frac{8\pi G}{c^4} T^{ij} + \Lambda ;_{ij} g^{ij} = 0
\]

\[
(i, j = 1, 2, 3, 4)
\]

which governs the variation of \( G \) & \( \Lambda \). This equation (4.31) is an additional equation required when we have variable \( G \) & \( \Lambda \). The experimental tests of General theory of Relativity are based on the Schwarzschild solution of the field equation which is valid around a spherically symmetric gravitating body. The space around the gravitating body is empty.

i.e. \( T_{ij} = 0 \)

and then in this case

\[
R_{ij} = \Lambda g_{ij}
\]

(4.32)

Therefore we have strong reason to believe on the variation of Gravitational constant ‘\( G \)’ and the Cosmological Constant ‘\( \Lambda \)’. During the study it has been noticed that cosmological models constructed by Rahman, Berman & Mishra et al. have the following fundamental properties:

i. They have all used the equation (4.31) as an additional requirement along with Einstein field equations.

ii. For an empty space, \( T_{ij} = 0 \) and then the equation (4.31) gives \( \Lambda = \text{constant} \).

So the effect of variation of \( G \) & \( \Lambda \) may be seen only in the non-empty solutions of the field equations.
The detail regarding the mathematical properties of cosmological models is presented in the next chapter.

4.7 COSMOLOGICAL MODEL WITH \( G \rho \propto t^{-2n} \) \& \( \Lambda \propto t^{-2n} \).

In the previous chapters/sections we have discussed the fundamental equations governing a homogeneous and isotropic cosmological model as proposed by Berman [132] & Rahman [136]. It has been shown that the simplest way to construct a cosmological model is to suppose the variation of Gravitational Constant ‘G’ and the Cosmological Constant ‘\( \Lambda \)’ as

\[
\frac{\Lambda c^2}{8\pi G \rho} = \text{constant}
\]

Here we assume the more generalized form keeping into consideration of the assumptions given by Berman [132].

\[
\Lambda \propto t^{-2n} \quad \& \quad G \rho \propto t^{-2n}
\]

(4.33)

for the construction of the cosmological model of the universe.

In this section we also study the possible homogeneous & isotropic cosmological models satisfying the relation (4.33).

The fundamental equations governing the cosmological model are given by

\[
3 \frac{\dot{R}}{R} = -4\pi G \left[ \frac{3p}{c^2} + \rho - \frac{\Lambda c^2}{4\pi G} \right]
\]

(4.34)

and

\[
\frac{3\dot{R}^2}{R^3} + \frac{3ke^2}{R^2} = 8\pi G \rho + \Lambda c^2
\]

(4.35)

The pressure-density relation is

\[
p = \beta \rho c^2
\]

(4.36)

The variation of density \( \rho \) is derived from the conservation equation

\[
\dot{\rho} = -3 (1 + \beta) \rho \frac{\dot{R}}{R}
\]

(4.37)
which on integration under the condition

\[ R = R_0, \quad \rho = \rho_0 \]  

(4.38)

gives

\[ \rho = \rho_0 \left( \frac{R}{R_0} \right)^{-\beta} \]  

(4.39)

The variation of G & \( \Lambda \) is governed by the equation

\[ 8\pi \dot{G} \rho + \dot{\Lambda} c^2 = 0 \]  

(4.40)

Also for a realistic cosmological model of the universe we must have

\[
\begin{cases}
G > 0 & \text{&} \quad \dot{G} < 0 \\
\Lambda < 0 & \text{&} \quad \dot{\Lambda} > 0
\end{cases}
\]  

(4.41)

during the expansion of the cosmological model. As a consequence of this result the equation (4.34) implies that

\[ \ddot{R} < 0 \]  

(4.42)

during the expansion of the cosmological model and hence it is a decelerating cosmological model, which in turn implies that the cosmological model is either expanding or oscillatory.

**The Possible Cosmological Model**

Keeping into consideration of Berman [132], let us consider the variation of \( G \) and \( \Lambda \) as:

\[ 8\pi G \rho = C_1 t^{-2n} \quad \& \quad \Lambda c^2 = -C_2 t^{-2n} \]  

(4.43)

where \( C_1 \) & \( C_2 \) are positive constants to be determined under the condition that

\[ t = t_0 ; \quad \rho = \rho_0, \quad G = G_0 \quad \& \quad \Lambda = \Lambda_0 \]  

(4.44)

and hence we have

\[ C_1 = 8\pi G_0 \rho_0 t_0^{-2n} \]  

(4.45)

\[ C_2 = -\Lambda_0 c^2 t_0^{-2n} \]  

(4.46)

the constant \( n \) is to be decided later on.
Differentiating the equations (4.43) w. r. t. time ‘\(t\)’, we get

\[8\pi\dot{G}\rho + 8\pi G \dot{\rho} = -2nC_1 t^{-2n-1}\]  
\[\dot{\Lambda}c^2 = 2nC_2 t^{-2n-1}\]  

(4.47)  
(4.48)

Adding the equations (4.47) & (4.48)

\[(8\pi\dot{G}\rho + \dot{\Lambda}c^2) + 8\pi G \dot{\rho} = 2n(C_2 - C_1) t^{-2n-1}\]

using the equation (4.40) in (4.49), we get

\[8\pi G \dot{\rho} = 2n(C_2 - C_1) t^{-2n-1}\]  

(4.49)  
(4.50)

The first equation of (4.43) and (4.50) together gives

\[\frac{\dot{\rho}}{\rho} = \frac{2n(C_2 - C_1)}{C_1} \left(\frac{1}{t}\right)\]

(4.51)

Integrating (4.51) under condition

\[t = t_0; \rho = \rho_0\]

we get

\[\rho = \rho_0 \left(\frac{t}{t_0}\right)^{-2n\frac{C_1 - C_2}{C_1}}\]

(4.52)  
(4.53)

Comparing the equation (4.39) & (4.53), we have

\[\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{\frac{2n(C_1 - C_2)}{3C_1(1 + \beta)}}\]

(4.54)

Differentiating the equation (4.54) w. r. t. ‘\(t\)’, we get

\[\frac{\dot{R}}{R} = \frac{2n(C_1 - C_2)}{3C_1(1 + \beta)} \left(\frac{1}{t}\right)\]

(4.55)

Putting the value of \(G \rho\) & \(\Lambda\) from equations (4.43) and using equation (4.54) & (4.55) in (4.35), it takes the form

\[\frac{4n^2}{3C_1^2(1 + \beta)^2} \frac{(C_1 - C_2)^2}{t^2} 1 + \frac{3k c^2}{R_0^2} \frac{4n(C_1 - C_2)}{3C_1(1 + \beta)} \left(\frac{t_0}{t}\right)^{-2n\frac{4n(C_1 - C_2)}{3C_1(1 + \beta)}} = \frac{C_1 - C_2}{t^{2n}}\]

(4.56)
Equation (4.55) must be true for all positive real values of ‘t’. For this we have the following possibilities:

**Possibilities:**

P₁) For \( n = 0 \)

a) \( \frac{3kc^2}{R_0^2} = C_1 - C_2 \), \( C_1 \neq C_2 \) for \( k \neq 0 \)

b) \( C_1 = C_2 \) for \( k = 0 \)

P₂) For \( n = 1 \)

a) \( k = 0 \), \( \frac{4(C_1 - C_2)}{3C_1^2(1 + \beta)^2} = 1 \)

b) \( C_1 \neq C_2 \), \( t = t_0 \)

and \( \frac{4(C_1 - C_2)^2}{3C_1^2(1 + \beta)^2} + \frac{3kc^2t_0^2}{R_0^2} = C_1 - C_2 \)

Please look A₃ also.

### 4.8 RESULT AND DISCUSSION

A₁) The possibility \( P_1(a) \) is not admissible due to the fact that for \( n = 0 \), the equation (4.54) reduces to

\[
R = R_0 = \text{constant}
\]

and hence the model is static, which is against the observed fact that the universe is expanding.

A₂) For the possibility \( P_1(b) \), again the equation (4.55) reduces to (4.57) and hence the case may not be admissible.

A₃) For the possibility \( P_2(a) \),

\[
C_1 = 2 \left[ \frac{1 \pm \sqrt{1 - 3(1 + \beta)^2} C_2}{3(1 + \beta)^2} \right]
\]

(4.58)
\[ C_1 \text{ & } C_2 \text{ are positive constants, many admissible solutions may occur for } \beta > -1. \]

For \( 0 \leq \beta \leq 1/3 \), we have many admissible solutions. We may take

\[ C_2 = \frac{1}{3(1 + \beta)^2} \text{ then } C_1 = \frac{2}{3(1 + \beta)^2}, \quad (4.58a) \]

For the possibility \( P_2(a) \), for \( n = 1 \text{ & } k = 0 \), we have

for \( \beta = 0 \), (i.e. in matter dominated era),

\[ 4(C_1 - C_2) = 3C_1^2 \quad (4.59) \]

for \( \beta = \frac{1}{3} \), (i.e. in radiation dominated era)

\[ 3(C_1 - C_2) = 4C_1^2 \quad (4.60) \]

So by above equation (4.59), we get

\[ C_2 = C_1 - \frac{3}{4} C_1^2 \quad (4.61) \]

The maxima or minima of \( C_2 \) exists when

\[ \frac{dC_2}{dC_1} = 1 - \frac{3}{2} C_1 = 0 \]

\[ \Rightarrow C_1 = \frac{2}{3}, \]

also

\[ \frac{d^2C_2}{dC_1^2} = -\frac{3}{2} < 0 \quad (4.62) \]

this shows that \( C_2 \) has the maximum value when \( C_1 = \frac{2}{3} \) and this maximum value of \( C_2 \) is obtained as:

\[ (C_2)_{\text{max}} = [C_1 - \frac{3}{4} C_1^2]_{C_1 = \frac{2}{3}} = \frac{1}{3} \quad (4.63) \]

As shown in the following

\[ C_1 = \frac{2 - 2\sqrt{1 - 3C_2}}{3} \text{ & } \frac{2 + 2\sqrt{1 - 3C_2}}{3} \quad (4.64) \]
Therefore we have two positive values of $C_1$ for a value of $C_2$ between 0 & 1/3, and out of the two values of $C_1$, which one is valid in the model is to be decided from other considerations. The conditions $C_2 = 1/3$, $C_1 = 2/3$ are valid for $\beta = 0$ (i.e. for the evolution of matter dominated era).

Similarly from (4.60) we have

$$C_1 = 3/8 \quad \& \quad C_2 = 3/16 \quad (4.65)$$

For $0 < C_2 < 3/16$,

$$C_1 = \frac{3 - \sqrt{9 - 48C_2}}{8} \quad \text{or} \quad C_1 = \frac{3 + \sqrt{9 - 48C_2}}{8}$$

which are the conditions imposed on the constraints $C_1$ & $C_2$ for $\beta = \frac{1}{3}$, for the evolution of the radiation dominated era.

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