CHAPTER 1
THEORY OF COSMOLOGY

1.1 INTRODUCTION

A human being is a part of the whole, called by us ‘universe’, a part limited in time and space. We experience ourselves, our thoughts and feelings, as something separate from the rest. The true understanding of the cosmos is a true understanding of ourselves. “All is one and interconnected” is an ancient wisdom which is the foundation of both metaphysics & philosophy. As Leibniz said, “Reality cannot be found except in one single source, because of the interconnection of all things with one another”. Today another scientific revolution is beginning, one that may change our view of the cosmos as radically as the last. It again seems likely that the effects of this revolution, both social and scientific, will be profound.

As per belief of human, the Earth was flat in the beginning. This may be think because the curvature of our planet surface is not immediately apparent. The Earth being flat, bring about the problems that it must end somewhere unless we imagine it to extend infinitely. What lies beyond these boundaries was largely unknown and open for all to speculations. People from all parts of the world created their own myths, inspired by the skies and the celestial bodies. Their cosmogonies can be seen as an attempt to explain their own place in the universe. As mentioned in the history, it was the Greeks who first put forward the idea that our planet is a sphere. Around 340 B.C., Aristotle made a few good points in favour of this theory. The influence of Aristotle was significance and in around 150 A.D., Claudius Ptolemaeus elaborated Aristotle ideas into a complete Mathematical description of universe (cosmological model). This system was later adopted by the Christian church and became the dominant cosmology until the 16th century. Later on this fascinating subject was developed by so many philosophers/cosmologists/researchers. Few of them are Copernicus, Galileo, Kepler, Newton & Immanuel Kant etc.

For better understanding of the development of cosmology we feel necessary to discuss the cosmology time line:
1.1.1 Cosmology Timeline

Various ancient cultures developed mythology based upon the cosmos, understanding the universe is said to begin with ancient Greeks as listed in the table below:

*Table-1.1 History of Cosmology*

<table>
<thead>
<tr>
<th>Period</th>
<th>Description</th>
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<tbody>
<tr>
<td>500BC to 300BC</td>
<td>• Pythagoras believed that the earth was in motion.&lt;br&gt;• Aristotle taught that rotating spheres carried the Moon, Sun, planets and stars around a stationary Earth.&lt;br&gt;• Greek Philosophers estimated the distance to the Moon and calculated the size of the finite universe.</td>
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<tr>
<td>300BC to 210BC</td>
<td>• Greek Mathematician Aristarchus was first person to propose a scientific heliocentric model of the solar system and placing the Sun at the center of the universe.&lt;br&gt;• He has also established the order of planets from the Sun.</td>
</tr>
<tr>
<td>200AD</td>
<td>• Ptolemy proposes an Earth centered universe, with Sun &amp; planets revolving around the Earth.&lt;br&gt;• Motion should be in circles &amp; epicycles had to be introduced.</td>
</tr>
<tr>
<td>1401AD to 1464AD</td>
<td>• Nicholas de Cusa suggests that the Earth is a nearly spherical shape that revolve around the Sun and that each star is itself a distant Sun.</td>
</tr>
<tr>
<td>1500AD</td>
<td>• Many Astronomers proposed a Sun-centered universe including Indian Mathematicians the Great Aryabhata, Bhaskara I &amp; Copernicus, a European Mathematician.</td>
</tr>
<tr>
<td>1576AD</td>
<td>• Thomas Digges modified the Copernican system proposing a multitude of stars extending to infinity.</td>
</tr>
<tr>
<td>1584AD</td>
<td>• Proposal of non-hierarchical cosmology by Giordano Bruno.&lt;br&gt;• With the assumption that the universe had its center everywhere and its circumference nowhere.</td>
</tr>
<tr>
<td>Period</td>
<td>Description</td>
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<tr>
<td>1600AD</td>
<td>• Tycho proposed a system in which the planets other than Earth orbited the Sun while the Sun orbited the Earth.</td>
</tr>
</tbody>
</table>
| 1609AD   | • Johannes Kepler used the dark night sky to argue for a finite universe.  
          • He told that planets moved in ellipse and not in perfect circles, about the Sun, known as the law of planetary motion.  
          • Newton later explained it by his inverse square law for the gravitational force.  
          • Galileo observed Moon of Jupiter in support of the heliocentric model. |
| 1687AD   | • Newton established Laws of motion. |
| 1791AD   | • Eramus Darwin given the first description of cyclic expanding and contracting universe. |
| 1848AD   | • Edgar Allan poe offers a solution to Olbers paradox in an essay that also suggests the expansion and collapse of the universe. |
| 1905AD   | • Albert Einstein published the Special theory of Relativity, pointing that the space and time are not separate continuums. |
| 1915AD   | • Einstein published the General theory of Relativity (GTR). |
| 1922AD   | • The Russian Mathematician Friedmann realized that Einstein’s equations could describe an expanding universe and published his paper entitled “Über die Krümmung des Raumes” (English Translation: On the curvature of Space).  
          • Einstein was reluctant; believing in a static (non-expanding) universe. |
<p>| 1927AD   | • Georges Lemaitre presented his idea of an expanding universe. He also derived Hubble’s law and provided the first observational estimation of the Hubble’s constant. |</p>
<table>
<thead>
<tr>
<th>Period</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>● Through the research contribution Lemaitre has added a completely new feature to the discussion of cosmology and he proposed that the universe began as a single lump of matter, a primeval atom that radioactively decays in an outrushing explosion. Due to this he may be called as ‘father of the Big-Bang’.</td>
</tr>
</tbody>
</table>
| 1929AD | ● The American Astronomer Hubble established that some nebulae were indeed distant galaxies comparable in size to our own Milky way.  
● Concept of expanding universe & cosmological constant. |
| 1950AD | ● The British astronomer Fred Hoyle coins the phrase “Big Bang”, (means: the universe had been born at one moment about ten thousand million years ago in the past and the galaxies were still travelling away from us after that initial burst. |
| 1965AD | ● Penzias & Wilson discovered a cosmic microwave background radiation (CMB). |
| 1970AD | ● Cosmologists have accepted the Hot Big Bang model. |
| 1986  | ● Dr. Milo Woltt discovered the wave structure of matter. |
| 1997  | ● Description of an infinite eternal space full of matter. |

### 1.2 THE BIRTH OF THE UNIVERSE AS WE BELIEVE

The universe we can observe is finite. It has a beginning in space & time, before which the concept of space & time has no meaning because space time itself is a property of the universe. It is clear from the above discussed cosmology time-line that according to the Big-Bang theory, the universe began about 10 billion years ago in a violent explosion. For an incomprehensibly small fraction of second, the universe was
an infinitely dense & infinitely hot fire ball known as ‘inflation’, which lasted for only 1 millionth of a second. A number of different observations corroborated the Big-Bang theory.

1.3 COSMOLOGY

Cosmology means the science of world. It is a natural complement of the special sciences. It begins where they leave off and its domain is quite distinct from theirs. The cosmologists/scientists determine the immediate cause of the phenomena observed in the mineral or organic world. They formulate their laws and build these into a synthesis with the help of certain general theories such as:

- Theory of Light
- Theory of Heat
- Theory of Electricity

The cosmologists, on the other hand seek the ultimate cause not of this or that class of beings or of phenomena, but of the whole material universe. They inquire into the constituent nature of corporeal beings, their density & first cause. Present thesis is a small attempt in this direction.

Cosmology is a branch of science in which we study the universe as whole. At the beginning of twentieth century, two conceptual revolutions occur in Science for Quantum Mechanics & General Theory of Relativity. Einstein laid the foundation stones of both of these revolutions in the single year of 1905. In this year Einstein also provided fundamental insight into two other emerging areas, the determination of molecular dimensions and with his analysis of the nature of Brownian motion, laid the foundation of an important piece of statistical understanding, which has had enormous implications in other fields. In this new framework our modern reason of the origin, the structure and the evolution of the universe has now been established. This new idea played a determining role in the development of our current vision of the universe from both a scientific and philosophical point of view. This influence clearly exceeds the medium of research in this area. In 1917, Albert Einstein produced a
model of the universe based on the Theory of General Relativity. As per this research, time is assumed as the fourth dimension, and gravitational attraction is equivalent to a curvature of this four-dimensional space [1]. An object travelling in a 'straight line' in four dimensional space would appear to follow a curve in three dimensions. This is very difficult to imagine but think of the path of the shadow compared to the path of the object. As most scientists assumed the universe is static, Einstein postulated the existence of a force of repulsion between galaxies that counterbalanced the gravitational force of attraction. He introduced a term called cosmological constant ‘Λ’ in his field equation. This 'Cosmological Constant' resulted in a static universe. He therefore missed the chance to predict the expansion of the universe by introducing an arbitrary constant. Einstein later called this the "biggest mistake of my life"[2]. Non-static models of the universe were developed in 1917 by the Dutch astronomer W. de Sitter [3] in 1922 by the Russian mathematician Alexander Friedman and in 1927 by the Belgian abbé, Georges Lemaître [4]. We belong to the family of peer researchers who believe that universe is expanding. My supervisor Dr. R. K. Mishra have constructed several cosmological models with variable cosmological constant Λ and variable gravitational constant G under different assumptions and agreed with the above referred researches that universe is expanding [5, 6, 7].

As a result of these investigations, we have a number of rival theories of cosmology, which differ from one another more on attitude than on outcome. The observational data are yet in a position to make a final choice among the various theories of gravity. The experiment is going on to establish the fact.

After formulating the General theory of relativity and testing it on the basis of the solar experiments, Einstein had applied the theory for the construction of cosmological model of the universe. He proposed that the universe is static like a static sphere filled with matter [1]. According to the General theory of Relativity, the construction of static cosmological model is possible when we add the cosmological term $\Lambda g_{ij}$ into the Einstein field equation, where $\Lambda$ is a constant known as cosmological constant.
1.4 MATHEMATICS OF THE UNIVERSE BASED ON PURE MATHEMATICAL DISCOVERY

We feel here necessary to discuss why the mathematicians like most Cosmology as physicists do and from literature survey we can say that there are a large number of ways that mathematics has supported our attempts to understand the universe as whole. Both Plato & Pythagoras influenced the first logical consistent cosmological world view developed in the 4th century B.C. This theory states that all matter in the universe is composed of some combination of four elements: Earth, Water, Fire and Air. (In ancient India the whole material world was classified into four elements; Earth (Prithvi), Fire (Agni), Air (Maya) and Water (Apa).

Figure-1.1: Properties of Matter in the Universe

These four elements arise from the working of the two properties of hotness and dryness upon an original unqualified or primitive matter. The possible combinations of these two properties of primitive matter give rise to the four elements or element
forms. It means a culture where people were involved with logic and geometry would put the concept of chemistry also in such a logical and geometrical form.

**Interpretation:**
From the figure 1.1, it is easy to see that Fire and water are obvious opposites and so are Earth & Air share no common property. There are four properties each shared by two non-opposite elements.

- Fire & Air : Share the property of hotness \((A \cap B)\).
- Water & Air : Share the property of wetness \((A \cap C)\).
- Water & Earth : Share the property of coldness \((C \cap D)\).
- Fire & Earth : Share the property of dryness \((B \cap D)\).

Since the four elements are two pairs of opposite elements, so also have the four properties – hotness being the opposite of coldness & wetness the opposite of dryness. Each of the four elements was held to exist in an ideal pure form which could not be actually found in Earth. The real things around us were considered impure or mixed form of these four ideal elements. **In the words of logic we can say that, the real or observed different kinds of the same elements are due to different degree of the same properties.** The elements could be changed into one another by removal of one property and addition of another. To these five elements, was added a fifth one; ether or Akasha. According to some scholars these five elements were also identified with the various senses of perception; earth with smell, air with feeling, fire with vision, water with taste and ether with sound. Whatever the validity behind these interpretations, it is true that since very ancient times, scholars had perceived that the material world is comprising of these five elements. A **Greek mathematician ‘Euclid’** proved that there are only five solid shapes that can be made from **simple polygons (triangles, squares & hexagons)** and called these five solids, the atoms of the universe. Plato was influenced by this pure mathematical discovery and revised the four element theory with the proposition that there were five elements to the universe (Earth, water, air, fire & quintessence) in corresponding with the five regular solids (mathematical shapes) [8,9].
Table 1.2: Mathematical discovery Chart

<table>
<thead>
<tr>
<th>Shape/Name</th>
<th>Proposed element of the universe</th>
<th>Nature</th>
</tr>
</thead>
</table>
| Cube       | Earth                            | • Regular solid that tessellates Euclidean space.  
|            |                                  | • Solids have a definite volume & shape and high density.  
|            |                                  | • Molecules are limited to move about fixed position.  |
| Icosahedron| Water                            | • Flows out of one’s hand when picked up.  
|            |                                  | • Molecules free to move throughout the liquid, giving it a definite volume but no definite shape.  |
| Octahedron | Air                              | • Its minuscule components are so smooth that one can barely feel it.  
|            |                                  | • It has no definite shape or volume and density is lower than liquids.  |
| Tetrahedron| Fire                             | • The heat of fire feels sharp and stabbing like tetrahedron.  |
| Dodecahedron| Quintessence                     | • The vacuum has a structure as well.  
|            |                                  | • It is made of electromagnetic energy, which can be organized in any of the other platonic structure.  |
1.5 EINSTEIN FIELD EQUATIONS

Einstein Field Equations describe the gravitational field resulting from the distribution of matter in the universe. It is expressed as:

\[ R_{ij} - \frac{1}{2} R g_{ij} = - \frac{8 \pi G}{c^4} T_{ij} \]  

(1.1)

where \( R_{ij} \) is the Ricci tensor, \( R \) is the curvature scalar, \( T_{ij} \) is the energy momentum tensor of the source producing the gravitational field and \( G \) is the Newton’s Gravitational constant. The Einstein field equations are used to determine the curvature of the space-time resulting from the presence of mass & energy. Because of that they determine the metric tensor of the space time for a given arrangement of stress energy in the space-time. The energy momentum tensor satisfies the following conditions:

i) \( T_{ij} \) is symmetric w. r. t. interchange of \( i \) & \( j \).

ii) \( T_{ij} \) is divergence-less for energy and momentum to be conserved.

\[ T^{ij}_{;i} = 0 \quad (i, j = 1,2,3,4) \]  

(1.2)

Here, \( T^{ij} = \rho u^i u^j \)  

(1.2a)

The symmetric tensor \( T^{ij} \) is the material energy momentum tensor.

we can define

\[ T^{ij}_{;i} = (\rho u^i u^j)_{;j} \]

\[ = (\rho u^j)_{;j} u^i + \rho u^i (u^j_{;j}) \]

\[ = \rho u^j u^i_{;j} \]  

(1.2b)

If, \( u^j \) is regarded as a field function (i.e. meaningful not just on one world line but a whole set of worldliness filling up all space time or a region thereof), we obtain

\[ \frac{du^j}{ds} = \left( \frac{\partial u^j}{\partial x^i} \right) \left( \frac{dx^i}{ds} \right) = u^i_{;j} u^j \]  

(1.2c)
we get by using other relations
\[ u^i_{;j} u^j + \Gamma^i_{jk} u^j u^k = (u^i_{;j} + \Gamma^i_{jk} u^k)u^j = 0 \]
\[ \text{(1.2d)} \]
with the use of (1.2b) and (1.2d), we can get
\[ T^y_{;i} = 0 \]
So that the tensor \( T^{ij} \) defined can be used in the right hand side of Einstein equations.
However the tensor given by \( T^{ij} = \rho u^i u^j \) is a special case, which occurs in
\[ T^{ij} = (\rho + \frac{p}{c^2}) u^i u^j - \frac{p}{c^2} g^{ij} \]
\[ \text{(1.2e)} \]
here \( \rho \) is the mass energy density being obtained from the later by setting \( p = 0 \).
This zero-pressure case obtains when there is no random motion of the material particles that is associated with pressure, so that the particles move solely under the influence of gravitation and so move along geodesics
\[ \frac{du^k}{ds} + \Gamma^k_{ij} u^i u^j = 0 \]
\[ \text{(1.2f)} \]
The Ricci tensor is defined by:
\[ R_{ij} = \frac{\partial^2 \log \sqrt{-g}}{\partial x^i \partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^l_{im} \Gamma^m_{jl} - \Gamma^k_{ij} \frac{\partial \log \sqrt{-g}}{\partial x^k} \]
\[ \text{(1.3)} \]
and
\[ R = g^{ij} R_{ij} \]
\[ \text{(1.4)} \]
where \( g_{ij} \) is the metric tensor & \( g \) is its determinant. \( \Gamma^i_{jk} \) are Cristoffel symbols related to \( g_{ij} \). The constant \( \frac{8 \pi G}{c^4} \) on the right hand side of the equation is obtained by the weak & static field approximation of the equation (1.1) and then comparing it with Poisson’s equation:
\[ \nabla^2 \phi = 4G \pi \rho \]
\[ \text{(1.5)} \]
governing the gravitational field in Newtonian mechanics. $\phi$ is the gravitational potential & $\rho$ is the density of the matter. Einstein proposed that universe is like a static sphere filled with matter and uniform density.

### 1.6 ROBERTSON WALKER METRIC

The distribution of matter & radiation in the observable universe is homogeneous & isotropic. Observational evidences of large scale homogeneity & isotropy of the universe provide motivation for cosmological principles [10]. The beauty of the homogeneous model is that they may be studied locally. Since in a homogeneous model, any part is the representative of the universe as a whole. The demand of isotropy may be relaxed but such relaxation leads to complications in the construction of cosmological models. Therefore according to the cosmological principle no cluster of galaxy thought to be in any way preferred. An important property of in spatial homogeneity of the universe is the existence of the cosmic time ‘$t$’, which reduces to the proper time at each fundamental particle. The mathematical formulation of the spatial homogeneity and isotropy of the universe gives the following results:

(i). The hyper surfaces with constant cosmic time are maximally symmetric subspaces of the whole space time.

(ii). Not only the metric $g_{ij}$ but all cosmic tensors such as the energy momentum tensor $T_{ij}$, are invariant w. r. t. isometry of subspaces. The metric of space with homogeneous & isotropic sections which being maximally symmetric, is the Robertson Walker Metric (R.W. metric). This has been explained as:

\[
\begin{align*}
\text{(1.6)} \\
&ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] 
\end{align*}
\]

where \(R(t)\) is an unknown function of time. This sets the scale of the geometry of the space, therefore is called the scale factor. $k$ is a constant and may be chosen to be -1, 0, 1 for spaces of constant negative (open), constant zero (spatial or Flat), constant positive (closed) curvature respectively and therefore it is called the curvature index.
The coordinates \((r, \theta, \phi, t)\) form a comoving coordinate system in the sense that the fundamental particles are at rest w. r. t. \((r, \theta, \phi)\).

### 1.7 VARIABLE CONSTANT OF GRAVITATION ‘G’.

During the investigations it has been noticed that there are mainly three different approaches leading towards a variable constant of Gravitation.

**First Approach:**
First approach is known as the Dirac large number hypothesis [11]. The ratio of the gravitational to the electrostatic force is of the order \(10^{-40}\). There is no convincing explanation of why such a small dimensionless number appears in the fundamental laws of physics. Dirac pointed out that a dimensionless no. of the order of \(10^{-40}\) may be constructed with \(G, h, c\) and the Hubble’s constant \(H_0\). It means if \(H\) is not a constant due to the expansion of the universe then the constant \(G\) may also vary with time.

**Second Approach:**
The second approach is based upon the Mach’s principle, which states that in a particular region of the space, the inertial frame is determined by the matter distribution around the region as pointed out by Lord [12]. In this context Jordan has introduced coupling parameter of the scalar field, to change energy conservation, as suggested by Dirac. Following the conform equivalence theory, multidimensional theories of gravity are conform equivalent to the theories of usual General Relativity in four dimensions with an additional scalar field. One case of this is given by Jordan’s theory [13], which, without breaking energy conservation is equivalent to the theory of C. Brans and R. Dicke of 1961. The Brans-Dicke theory follows the idea of modifying Hilbert-Einstein theory to be compatible with Mach’s principle. For this, Newton’s Gravitational constant had to be variable, dependent of mass distribution in the universe, as a function of a scalar variable, coupled as a field in Lagrangian.
To incorporate the Mach’s principle in the theory of Relativity, Brans-Dicke [14] proposed the existence of a long range scalar field which is related to the constant of Gravitation ‘$G$’. Brans-Dicke theory is now commonly known as Jordan-Brans-Dicke theory.

**Third Approach:**

The third approach is based upon the strong gravity theory proposed by Isham et al. [15], in the Einstein field equation (1.1) the constant $\frac{8 \pi G}{c^4}$ is obtained through the Newtonian approximation of the field equation (1.1), but the existence of strongly interacting massive spin-2 mesons particles in nature suggest that for the determination of material properties the space time near an elementary particle, one has to consider $\frac{8 \pi G_f}{c^4}$ in the Einstein’s field equation (1.1) [16],

where $G_f = 10^{39} G$  \hspace{1cm} (1.7)

In the Einstein field equation

$$R_{ij} - \frac{1}{2} R g_{ij} = - K T_{ij}$$  \hspace{1cm} (1.8)

the constant $K$ (This $K$ is different from the $k$ used in the thesis.) is related to the Newtonian gravitational constant $G$ as $\frac{1}{2} K \rho c^4 = 4 \pi G \rho$

\hspace{1cm} i.e. $K = \frac{8 \pi G}{c^4}$  \hspace{1cm} (1.9)

through the Newtonian approximation of the field equation (1.8). It would be worthwhile to point out that the field equation (1.8) places no restriction whatsoever on the numerical value of the constant $K$. In the derivation of the field equation (1.8) from the action principle with the Lagrangian density

$$I = K^{-1} R \sqrt{-g} + I_m$$  \hspace{1cm} (1.10)

where $I$ is the Lagrangian density & $I_m$ is the mean mass Langrangian density.
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$K$ is a factor of dimension $g^{-1}cm^{-1}sec^{-2}$, whose numerical value is undetermined at this stage. Since Einstein used the field equation (1.8) to describe the theory of gravitation and to incorporate the Newtonian theory of Gravitation

$$\nabla^2 \phi = 4\pi G \rho$$

(1.11)
as a special case of field equation (1.8), he proposed the value of $K$ equal to $\frac{8 \pi G}{c^4}$.

Following this, it has become customary to always relate $K$ to the Newtonian constant of Gravitation ‘$G$’ via equation (1.9). While the existence of the strongly interacting massive spin-2 mesons states in nature which determine the space time geometry in the region near an elementary particle, suggest that for the description of the space time geometry in a region near an elementary particle via the field equations (1.8), we must have

$$K = \frac{8\pi G_f}{c^4}$$

(1.12)

where

$$G_f = 10^{39} G$$

(1.13)

For the purpose of the numerical estimation we take the value of $G$ as:

$$G = 6.7 \times 10^{-8} c.g.s. \text{ units}$$

(1.14)

and the value of $G_f$ is taken as

$$G_f = 6.7 \times 10^{31} c.g.s. \text{ units}$$

(1.15)

As a result of the above discussion, we will use the value of $G$ as given below during construction of the cosmological models.

$$G = G_f = 6.7 \times 10^{31} c.g.s. \text{ units at } t = 0$$

(1.16)

and

$$G = 6.7 \times 10^{-8} c.g.s \text{ units at } t = 2 \times 10^{10} \text{ years}$$

(1.17)
1.8 THE EXPANSION OF THE UNIVERSE AND ITS PROPERTIES

In 1929, Edwin Hubble working at the Carnegie observatories in Pasadena, California, measured the red shift of a number of distant galaxies; he also measured their relative distances by measuring the apparent brightness of variable stars called cepheids in each galaxy. When he plotted red shift against relative distance he found that the red shift of distant galaxies increased as a linear function of their distance. The only explanation for this observational experiment is that the universe was expanding. Here in this study Hubble [17] pointed out that the galaxies are receding from us. The speed $v$ of the recession of the galaxies is proportional to its distance ‘$D$’ from us. In this way Hubble proposed his famous law as:

$$ v = HD $$

where $H$ is the Hubble’s constant. The constant ‘$H$’ is one of the important parameter in all cosmological theories. Several ways have been proposed to measure the Hubble’s constant $H$.

The expanding universe is finite in both time & space. The reason that the universe didn’t collapse as Newton’s and Einstein’s equations said it might is, that it had been expanding from the moment of its creation. The expanding universe is a new idea of modern cosmology. The three possible type of expanding universe are called open, flat and closed universe. If the universe were open, it would expand forever. If the universe were flat it would also expand forever but the expansion rate would be slow to zero after an infinite amount of time. If the universe were closed, it would eventually stop expanding and recollapse on itself, possibly leading to another big bang. In all the above three cases, the expansion and the force that causes the slowing rate is gravity.

Following the equation (1.6), let us consider a fundamental particle at the origin $r = 0$ and another particle at ‘$r$’ then the proper distance ‘$D$’ between the two particles at a time ‘$t$’ is given
Therefore, the proper distance ‘D’ is proportional to the scale ‘R(t)’. The proper velocity ‘v’ of the particle at ‘r’ relative to the particle at the origin is obtained by differentiating ‘D’ w. r. t. ‘t’ realising that ‘r’ remains constant because it is a co-moving coordinate.

Thus, we have

\[ v = \dot{D} = \dot{R} \int_0^r \frac{dr}{\sqrt{1 - kr^2}} = \frac{\dot{R}}{R} D \]  

(1.20)

where a dot on the symbol means differentiation with respect to ‘t’. This tells us that at any time ‘t’ the speed ‘v’ is proportional to the proper distance ‘D’. Comparing this result with the Hubble’s law (1.18),

we get,

\[ H = \frac{\dot{R}}{R} \]  

(1.21)

In fact, we cannot stretch the measuring tape between the particles, therefore, the proper distance ‘D’ is not a measurable quantity. However, the great advantage of the relativistic formulation is that, it gives relationship between quantities such as red-shifts, apparent magnitudes, number counts etc. which can be measured. The important point is that, they all involve only the scale factor R(t).

In a similar way \( \ddot{R} \) & \( \dddot{R} \) are treated as measure of the velocity and the acceleration of the fundamental particles related to the origin respectively.

### 1.9 IMPORTANCE OF VARIABLE COSMOLOGICAL CONSTANT

The cosmological constant is an extra term in the Einstein equation of General theory of Relativity which physically represents the possibility of density and pressure associated with ‘empty’ space. It is denoted by ‘Λ’ (Lambda). The inclusion of this
vacuum energy term can greatly affect the cosmological theories. As pointed out by Linde [18], the spontaneous symmetry breaking result is a change in the temperature of the medium that decreases with the increase of the energy and vanishes at a certain critical temperature. This shows the dependence of the cosmological constant on temperature and hence in an expanding hot big bang model. This also implies a dependence of ‘Λ’ on time as may be seen in many models including Weinberg-Salam model [19]. The General theory of Relativity has two distinct kind of correspondence with the Special theory of Relativity. The first is the limit of vanishing gravitational field locally. It is the demand of the equivalence principle. According to which the metric tensor \( g_{ij} \) of a gravitational field satisfies the conditions that

\[
g_{ij} = \eta_{ij} \quad (1.22)
\]

and

\[
\frac{\partial g_{ij}}{\partial x_k} = 0, \quad (i, j, k = 1, 2, 3, 4)
\]

in a local inertial frame.

Therefore the L.H.S. of the Einstein field equation

\[
R_{ij} - \frac{1}{2} R g_{ij} = - \frac{8\pi G}{c^4} T_{ij}, \quad (i, j = 1, 2, 3, 4)
\]

do not vanish in the local inertial frame.

The field equation reduces to

\[
\frac{1}{2} \left[ \Box^2 g_{ij} - \frac{\partial^2 g^k_j}{\partial x^k} + \frac{\partial^2 g^k_i}{\partial x^k} + \frac{\partial^2 g^k_l}{\partial x^k \partial x^l} \right] - \frac{1}{2} \eta_{ij} \left[ \Box^2 g^k_l - \frac{\partial^2 g^k_n}{\partial x^k \partial x^n} \right] = - \frac{8\pi G}{c^4} T_{ij} \quad (1.25)
\]

\((i, j, k, n = 1, 2, 3, 4)\)

in a local inertial frame. The result obtained from the equation (1.25) may be physically interpreted in a local inertial frame. In fact the space-time of a local inertial frame is not a flat space-time, unless

\[
\frac{\partial^2 g_{ij}}{\partial x^i \partial x^j} = 0, \quad (i, j, k, n = 1, 2, 3, 4) \quad (1.26)
\]
Therefore the geometrical object characterising the gravity does not vanish in a local inertial frame.

If the Einstein tensor vanishes in a local inertial frame, it must vanish everywhere and in every coordinate system. This is the case of second kind of Special Relativity limit of the General theory of Relativity. Therefore the field equation (1.24) yields the vanishing of the energy momentum tensor and hence the space-time under consideration is empty, but it is not true, because, it is possible to define the perfect fluid distribution of matter in a Minkoski space-time.

If we introduce the cosmological term $\Lambda g_{ij}$ in the Einstein field equation. i.e.

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = - \left( \frac{8\pi G}{c^4} \right) T_{ij}$$

(1.27)

$$R_{ij} - \frac{1}{2} R g_{ij} = (i, j, k, n = 1, 2, 3, 4)$$

Since the space-time under consideration is flat, the quantity $R_{ij} - \frac{1}{2} R g_{ij}$ vanish identically in the space-time. On substituting the values from the equation

$$T_{ij} = \left( \frac{p}{c^2} + \rho \right) u^i u^j - \frac{p}{c^2} g^{ij}$$

(1.27a) into (1.27), we get

$$0 = - \frac{8\pi G}{c^4} \left[ \left( \frac{p}{c^2} + \rho \right) u^i u^j \right], \quad (i, j = 1, 2, 3)$$

(1.27b)

$$- \Lambda = - \frac{8\pi G}{c^4} \left[ \left( \frac{p}{c^2} + \rho \right) u^i u^j + \frac{p}{c^2} \right], \quad (i, j = 1, 2, 3)$$

(1.27c)

$$0 = - \frac{8\pi G}{c^4} \left[ \left( \frac{p}{c^2} + \rho \right) u^i u^j \right], \quad (i, j = 1, 2, 3; \quad i \neq j)$$

(1.27d)

$$\Lambda = - \frac{8\pi G}{c^4} \left[ \left( \frac{p}{c^2} + \rho \right) u^i u^j - \frac{p}{c^2} \right], \quad (i, j = 1, 2, 3; \quad i \neq j)$$

(1.27e)

From the equation (1.27b), we get

$$u^i = 0, \quad (i = 1, 2, 3)$$

(1.27f)
which in combination with the equation
\[ u^i u_i = c^2 \quad , \quad (i = 1, 2, 3, 4) \quad (1.27g) \]

obtained from the metric of the space-time, we get
\[ u_4 = 0 \quad (1.27h) \]
equations (1.27c) & (1.27e) gives
\[ p = -\frac{\Lambda c^4}{8\pi G}, \quad \rho = \frac{\Lambda c^2}{8\pi G} \quad (1.28) \]
which shows that the density and pressure of perfect fluid are of opposite sign and hence, the result is either physically admissible or trivial. In an attempt to construct a model of the universe Guth [20] has suggested that the early universe had gone through a period of rapid expansion.

For increasing rate of expansion one need anti gravitational force in the model, this may be produced by considering the cosmological term \( \Lambda g_{ij} \) in the Einstein field equations.

### 1.10 OBSERVATIONAL CONSTRAINTS ON THE COSMOLOGICAL CONSTANT

Observations provide various methods for constraining the value of the cosmological constant in our universe because both the spatial geometry and past evolution of the universe are affected by the presence of a cosmological constant. As per our investigation a brief report regarding various techniques and their results are presented in the following table:

<table>
<thead>
<tr>
<th>Techniques</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Redshift Object</td>
<td>Bouncing universe can be ruled out due to the existence of high redshift objects &amp; the cosmic microwave background radiation (CMB).</td>
</tr>
</tbody>
</table>
The Age Problem

One of the important evidence for the existence of cosmological constant has been the fact that the age derived for a universe without cosmological constant is younger than the age derived for the older star. Recently the age problem has found another solution from reports of the Hippocrates satellite.

Statistics of Gravitational Lensing

Lensing results in multiple images of the same object appearing on the sky. A cosmological constant affects the geometry & evolution of the universe and makes the statistics of lensing a powerful technique in putting limits on the value of the cosmological constant in our universe.

High Redshift Supernovae

One of the effects of a cosmological constant is to change the relations of physical distance to redshift. One set of objects exists that seems to be free of evolutionary effects, they are the Ia Supernovae.

Following is the summary of Observational Constraints on the Cosmological Constant:

<table>
<thead>
<tr>
<th>Year</th>
<th>Evidence</th>
<th>Technique</th>
<th>Value of Cosmological Constant</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1996</td>
<td>Kochanek [21]</td>
<td>Gravitational lensing statistics</td>
<td>NA</td>
<td>&lt; 0.66 (95 % confidence)</td>
</tr>
<tr>
<td>1997</td>
<td>Myungshin et al. [22]</td>
<td>Gravitational lensing statistics</td>
<td>0.64 (+0.15, -0.26)</td>
<td>NA</td>
</tr>
<tr>
<td>1998</td>
<td>Riess et al. [23]</td>
<td>High redshift in Supernovae</td>
<td>0.68 ± 0.10</td>
<td>&gt; 0 (98 % confidence)</td>
</tr>
<tr>
<td>1999</td>
<td>Chiba &amp; Yoshi [24]</td>
<td>High redshift in Supernovae</td>
<td>0.7 (+0.1, -0.2)</td>
<td>NA</td>
</tr>
</tbody>
</table>
1.11 WORLD-METRIC

The General Theory of Relativity and Kinematic Theory of Relativity being continuations of Einstein’s Special Theory of Relativity, this extend the Einstein theory in two different directions from logical view point the theories exclude one another, from physical view point they are non equivalent absolutely. The term ‘relativistic’ will always show a relationship to Einstein’s General Theory of Relativity. In this theory there are two suppositions:

Einstein’s equations of gravitation are applicable to the universe as a whole, this consider the Einstein’s equations, with $K$ as defined in equation (1.9)

$$G_{ij} = -K (T_{ij} - \frac{1}{2} g_{ij} T) - \Lambda g_{ij} \quad (i, j = 0, 1, 2, 3)$$

and their applicability in different scales. This also defines the theory of homogeneous universe. Robertson has introduced more mathematically sound statement. This supposition contains the Einstein’s equations with the cosmological constant ‘$\Lambda$’, which sign in an open problem in the contemporary cosmology. In general, $\Lambda$ may be negative, zero or positive. $\Lambda > 0$ had been introduced by Einstein in his theory of static universe. After Friedmann & Hubble’s research, Einstein himself had excluded the constant from the equation because the relativistic gravitational equation of second order filled into their general form.

The contemporary cosmology has a peculiarity to identify the galaxy, Metagalaxies, assume as infinite, with the universe as a whole. For this reason we use the terms ‘neighbourhood’ and ‘large scale’ as second supposition accordingly we assume volumes elementary, of the volumes contains so much galaxies that a matter inside the volumes can be assumed cont. distributed. The assumptions considering relativistic equation of motion and other related results give a possibility to take a coordinate frame, which being at rest in respect to the matter, counts the cosmic universal time which satisfies the conditions:

$$g_{00} = 1, \quad g_{0i} = 0$$
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\( \left( i = 1, 2, 3 \right) \)

The space is a constant curvature space i.e. \( \frac{C}{3} \) as:

\[ C = 3 \frac{k}{R^2}, \quad (k = 0, \pm 1) \quad (1.31) \]

\[ R = R(t) \quad (1.32) \]

The Euclidean spatial coordinates has the metric \( (ds^2) \) known as world-metric and defined as:

\[
ds^2 = c^2 dt - R^2 \left[ \frac{dx^2 + dy^2 + dz^2}{1 + \frac{k}{4} (x^2 + y^2 + z^2)^2} \right] \quad (1.33)\]

1.12 PROPOSED LINK BETWEEN EVOLUTION SCENARIOS OF COSMOLOGICAL MODEL & SPACE CURVATURE.

If \((r, \theta, \phi)\) are the polar coordinates of an arbitrary point then we obtain the condition for homologous state

\[
\frac{\dot{r}}{r} = f(t) \quad (1.34)
\]

with condition of continuity

\[
\dot{\rho} + 3 \frac{\dot{r}}{r} \rho = 0 \quad (1.35)
\]

The equation of motion is

\[
3 \frac{\ddot{r}}{r} = -4\pi \gamma' \rho + \Lambda c^2 \quad (1.36)
\]

where \( \rho \) is its density \& \( \gamma' \) is Gauss constant \( (\gamma' = \frac{Ke^2}{8\pi}) \)

and the integral of energy is
where \( c_i \) is an integration constant.

Introducing the real function of time \( R(t) \), which satisfies the condition

\[
\frac{\dot{R}}{R} = \frac{\dot{r}}{r}, \quad \frac{k}{R^2} = -2 \frac{c_i}{c^2 r^2}, \quad k = 0, \pm 1
\]

under the condition \( p = 0 \), the link between the dynamics and the geometry of the relativistic homogeneous universe may be considered.

As discussed above for \( p = 0 \), the link between the evolution scenario of the models and the space curvature has been shown under the condition mentioned in equation (1.38), which is possible in a homogeneous universe for \( \Lambda > 0 \). The detail has been discussed in chapter 5 (under section 5.2).

### 1.13 N-DIMENSIONAL EINSTEIN’S EQUATIONS

The metric for an N-dimensional space time with flat \( N = n - 1 \) dimensional foliation is:

\[
ds^2 = g_{ij} dx^i dx^j
\]

\[
g_{ij} = [\alpha^2(y) \beta_{kl} \pm 1]
\]

where \( \beta_{kl} \) is the n-dimensional metric for flat Euclidian/Minkowski space & \( y \) is a spacelike/timelike Gaussian normal coordinate for upper/lower signs so that \( ds^2 < 0 \) is the timelike case. Here \( \alpha \) is different from fine structure constant as taken in chapter-2.

The Christoffel symbols are given by

\[
\Gamma^a_{bi} = \frac{1}{2} g^{ae} (g_{eb,c} + g_{ci,b} - g_{bi,c})
\]

so that using metric in equation (1.38), we have
\[ \Gamma^a_{mn} = 0, \quad \Gamma^a_{nk} = 0, \quad \Gamma^k_{nn} = 0 \]
\[ \Gamma^k_{in} = \frac{\alpha'}{\alpha} \beta^k_i, \quad \Gamma^k_{ln} = 0 \]  
(1.43)
\[ \Gamma^a_{kl} = \pm \alpha' \alpha \beta_{kl} \]
and
\[ \Gamma^a_{kl, a} = [\pm \alpha' \alpha \pm \alpha'^2] \beta_{kl} \]  
(1.44)
\[ \Gamma^k_{ln, n} = \left[ \frac{\alpha''}{\alpha} - \frac{\alpha'^2}{\alpha^2} \right] \beta^l_k \]  
(1.45)
where dash represents derivative w. r. t. ‘y’

Here Riemann curvature tensor is defined by:
\[ R^a_{iu} = \Gamma^a_{iv, u} - \Gamma^a_{iu, v} + \Gamma^a_{ib} \Gamma^b_{ij} - \Gamma^a_{bj} \Gamma^b_{iu} \]  
(1.46)
so that
\[ R^a_{iml} = N[\pm \alpha' \alpha \pm \alpha'^2] \beta_{il} \pm N \alpha'^2 \beta_{ab} \pm 2 \alpha'^2 \beta_{il} \]  
(1.47)
\[ R^b_{ij, n} = -N \left[ \frac{\alpha''}{\alpha} - \frac{\alpha'^2}{\alpha^2} \right] \beta^k_i - N \left( \frac{\alpha'}{\alpha} \right)^2 \beta^k_i = -N \frac{\alpha''}{\alpha} \beta^k_i \]  
(1.48)
The Ricci tensor is the contraction of the curvature tensor on the first and third indices, derived as
\[ R_{ij} = R^a_{iaj} = R_{ji} \]  
(1.49)
using these relations we have
\[ R^k_i = g^{kn} R_{mj} = \frac{1}{\alpha^2} \beta^{kn} R_{ml} = \left[ \pm \left( \frac{\alpha'}{\alpha} \right)' \pm N \left( \frac{\alpha'}{\alpha} \right)^2 \right] \delta^k_i \]  
(1.50)
\[ R^a_n = R_{na} g^{mn} = \pm N \frac{\alpha''}{\alpha} \]  
(1.51)
The Ricci scalar is defined as:
\[ R = g^{ij} R_{ij} \]  
(1.52)
using above relation we have
\[ R = \pm 2 N \left( \frac{\alpha'}{\alpha} \right)' \pm N (N + 1) \left( \frac{\alpha'}{\alpha} \right)^2 \]  
(1.53)
since the Einstein tensor is defined as:
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\[ G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \]  (1.54)

so,

\[ G_i^k = R_i^k - \frac{1}{2} g_i^k R \]  (1.55)

Due to symmetry,

\[ G_1^i = G_2^i = \ldots = G_n^i \]  (1.56)

The energy momentum tensor for a scalar field \( \phi \) and potential \( V(\phi) \) is given by

\[ T_j^i = \nabla_i \phi \nabla_j \phi - \frac{1}{2} [\nabla^p \phi \nabla_p \phi + 2V] \delta_j^i \]  (1.57)

so that for \( \phi = \phi(y) \),

\[ T_i^k = -\frac{1}{2} [\pm \phi'^2 + 2V] \delta_i^k \]  (1.58)

**************************************************************************