APPENDIX B
FOUR-ELEMENT VISCOELASTIC MODEL

The four-element viscoelastic model as shown in Fig. 3.1 consists of a spring, a dashpot and a maxwell element in parallel. The stress-strain relation for this model is obtained as follows:

\[ \sigma = \sigma_1 + \sigma_2 + \sigma_3 \]  

(B.1)

where \( \sigma_1 \), \( \sigma_2 \), \( \sigma_3 \) are the stress components in the three elements respectively and \( \sigma \) is the total stress.

Strain \( \varepsilon \) is the same through each of these units, giving

\[ \sigma = (k_1 + c_2q + \frac{k_2c_3q}{k_3 + c_3q})\varepsilon \]  

(B.2)

where \( q = \frac{\partial}{\partial t} \), \( k_1 \) and \( k_3 \) are the moduli of springs and \( c_2 \), \( c_3 \) are viscosities of the dashpots.

The dynamic modulus \( Y (= \sigma/\varepsilon) \) of the viscoelastic material given by Eq. (B.2) can be written as:

\[ Y = \frac{\beta + \gamma q + \omega q^2}{1 + \alpha q} \]  

(B.3)

where \( \alpha = c_3/k_3 \), \( \beta = k_1 \), \( \gamma = c_2 + c_3 + k_1c_3/k_3 \) and \( \omega = c_2c_3/k_3 \).

If the time dependence of both stress and strain is taken in the form of \( e^{i\omega t} \), operator \( q \) can be replaced by \( j\omega \), where \( j = (-1)^{1/2} \) and \( \omega \) is the angular frequency of sinusoidal excitation. This substitution in Eq. (B.2) yields:
Separating real and imaginary parts and putting $Y = Y_S + jY_L$ (where $Y_S$ and $Y_L$ are respectively the storage and loss moduli of the viscoelastic material), one obtains:

$$Y_S = k_1 + k_3 - \frac{j\omega c_2}{k_3 + j\omega c_3}$$  \hspace{1cm} (B.5)

and

$$Y_L = \omega (c_2 + \frac{c_2 k_3^2}{k_3^2 + \omega^2 c_3^2})$$  \hspace{1cm} (B.6)

The loss factor $\eta (= Y_L/Y_S)$ of the viscoelastic material is given by

$$\eta = \frac{\omega [k_3^2 (c_2 + c_3) + \omega^2 c_2 c_3^2]}{[\omega^2 c_3^2 (k_1 + k_3) + k_1 k_3^2]}$$  \hspace{1cm} (B.7)

Numerical values of the viscoelastic model constants used for the evaluation of results in Chapter 3 are substituted in Eqs. (B.5) and (B.6) to obtain the characteristics of storage and loss moduli at different frequencies of vibration. These characteristics are plotted in Fig. B.1 which shows that the storage and loss moduli increase as the frequency of vibration increases. The general behaviour of the four-element model is typical of that exhibited by many viscoelastic materials used in practice.
$c_2 = 1176.84 \text{ Ns/m}^2$
$c_3 = 392.26 \text{ Ns/m}^2$
$k_3 = 10.7877 \text{ MN/m}^2$
$k_1 = \text{ELEMENT OF MODEL (FIG. 3.1)}$
$Y_S = \text{STORAGE MODULUS}$
$Y_L = \text{LOSS MODULUS}$

FIG. B.1 VARIATION OF $(Y_S - k_1)$ AND $Y_L$ WITH $\omega$