Introduction

The introduction of the concept of a fuzzy subset of a nonempty set $X$ as a function of $X$ into the unit interval $[0, 1]$ by L. Zadch [36] initiated several algebraists to take up the study of fuzzy subalgebras of various algebraic systems such as groups, rings, modules, lattices etc. Rosenfeld [23] defined the notion of a fuzzy subgroup of a group and since then several researchers worked on fuzzy subrings and ideals of rings (Malik and Mordeson [19] and [20]), fuzzy ideals of lattices (Attallah [2], Lehmke [18] and U.M.Swamy and D.V.Raju [28]), fuzzy semi prime ideals of a semi ring (Dutta and Biswas [7]), fuzzy maximal ideals of Gamma near-rings (Jun, Kim and Oztirk [12]), fuzzy prime ideals of lattices and hyper lattices (Koguep, N kuimi and Lele [14] and [15]), fuzzy prime ideals of rings (U.M.Swamy and K.L.N.Swamy [25]), algebraic fuzzy systems and irreducibility (U.M.Swamy and D.V.Raju [26] and [27]), fuzzy ideals of a ring (Mukharjee and sen [22]), fuzzy groups and level subgroups (Das [5]), fuzzy pseudo ideals in semi groups (Dutta [8]), fuzzy vector spaces and fuzzy topological vector spaces (Katsaras and Liu [13]), etc.

In most of the works mentioned above, the fuzzy statements take truth values in the interval $[0,1]$ of real numbers, while the conventional (or crisp) state-
ments take truth values in the two-element set \( \{F, T\} \) or \( \{0, 1\} \), where \( F \) and 0 stand for 'false' and T or 1 stand for 'true'. However, Gougen [10] realized that the unit interval \([0, 1]\) is insufficient to have the truth values of certain fuzzy statements. For example, consider the statement ‘Andhra University is a good university’. The truth value of this statement may not be a real number in \([0, 1]\). Being good is fuzzy. Being good university may have several components; good in instructional facilities, good with respect to climatic conditions, good faculty, good for research facilities, good living conditions, having a good library and laboratories, good in hostel facilities, good in cleanliness, etc. The truth value corresponding to each component may be a real number in \([0,1]\). If \( n \) is the number of such components under consideration, the truth of the said statement is a \( n \)-type of real numbers in \([0, 1]\); that is, an element of \([0,1]^n\), which is not a totally ordered set, when \( n > 1 \), under the coordinate wise ordering, where\([0,1]\) is considered with the usual ordering of real numbers.

Though \([0,1]^n\) is not totally ordered, when \( n > 1 \), it has certain rich lattice theoretic properties. For example, \([0,1]^n\) is a complete lattice satisfying the infinite distributivity; that is, every subset of \([0,1]^n\) has greatest lower bound (glb) and least upper bound (lub) and, for any element \( a \) and for any subset \( X \) of \([0,1]^n\),

\[
a \land (lub X) = lub \{a \land x \mid x \in X\}
\]

where \( \land \) denotes the glb. To make an abstract study, we consider a general complete
lattice satisfying the infinite meet distributivity to have truth values of fuzzy statements. This type of lattice is called a frame. If \( U \) denotes the set of all universities on this globe, then the collection \( G \) of good universities is actually not a subset of \( U \), but it is a fuzzy subset of \( U \), since being good is fuzzy; that is, \( G \) can be considered as a function from \( U \) into a lattice \( L \) of the type discussed above, such a fuzzy subset of \( U \) is called an \( L \)-fuzzy subset of \( U \). In particular, when \( U \) is an algebra of given type \( \mathcal{F} \) and \( L \) is a frame, an \( L \)-fuzzy subset \( F \) of \( U \) is called an \( L \)-fuzzy subalgebra of \( A \) if \( F \) is compatible, in same sense, with all the fundamental operations on the algebra \( A \).

Let us recall that a language (or type) of algebras is a set \( \mathcal{F} \) together with a mapping \( a \) of \( \mathcal{F} \) into the set of nonnegative integers, \( a \) is called the arity map and, for any \( f \in \mathcal{F} \), the integer \( a(f) \) is called the arity of \( f \). If \( a(f) = n \), then \( f \) is called \( n \)-ary. The members of \( \mathcal{F} \) are called operation symbols. 0-ary symbols are called nullary, 1-ary symbols unary, 2-ary symbols binary, etc. For any type \( \mathcal{F} \) of algebras, an universal algebra or, simply an algebra, is a nonempty set together with a set \( \{ f^A \mid f \in \mathcal{F} \} \) of operations on \( A \) such that \( f^A \) is a \( n \)-ary operation (that is, a mapping of \( A^n \) into \( A \)) on \( A \), where \( n \) is the arity of \( f \). The operations \( f^A, f \in \mathcal{F} \), are called the fundamental operations on \( A \). If \( n = 0 \), \( A^0 \) is logically a singleton set and hence, for any nullary \( f \in \mathcal{F}, f^A \) can be identified with an element of \( A \).

In most of the results in this thesis, we consider algebras with at least one
nullary fundamental operation. This is the case with almost all the algebras we are familiar with, like groups, rings, modules, vector space etc. For any algebra $A$, the underlying set $A$ is called the universe of $A$ and a nonempty subset $B$ of $A$ is called a subalgebra (or subuniverse) of $A$ if the following are satisfied.

(1) $f^A \in B$ for all nullary of $f \in \mathcal{F}$ and

(2) for any $n$-ary $f \in \mathcal{F}$, $n > 0$, and $b_1, b_2, ..., b_n \in B$,

$$f^A(b_1, b_2, ..., b_n) \in B.$$ 

For any algebra $A$ of any type $\mathcal{F}$ and for any frame $L$, an $L$-fuzzy subset $F$ of $A$ is called an $L$-fuzzy subalgebra of $A$ if $F(f^A) = 1$, the largest element of $L$, for all nullary $f \in \mathcal{F}$ and, for any $n$-ary $f \in \mathcal{F}$, $n > 0$,

$$F(a_1) \land ... \land F(a_n) \leq F(f^A(a_1, ..., a_n))$$

for all $a_1, ..., a_n \in A$. In this thesis we make an extensive study of $L$-fuzzy subalgebras of an arbitrary algebra, when $L$ is an arbitrary frame.

The Thesis is broadly divided into four chapters 0, 1, 2, and 3. Chapter 0 is devoted to collect all the necessary preliminaries which will be useful in our discussions in the main text of the thesis. Even though these preliminaries are well known for those working in Universal Algebra and/or Lattice Theory, it will be convenient for others to have all these elementary notions and results in the
beginning of the thesis for the sake of ready reference and completeness of the thesis. The proofs of most of the results presented in chapter 0 are either straightforward verifications or well known and hence we simply state the results and skip the proofs.

The main text of the thesis is in chapters 1, 2 and 3. Chapter 1 is on general fuzzy subalgebras and is subdivided into three sections. In section 1.1, we discuss on distributivities in lattice. As mentioned above, the interval [0, 1] is insufficient to contain the truth values of general fuzzy statements and it is necessary to consider a more general class of lattices in place of [0, 1]. U.M. Swamy and others in [25] through [33] used a complete lattice satisfying the infinite meet distributivity, which are called frames, to have the truth values of general fuzzy statements. For this reason, we make a thorough discussion on various types of distributivities in lattices in section 1.1. Section 1.2 is devoted to a study $L$-fuzzy subsets of an arbitrary set, where $L$ is an arbitrary frame. In section 1.3, we consider an (universal) algebra $A$ and introduce the notion of an $L$-fuzzy subalgebra of $A$ and discuss certain general properties of these, by observing that these form an algebraic fuzzy set system.

Chapter 2 is on prime and maximal fuzzy subalgebras of an algebra $A$ and is subdivided into six sections. In section 2.1, we discuss irreducible and prime elements in general lattices in order to investigate these in the lattices of (crisp) subalgebras and fuzzy subalgebras. Section 2.2 is devoted to a study of certain
general properties of prime and maximal subalgebras of a given algebra.

In section 2.3, we introduce the notion of a prime $L$-fuzzy subalgebra of an algebra $A$ as simply a prime element in the lattice of $L$-fuzzy subalgebras; that is, an $L$-fuzzy subalgebra $P$ of $A$ is called prime if $P$ is not the constant map $\top$ and, for any $L$-fuzzy subalgebras $F$ and $G$ of $A$,

$$F \land G \leq P \Rightarrow \text{either } F \leq P \text{ or } G \leq P.$$ 

Here, we have mainly proved a characterization theorem for prime $L$-fuzzy subalgebras which states that an $L$-fuzzy subalgebra $P$ of $A$ is prime if and only if there is a prime subalgebra $B$ of $A$ and a prime element $\alpha$ in $L$ such that

$$P(a) = \begin{cases} 
1 & \text{if } a \in B \\
\alpha & \text{if } a \notin B.
\end{cases}$$

This proves that there is a one-to-one correspondence between prime $L$-fuzzy subalgebra of $A$ and the pairs $(B, \alpha)$ where $B$ is a prime subalgebra of $A$ and $\alpha$ is a prime element in the frame $L$.

In section 2.4, we discuss the $L$-fuzzy subsets $F$ of an algebra $A$ for which each $\alpha$-cut is either $A$ or a prime subalgebra of $A$ and there are termed as $L$-fuzzy prime subalgebras. We have proved that every prime $L$-fuzzy subalgebra is an $L$-fuzzy prime subalgebra and that the converse is not true, by providing counter examples. Also we have proved that a proper $L$-fuzzy subalgebra $P$ of $A$ is an $L$-fuzzy prime subalgebra if and only if $P(A)$ is a chain in $L$ and, for any elements $a$ and $b$ in $A$,

$$P(a) \lor P(b) = \text{Inf}\{P(x) \mid x \in <a> \cap <b>\}$$
where $< a >$ and $< b >$ are subalgebras of $A$ generated by $a$ and $b$ respectively. Further, we have proved that the 1-cut of an $L$-fuzzy prime subalgebra of $A$ is a prime subalgebra and that the converse is not true, by exhibiting a countable example.

In section 2.5, we discuss maximal $L$-fuzzy subalgebras of an algebra $A$ which are precisely maximal members in the set of proper $L$-fuzzy subalgebras of $A$ or, equivalently, the dual atoms in the lattice $\mathcal{F}\mathcal{S}_L(A)$ of $L$-fuzzy subalgebras of $A$. We have proved that $F$ is a maximal $L$-fuzzy subalgebra of $A$ if and only if there exist a maximal subalgebra $M$ of $A$ and a dual atom $\alpha$ in $L$ such that

$$F(a) = \begin{cases} 1 & \text{if } a \in M \\ \alpha & \text{if } a \notin M. \end{cases}$$

In such a case, $F$ is denoted by $\alpha_M$.

This yields a one-to-one correspondence between maximal $L$-fuzzy subalgebras of $A$ and pairs $(M, \alpha)$, where $M$ is a maximal subalgebra of $A$ and $\alpha$ is a dual atom in $L$. When $L$ is no dual atoms (for example, $[0,1]^n$ is one such, for any positive integer $n$), then any algebra $A$ has no maximal $L$-fuzzy subalgebras.

Finally, in section 2.6, we introduce the notion of an $L$-fuzzy maximal subalgebra as a proper $L$-fuzzy subalgebra each of whose $\alpha$-cuts are either a maximal subalgebra of $A$ or the whole of $A$. We have proved that $F$ is an $L$-fuzzy maximal subalgebra if and only if $F = \alpha_M$ for some maximal subalgebra $M$ of $A$ and $\alpha < 1$ in $L$. From this, we have derived that every maximal $L$-fuzzy subalgebra of $A$ is an $L$-fuzzy maximal subalgebra. Counter examples are provided to disprove the
Chapter 3 is on $L$-fuzzy congruences. For any algebra $A$ of type $\mathcal{F}$, an equivalence relation on $A$ is called a congruence if it is compatible with all the fundamental operations on $A$. It is known that congruences on a group (ring or $R$-module) are in one-to-one correspondence with the normal subgroups (ideals or $R$-submodules respectively). Further congruences on any algebra $A$ of type $\mathcal{F}$ are precisely same as subalgebras the augmented algebra $A \times A$ of $\mathcal{F}'$, where $\mathcal{F}'$ is $\mathcal{F}$ together a nullary operation corresponding to each $(a, a) \in A \times A$, a unary operation $s$ defined by $s(a, b) = (b, a)$ for all $(a, b) \in A \times A$ and a binary operation $t$ on $A \times A$ defined by

$$t((a, b), (c, d)) = \begin{cases} (a, d) & \text{if } b = c \\ (a, b) & \text{if } b \neq c \end{cases}$$

with this in view, congruences on algebra $A$ can be treated as subalgebras of some algebraic structure on $A \times A$ and hence all the concepts and results on subalgebras or $L$-fuzzy subalgebras can be easily extended to those on congruences and $L$-fuzzy congruences. In section 3.1, we formally define the notion of an $L$-fuzzy congruence on an algebra and state certain equivalent formulations. In section 3.2, we introduce the notions of prime $L$-fuzzy congruences and $L$-fuzzy prime congruences and state the results on these analogous to those of $L$-fuzzy subalgebras in sections 2.3 and 2.4. Since the proofs of these are analogous to those in sections 2.3 and 2.4, we simply state these and skip the proofs. In section 3.3, we briefly discuss the notions and results on maximal $L$-fuzzy congruences and $L$-fuzzy maximal congruences on an
algebra and simply state the results by skipping the proof, since these are analogous to those corresponding results in sections 2.5 and 2.6.