Chapter 5

Connectivity of es-Splitting Binary Matroids

In this chapter, we explore the connectivity of es-Splitting binary matroids. We provide a sufficient condition for a 3-connected binary matroid which yields a 3-connected binary matroid by the es-splitting operation. Further, given an $n$-connected binary matroid $M$ and $X \subseteq E(M)$, we investigate $n$-connected minors of the es-splitting matroid $M_X^e$.

5.1 Introduction

As mentioned earlier, Slater [38] specified the $n$-line splitting operation on a graph on the way to his classification of 4-connected graphs. He proved that, an $n$-connected graph yields an $n$-connected graph by the $n$-line splitting operation. Shikare and Azanchiler[14] extended $n$-line splitting operation from graphs to binary matroids. Azanchiler[15]
proved that $es$-splitting matroid arising from a connected binary matroid is a connected binary matroid. In fact, he proved the following theorem.

**Theorem 5.1.1.** Let $M$ be a connected binary matroid and $X \subset E(M)$. Then $M_X^e$ is a connected binary matroid.

Note that the $es$-splitting operation on a 3-connected binary matroid may not yield a 3-connected binary matroid.

In the second section of this chapter, we provide a sufficient condition for a 3-connected binary matroid which yields a 3-connected binary matroid by the $es$-splitting operation. We derive splitting lemma for 3-connected matroids from the results obtained in the process.

For $n \geq 4$, the $es$-splitting operation on an $n$-connected binary matroid need not produce an $n$-connected binary matroid because it contains a triangle $\{a, e, \gamma\}$. But it may contain $n$-connected minors. In the third section, we provide a procedure to obtain $n$-connected minors of rank $(r + 1)$ of the $es$-splitting matroid where the original matroid is an $n$-connected matroid of rank $r$. The minors have the property that each of them has exactly one element more than $M$.

### 5.2 Splitting Lemma for $es$-Splitting Matroids

It is observed that the $es$-splitting operation on a 3-connected binary matroid may not yield a 3-connected binary matroid. In the following result, we provide a sufficient condition under which the $es$-splitting operation on a 3-connected binary matroid yields a 3-connected binary
5.2 Splitting Lemma for es-Splitting Matroids

Let $M$ be a 3-connected binary matroid, $X \subseteq E(M)$ and $e \in X$. Suppose that $M$ has an $OX$-circuit not containing $e$. Then $M^e_X$ is a 3-connected binary matroid.

**Proof.** By Theorem 5.1.1, $M^e_X$ is connected. Suppose that $M^e_X$ is not 3-connected. Then $E(M^e_X)$ has a 2-separation $(A, B)$ such that $\min\{|A|, |B|\} \geq 2$ and

$$r'(A) + r'(B) - r'(M^e_X) \leq 1. \quad (5.2.1)$$

Now one of the following two cases occurs.

**Case (I)** $a \in A$ and $\{a, \gamma\} \not\subset A$.

Let $A' = A \setminus a$ and $B' = B \setminus \gamma$. Then one of the following three subcases occurs.

**Subcase (i)** $|A| = 2$ and $A = \{a, x\}$ for some $x \in E(M)$. Then, by Lemma 4.1.1 (2), $r'(A) = r(A') + 1$. If $x = e$, then the $OX$-circuit $C$ of $M$ not containing $e$ is contained in $B$. So, by Lemma 4.1.1 (3), $r'(B) = r(B') + 1$. Further, if $x \neq e$, then $e \in B'$ and, by Lemma 4.1.1 (3), $r'(B) = r(B') + 1$. By inequality (5.2.1), $r(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. This implies that, $(A', B')$ is a 1-separation of $M$, a contradiction.

**Subcase (ii)** $|B| = 2$ and $B = \{\gamma, x\}$ where $x \in E(M)$. Then, by Lemma 4.1.1 (2) and (3), $r'(A) = r(A') + 1$ and $r'(B) \geq r(B') + 1$. Thus, by inequality (5.2.1), we have $r(A') + r(B') - r(M) \leq 0$ and $|A'|, |B'| \geq 1$. Again $(A', B')$ is a 1-separation of $M$, a contradiction.

**Subcase (iii)** $\min\{|A|, |B|\} > 2$. Then $|A'|, |B'| \geq 2$. So, by Lemma
4.1.1 (2) and (3), $r'(B) \geq r(B')$ and $r'(A) \geq r(A') + 1$. Then, inequality (5.2.1), implies that $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. We conclude that $(A', B')$ is a 2-separation of $M$; a contradiction.

**Case (II) $\{a, \gamma\} \subset A$.**

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. Then one of the following two sub cases occurs.

**Subcase (i) $A = \{a, \gamma\}$.**

Then $r'(A) = 2$ and $A' = \phi$. Since $B = B' = E(M)$ and $M$ contains a circuit $C$ containing odd number of elements of $X \setminus e$, $C \subseteq B$. Then, by Lemma 4.1.1 (1), $r'(B) = r(B') + 1 = r(M) + 1$. Consequently, by inequality (5.2.1), we get $r(M) \leq r(M) - 1$; a contradiction.

**Subcase (ii) $A = \{a, \gamma, x\}$ where $x \in E(M)$.**

Note that $A' = \{x\}$ and $cl(A') = A'$. Then, by Lemma 4.1.1 (4), we have

\[
 r'(A) = \begin{cases} 
 r(A') + 1, & \text{if } e \in cl(A') \\
 r(A') + 2, & \text{if } e \notin cl(A') 
\end{cases} 
\]

If $x = e$, then $r'(A) = r(A') + 1$. Since, $M$ has an $OX$-circuit $C$ not containing $e$, $C \subseteq B$. Hence, by Lemma 4.1.1 (1), $r'(B) = r(B') + 1$ and, by inequality (5.2.1), we have $r(A') + r(B') - r(M) \leq 0$ with $|A'|, |B'| \geq 1$. Thus, $(A', B')$ is a 1-separation of $M$, a contradiction.

If $x \neq e$, then, by (1) and (4) of Lemma 4.1.1, $r'(A) = r(A') + 2$ and $r'(B) \geq r(B')$. Consequently, by inequality (5.2.1), $r(A') + r(B') - r(M) \leq 0$ where $|A'|, |B'| \geq 1$. This implies that $(A', B')$ gives a 1-separation of $M$; a contradiction.

**Subcase (iii) $|A| > 3$.**
5.3 Effect of the es-Splitting Operation on Wheels

Then $|A'|, |B'| \geq 2$. If $B$ contains an $OX$-circuit of $M$, then by (1) and (4) of Lemma 4.1.1, $r'(B) = r(B') + 1$ and $r'(A) \geq r(A') + 1$. So, by inequality (5.2.1), $r(A') + r(B') - r(M) \leq 0$ where $|A'|, |B'| \geq 1$. Hence $(A', B')$ is a 1-separation of $M$; a contradiction.

If $B$ contains no $OX$-circuit of $M$, then, by Lemma 4.1.1 (1) and (4), $r'(B) = r(B')$ and $r'(A) \geq r(A') + 1$. Consequently, by inequality (5.2.1), $r(A') + r(B') - r(M) \leq 1$ where $|A'|, |B'| \geq 2$. We conclude that $M$ has a 2-separation; a contradiction.

The above facts imply that $M_X^c$ cannot have a 2-separation. Hence, $M_X^c$ is 3-connected.

As an immediate consequence of Theorem 5.2.1, we obtain a Splitting Lemma for 3-connected binary matroids when $X = \{x, y\} \subseteq E(M)$.

**Corollary 5.2.2. (Splitting Lemma):** If $M$ is a 3-connected binary matroid and $\{x, y\} \subseteq E(M)$, then $M_{x,y}^c$ is a 3-connected binary matroid.

**Proof.** $M$ is a 3-connected binary matroid so no two elements of $M$ are in a series. Thus, there is a circuit of $M$ containing $x$ or $y$ but not both. Hence, by Theorem 5.2.1, $M_{x,y}^c$ is 3-connected.

5.3 Effect of the es-Splitting Operation on Wheels

Two particularly important families of 3-connected matroids are the wheels and the whirls. For $r \geq 2$, the wheel $W_r$ of rank $r$ is a graph having $r + 1$ vertices, $r$ of which lie on a cycle (the rim); the remaining vertex is joined by a single edge (a spoke) to each of the other vertices.
The rank-\(r\) whirl \(W^r\) is the matroid on \(E(W_r)\) that has its circuits all cycles of \(W_r\) other than the rim as well as all sets of edges formed by adding a single spoke to the edges of the rim. The smallest 3-connected whirl is \(W^2\), which is isomorphic to \(U_{2,4}\); the smallest 3-connected wheel is \(M(W_3)\), which is isomorphic to \(M(K_4)\).

The es-splitting operation on a wheel may not yield a wheel. In Theorem 5.3.3, we give a sufficient condition for the es-splitting operation to yield a wheel from a wheel. In the proof of Theorem 5.3.3, we use Seymour’s Splitter Theorem [36] and a corollary of Theorem 4.3.14.

**Theorem 5.3.1.** (The Splitter Theorem) Let \(N\) be a connected, simple, cosimple minor of a 3-connected matroid \(M\). Suppose that if \(N\) is a wheel, then \(M\) has no larger wheel as a minor, while if \(N\) is a whirl, then \(M\) has no larger whirl as a minor. Then either \(M = N\), or \(M\) has a connected simple, cosimple minor \(M_1\) such that some single element deletion or some single element contraction of \(M_1\) is isomorphic to \(N\). Moreover, if \(N\) is 3-connected, so too is \(M_1\).

We denote by \(C^*(M)\) the set of all cocircuits of \(M\). In Chapter 4, we described the cocircuits of \(M_X^c\) in terms of the cocircuits of \(M\). The following corollary of Theorem 4.3.14 is used in the proof of Theorem 5.3.3.

**Corollary 5.3.2.** Let \(M\) be a binary matroid on a set \(E\) and \(M_X^c\) be the splitting matroid of \(M\) with respect to a set \(X = \{x, y\} \subseteq E\). Suppose \(X\) does not contain any cocircuit of \(M\). Then \(X \cup \{a, \gamma\}\) and the members of the collection \(\{(C^* - X) \cup \{a, \gamma\} \mid C^* \in C^*(M)\ \text{and} \ X \subseteq C^*\}\) are cocircuits of \(M_X^c\).
**Theorem 5.3.3.** Let $M$ be a wheel and $\{x, y, z\}$ be a triad in $M$. If $x$ and $y$ are rim elements and $z$ is a spoke element, then each of the matroids $M_{x,z}^e$ and $M_{y,z}^e$ is a wheel.

**Proof.** Let $M$ be a wheel and $\{x, y, z\}$ be a triad in $M$. If $x$ and $y$ are rim elements and $z$ is a spoke element, then by Corollary 5.2.2, each of the matroids $M_{x,z}^e$ and $M_{y,z}^e$ is a 3-connected binary matroid. On the contrary, suppose the matroids $M_{x,z}^e$ and $M_{y,z}^e$ are not wheels. In $M_{x,z}^e$, if $z = e$, then, by Corollary in 5.3.2, $\{x, z, a\}$ and $\{a, \gamma, y\}$ are cocircuits and $\{a, \gamma, z\}$ is a triangle of $M_{x,z}^e$. Therefore, by deletion or contraction of $a$ or $\gamma$ from $M_{x,z}^e$ does not give a 3-connected matroid. Hence, every $p \in E(M)$ is in some triangle as well as in a triad of $M_{x,z}^e$. Therefore, neither $M_{x,z}^e \setminus p$ nor $M_{x,z}^e/p$ is 3-connected. We conclude that there does not exist a 3-connected minor $M'$ of $M_{x,z}^e$ such that for some $q \in E(M')$, $M' \setminus q$ or $M'/q$ is 3-connected and isomorphic with $M$. In view of Splitter Theorem, this is a contradiction to the existence of such a matroid $M'$. We conclude that $M_{x,z}^e$ is a wheel. By similar arguments, we can show that $M_{y,z}^e$ is a wheel. \[\square\]

### 5.4 $n$-Connected Minors of es-Splitting Matroids

In this section, we prove that given an $n$-connected binary matroid $M$ ($n \geq 3$) of rank $r$, $M_X^e \setminus a$, $M_X^e \setminus e$ and $M_X^e \setminus \gamma$ are the only $n$-connected minors of rank $r + 1$ of the es-splitting matroid $M_X^e$.

Let $M$ be an $n$-connected binary matroid ($n \geq 3$), $X \subseteq E(M)$ and $e \in X$. Suppose that $M$ has an $OX$-circuit not containing $e$. Then, by
Theorem 5.2.1, the binary matroid $M^e_X$ is 3-connected. Note that the matroid $M^e_X$ contains a triangle $\triangle = \{a, e, \gamma\}$. Hence, by Proposition 1.2.46, $M^e_X$ is not 4-connected. We observe that for any $x \in E(M^e_X)$, $M^e_X/x$ contains a 2-circuit or a triangle and thus it is not 4-connected. Further, for any $x \in (E(M^e_X) - \triangle)$, the minor $M^e_X \setminus x$ contains a triangle and therefore, it is not 4-connected. Thus, the possible 4-connected minors of $M^e_X$ are (i) $M^e_X \setminus a$ (ii) $M^e_X \setminus e$ and (iii) $M^e_X \setminus \gamma$. In this section, we provide a sufficient condition for an $n$-connected binary matroid $M$ so that $M^e_X \setminus a$, $M^e_X \setminus e$ and $M^e_X \setminus \gamma$ are $n$-connected minors of $M^e_X$.

Initially, in the following lemma, we provide a sufficient condition for a 3-connected binary matroid $M$ for which $M^e_X \setminus a$ is a 3-connected minor of $M^e_X$.

**Lemma 5.4.1.** Let $M$ be a 3-connected binary matroid and $|E(M)| \geq 4$. Let $X \subset E(M)$ with $|X| \geq 3$. Suppose for $x \in E(M)$ there is an $OX$-circuit of $M$ not containing $x$ and there is no $OX$-circuit containing $e$. Then $M^e_X \setminus a$ is a 3-connected binary matroid.

**Proof.** If $x = e$, then, by Theorem 5.2.1, $M^e_X$ is 3-connected. Now, $M^e_X \setminus a$ is at least 2-connected. Suppose $M^e_X \setminus a$ is not 3-connected and let $(A, B)$ be a 2-separation of $E(M^e_X \setminus a)$. Then, min \{$|A|, |B|$\} $\geq 2$ and

$$r'(A) + r'(B) - r'(M^e_X \setminus a) \leq 1. \quad (5.4.1)$$

Let $\{\gamma\} \subset A, A' = A \setminus \gamma$ and $B' = B$. Now one of the following two cases occurs.
**Case (I) |A| = 2.**

Suppose $A = \{z, \gamma\}$ and $A' = \{z\}$ where $z \in E(M)$. Therefore, by Lemma 4.1.1 (3), $r'(A) = r(A') + 1$. Now $M$ contains an $OX$-circuit $C$ of $M$ and $\{z\} \cap C = \phi$, implies $C \subseteq B$. Then, by Lemma 4.1.1 (1), $r'(B) = r(B') + 1$. Thus, by (5.4.1), we have

$$r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 1.$$  

That is, $r(A') + r(B') - r(M) \leq 0$, and $|A'|, |B'| \geq 1$. This implies that $(A', B')$ is a 1-separation of $M$, a contradiction.

**Case (II) |A| > 2.**

Since, no $OX$-circuit of $M$ contains $e$, by Lemma 4.1.1 (3), $r'(A) \geq r(A') + 1$ and, by Lemma 4.1.1 (1), $r'(B) \geq r(B')$. Consequently, by inequality (5.4.1), we get $r(A') + 1 + r(B') - r(M) - 1 \leq 1$, with $|A'|, |B'| \geq 2$. This leads to a 2-separation of $M$, again a contradiction. Thus, $M_X^e \setminus a$ has no 2-separation. We conclude that $M_X^e \setminus a$ is a 3-connected binary matroid.

We extend the above lemma to $n$-connected binary matroids. In fact, the following theorem provides a sufficient condition for an $n$-connected binary matroid where $M_X^e \setminus a$ is $n$-connected.

**Theorem 5.4.2.** Let $M$ be an $n$-connected binary matroid for integer $n \geq 4$, $|E(M)| \geq 2(n - 1)$ and let $X \subset E(M)$ with $|X| \geq n$. Suppose that for any $(n - 2)$-element subset $S$ of $E(M)$ there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \phi$ and there is no $OX$-circuit containing $e$. Then $M_X^e \setminus a$ is $n$-connected.
Proof. The proof is by induction on $n$. First we prove the case $n = 4$. By Lemma 5.2.1, the matroid $M^e_X \setminus a$ is 3-connected. To prove that $M^e_X \setminus a$ is 4-connected, it is enough to show that $M^e_X \setminus a$ has no 3-separation. On the contrary, suppose that $(A, B)$ be a 3-separation of $M^e_X \setminus a$. Then, $\min \{|A|, |B|\} \geq 3$ and

$$r'(A) + r'(B) - r'(M^e_X \setminus a) \leq 2.$$  \hfill (5.4.2)

Suppose $\{\gamma\} \subset A$. Let $A' = A \setminus \gamma$ and $B' = B$. Now one of the following two cases occurs.

**Case (I) $|A| = 3$.**

Let $A = \{x, y, \gamma\}$ where $x, y \in E(M)$. Then, by Lemma 4.1.1 (3), $r'(A) = r(A') + 1$. $B'$ contains an $OX$-circuit $C$ of $M$ such that $\{x, y\} \cap C = \emptyset$. Then $C \subset B$ and by Lemma 4.1.1 (1), $r'(B) = r(B') + 1$. Hence, by inequality (5.4.2), $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 2$. Consequently, $r(A') + r(B') - r(M) \leq 1$ and $|A'|, |B'| \geq 2$. We conclude that $(A', B')$ is a 2-separation of $M$; a contradiction.

**Case (II) $|A| > 3$.**

Since, no $OX$-circuit of $M$ contains $e$, by Lemma 4.1.1 (3), $r'(A) \geq r(A') + 1$ and, by Lemma 4.1.1 (1), $r'(B) \geq r(B')$. Consequently, by inequality (5.4.2), we get $r(A') + 1 + r(B') - r(M) - 1 \leq 2$. That is, $r(A') + r(B') - r(M) \leq 2$ and $|A'|, |B'| \geq 3$. This leads to a 3-separation of $M$; a contradiction.

Therefore, $M^e_X \setminus a$ has no 3-separation. We conclude that $M^e_X \setminus a$ is a 4-connected binary matroid.

Now we assume that the result is true for $k = 4$ and prove that the result is true for $k + 1$. 
Let $M$ be a $(k + 1)$-connected binary matroid and $M^c_X$ be the es-splitting matroid of $M$ and any $(k - 1)$-element subset $S$ of $E(M)$ there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \phi$. Note that $M^c_X \setminus a$ is $k$-connected by induction hypothesis. Thus, it is enough to show that $M^c_X \setminus a$ has no $k$-separation.

On the contrary, suppose $M^c_X \setminus a$ is not $(k+1)$-connected. Let $(A, B)$ be a $k$-separation of $E(M^c_X \setminus a)$. Then, $\min \{|A|, |B|\} \geq k$, and

$$r'(A) + r'(B) - r'(M^c_X \setminus a) \leq k - 1.$$ (5.4.3)

Suppose $\{\gamma\} \subset A, A' = A \setminus \gamma$ and $B' = B$. Then one of the following two cases occurs.

**Case (I) $|A| = k$.**

Let $A = \{x_1, x_2, \ldots, x_{k-1}, \gamma\}$ where $x_i \in E(M)$ for $i = 1$ to $k - 1$. As $M$ is $(k + 1)$-connected, $A'$ does not contain any circuit. Then, by Lemma 4.1.1 (3), we have $r'(A) = r(A') + 1$. Let $S = \{x_1, x_2, \ldots, x_{k-1}\}$. So, there is an $OX$-circuit $C$ of $M$ such that $S \cap C = \phi$. This implies $C \subset B$ and then, by Lemma 4.1.1 (1), $r'(B) = r(B') + 1$. Hence, by inequality (5.4.3), we have $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq k - 1$. That is, $r(A') + r(B') - r(M) \leq k - 2$ and $|A'|, |B'| \geq k - 1$. This implies that $(A', B')$ is a $(k - 1)$-separation of $M$, a contradiction.

**Case (II) $|A| > k$.**

Since, no $OX$-circuit of $M$ contains $e$, by Lemma 4.1.1 (3), $r'(A) \geq r(A') + 1$ and, by Lemma 4.1.1 (1), $r'(B) \geq r(B')$. Consequently, by inequality (5.4.2), we get $r(A') + 1 + r(B') - r(M) - 1 \leq k - 1$. That is, $r(A') + r(B') - r(M) \leq k - 1$ and $|A'|, |B'| \geq k$. Hence we get a $k$-separation of $M$, a contradiction.
Thus, $M_X^e \setminus a$ has no $k$-separation. We conclude that $M_X^e \setminus a$ is $(k+1)$-connected. Thus, by principle of mathematical induction, the result is true for all $n \geq 4$.

Now we provide a sufficient condition for a 3-connected binary matroid so that $M_X^e \setminus e$ is a 3-connected minor of $M_X^e$.

**Lemma 5.4.3.** Let $M$ be a 3-connected binary matroid, $|E(M)| \geq 4$ and let $X \subset E(M)$, where $|X| \geq 3$. Suppose for $x \in E(M)$ there is an $OX$-circuit of $M$ not containing $x$ and there is no $OX$-circuit containing $e$. Then $M_X^e \setminus e$ is a 3-connected binary matroid.

**Proof.** Suppose that $M$ contains an $OX$-circuit $C$ of $M$ and $e \notin C$. Then, by Theorem 5.2.1, $M_X^e$ is 3-connected. Therefore, $M_X^e \setminus e$ is at least 2-connected. It is enough to show that $M_X^e \setminus e$ has no 2-separation. On the contrary, suppose $(A, B)$ is a 2-separation of $E(M_X^e) \setminus e$. So, \[ \min \{|A|, |B|\} \geq 2 \text{ and } r'(A) + r'(B) - r'(M_X^e) \leq 1. \] (5.4.4)

Now one of the following two cases concerning $a$ and $\gamma$ occurs.

**Case (I) $a \in A$ and $\gamma \in B.$**

Let $A' = A \setminus a$ and $B' = B \setminus \gamma$. Then, by Lemma 4.1.1 (2), $r'(A) = r(A') + 1$. Since, no $OX$-circuit of $M$ contains $e$, by Lemma 4.1.1 (3), $r'(B) \geq r(B') + 1$. By inequality (5.4.4), we have $r(A') + r(B') + 1 - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$ where $|A'|, |B'| \geq 1$. This implies $(A', B')$ is a 1-separation of $M$, which is a contradiction.

**Case (II) $\{a, \gamma\} \subset A.$**

Let $A' = A \setminus \{a, \gamma\}$ and $B' = B$. We have the following three subcases.
Subcase (i) \(|A| = 2\) and \(A = \{a, \gamma\}\).
Then, \(r'(A) = 2\) and \(A' = \phi\). Since \(M\) contains an \(OX\)-circuit \(C\) not containing \(e\), we get \(C \subseteq B\). Then, by Lemma 4.1.1 (1), \(r'(B) = r(B') + 1\). Thus, by inequality (5.4.4), \(2 + r(B') + 1 - r(M) - 1 \leq 1\). That is, \(r(B') - r(M) \leq -1\) or \(r(B') \leq r(M) - 1\). This is a contradiction.

Subcase (ii) \(|A| = 3\) and \(A = \{a, \gamma, x\}\) where \(x \in E(M) \setminus e\).
If \(e \in Cl(A')\), then \(\{x, e\}\) forms a 2-circuit of \(M\). This is not possible, since \(M\) is 3-connected. We conclude that \(e \notin Cl(A')\). Consequently, by Lemma 4.1.1 (4), \(r'(A) = r(A') + 2\). And by Lemma 4.1.1 (1), \(r'(B) \geq r(B')\). Thus, by inequality (5.4.4), \(r(A') + 2 + r(B') - r(M) - 1 \leq 1\). That is, \(r(A') + r(B') - r(M) \leq 0\) and \(|A'|, |B'| \geq 1\). This gives a 1-separation of \(M\) which is a contradiction.

Subcase (iii) \(|A| > 3\).
Applying Lemma 4.1.1 to \(A\) and \(B\), we get \(r'(A) \geq r(A) + 1\) and \(r'(B) \geq r(B)\). Then, by inequality (5.4.4), we get \(r(A') + 1 + r(B') - r(M) - 1 \leq 1\). That is, \(r(A') + r(B') - r(M) \leq 1\) and \(|A'|, |B'| \geq 2\). This leads to a 2-separation of \(M\); a contradiction.

Thus, \(M^e_X\) has no 2-separation. We conclude that \(M^e_X \setminus e\) is 3-connected.

In the next theorem, we provide a sufficient conditions for a 4-connected matroid \(M\) where \(M^e_X \setminus e\) is a 4-connected minor of \(M^e_X\).

**Theorem 5.4.4.** Let \(M\) be a 4-connected matroid, \(|E(M)| \geq 6\) and let \(X \subset E(M)\) with \(|X| \geq 4\). Suppose that for any 2-element subset \(S\) of \(E(M)\) there is an \(OX\)-circuit \(C\) of \(M\) such that \(S \cap C = \phi\) and there is no \(OX\)-circuit containing \(e\). Then \(M^e_X \setminus e\) is 4-connected.
Proof. The matroid $M X \setminus e$ is 3-connected by Lemma 5.4.3. To prove that $M X \setminus e$ is 4-connected, it is enough to show that it has no 3-separation. On the contrary, suppose $(A, B)$ forms a 3-separation of $M X \setminus e$. Then \( \min \{|A|, |B|\} \geq 3 \) and
\[
 r'(A) + r'(B) - r'(M X \setminus e) \leq 2. \tag{5.4.5}
\]

Now one of the following two cases occurs.

**Case (I)** \( a \in A \) and \( \gamma \in B \).

Let \( A' = A \setminus a \) and \( B' = B \setminus \gamma \). Then, by (2) and (3) of Lemma 4.1.1, \( r'(A) = r(A') + 1 \) and \( r'(B) \geq r(B') + 1 \). By inequality (5.4.5), we have \( r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 2 \). That is, \( r(A') + r(B') - r(M) \leq 1 \) where \( |A'|, |B'| \geq 2 \). Thus, \((A', B')\) is a 2-separation of \( M \) and this is a contradiction.

**Case (II)** \( \{a, \gamma\} \subset A \).

Let \( A' = A \setminus \{a, \gamma\} \) and \( B' = B \). We have the following three subcases.

**Subcase (i)** \( |A| = 3 \) and \( A = \{a, \gamma, x\} \) where \( x \in E(M) \setminus e \).

If \( e \in Cl(A') \), then \( \{x, e\} \) forms a 2-circuit of \( M \). But this is not possible, since \( M \) is 4-connected. Thus, \( e \notin Cl(A') \) and by Lemma 4.1.1 (4), \( r'(A) = r(A') + 2 \). Since, there is an \( OX \)-circuit \( C \) of \( M \) not containing \( x \), \( C \subset B' \). Therefore, \( r'(B) = r(B') + 1 \). Consequently, by inequality (5.4.5),
\[
 r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.
\]

That is, \( r(A') + r(B') - r(M) \leq 0 \) and \( |A'|, |B'| \geq 1 \). This implies that \( M \) has a 1-separation; a contradiction.

**Subcase (ii)** \( |A| = 4 \) and \( A = \{a, \gamma, x, y\} \) where \( x, y \in E(M) \setminus e \).
If \( e \in Cl(A') \), then the set \( \{x, y, e\} \) itself is a 3-circuit or contains a 2-circuit of \( M \). This is not possible, since \( M \) is 4-connected. If \( e \notin Cl(A') \), then, by Lemma 4.1.1 (4), \( r'(A) = r(A') + 2 \). Since, there is an \( OX \)-circuit \( C \) of \( M \) not containing \( x \) and \( y \), \( C \subseteq B' \). Then \( r'(B) = r(B') + 1 \).

Consequently, by inequality (5.4.5),
\[
r(A') + 2 + r(B') + 1 - r(M) - 1 \leq 2.
\]
That is, \( r(A') + r(B') - r(M) \leq 0 \) and \( |A'|, |B'| \geq 1 \). This implies that \( M \) has a 1-separation, a contradiction.

**Subcase (iii) \(|A| > 4\).**

Now by (1) and (4) of Lemma 4.1.1, \( r'(B) \geq r(B) \) and \( r'(A) \geq r(A) + 1 \).

By inequality (5.4.5), we get
\[
r(A') + 1 + r(B') - r(M) - 1 \leq 2.
\]
That is, \( r(A') + r(B') - r(M) \leq 2 \) and \( |A'|, |B'| \geq 3 \). This leads to a 3-separation of \( M \), a contradiction.

Thus, \( M^e_X \) has no 3-separation. We conclude that \( M^e_X \setminus e \) is 4-connected.

Now we generalize Theorem 5.4.4 to \( n \)-connected binary matroids. In the following theorem, we give a sufficient condition for an \( n \)-connected binary matroid so that \( M^e_X \setminus e \) is an \( n \)-connected minor of \( M^e_X \).

**Theorem 5.4.5.** Let \( M \) be an \( n \)-connected binary matroid where \( n \geq 4 \), \( |E(M)| \geq 2(n - 1) \) and let \( X \subset E(M) \) with \( |X| \geq n \). Suppose that for any \( (n - 2) \)-element subset \( S \) of \( E(M) \) there is an \( OX \)-circuit \( C \) of \( M \) such that \( S \cap C = \phi \) and there is no \( OX \)-circuit containing \( e \). Then \( M^e_X \setminus e \) is \( n \)-connected.
The proof of Theorem 5.4.5 is by induction on $n$ by arguments similar as given for the proof of Theorem 5.4.2.

In the following lemma, we provide a sufficient condition for a 3-connected binary matroid $M$ so that $M_X^e \setminus \gamma$ is a 3-connected minor of the es-splitting matroid $M_X^e$.

**Lemma 5.4.6.** Let $M$ be a 3-connected binary matroid, $|E(M)| \geq 4$. Let $X \subset E(M)$ with $|X| \geq 3$. Suppose for $x \in E(M)$ there is an $OX$-circuit of $M$ not containing $x$. Then, $M_X^e \setminus \gamma$ is a 3-connected binary matroid.

**Proof.** If $x = e$, then by hypothesis there is an $OX$-circuit of $M$ not containing $x$. So, by Theorem 5.2.1, $M_X^e$ is 3-connected and $M_X^e \setminus \gamma$ is connected. Suppose $M_X^e \setminus \gamma$ is not 3-connected and let $(A, B)$ be a 2-separation of $E(M_X^e \setminus \gamma)$. Then $\min \{|A|, |B|\} \geq 2$ and

$$r'(A) + r'(B) - r'(M_X^e \setminus \gamma) \leq 1. \quad (5.4.6)$$

Assume that $\{a\} \subset A$. Let $A' = A \setminus a$ and $B' = B$. Then, by Lemma 4.1.1, $r'(A) = r(A') + 1$ and $r'(B) \geq r(B')$.

Now one of the following two cases occurs.

**Case (I) $|A| = 2$.**

Suppose $A = \{z, a\}$ and $A' = \{z\}$ where $z \in E(M)$. Then, by Lemma 4.1.1 (2), $r'(A) = r(A') + 1$. Now $M$ contains an odd circuit $C$ of $M$ and $\{z\} \cap C = \phi$, implies $C \subset B$. Then, by Lemma 4.1.1 (1), $r'(B) = r(B') + 1$. Thus, by inequality (5.4.6), $r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 1$. That is, $r(A') + r(B') - r(M) \leq 0$, and $|A'|, |B'| \geq 1$. This implies that $(A', B')$ is a 1-separation of $M$, a contradiction.
5.4 \(n\)-Connected Minors of es-Splitting Matroids.

Case (II) \(|A| > 2\).

By (1) and (2) of Lemma 4.1.1, \(r'(A) \geq r(A') + 1\) and \(r'(B) \geq r(B')\). Then, by inequality (5.4.6), \(r(A') + 1 + r(B') - r(M) - 1 \leq 1\). That is, \(r(A') + r(B') - r(M) \leq 1\) and \(|A'|, |B'| \geq 2\). This leads to a 2-separation of \(M\), a contradiction. Thus, \(M_X \setminus \gamma\) has no 2-separation. We conclude that \(M_X \setminus \gamma\) is a 3-connected binary matroid.

In the next theorem, we give a sufficient condition so that, if \(M\) is 4-connected, then \(M_X \setminus \gamma\) is a 4-connected minor of \(M_X^c\).

**Theorem 5.4.7.** Let \(M\) be a 4-connected matroid, \(|E(M)| \geq 6\). Let \(X \subset E(M)\) with \(|X| \geq 4\). Suppose that for a 2-element subset \(S\) of \(E(M)\) there is an \(OX\)-circuit \(C\) of \(M\) such that \(S \cap C = \emptyset\). Then \(M_X^c \setminus \gamma\) is 4-connected.

**Proof.** By Lemma 5.4.6, the matroid \(M_X^c \setminus \gamma\) is 3-connected. To prove that \(M_X^c \setminus \gamma\) is 4-connected, it is enough to show that \(M_X^c \setminus \gamma\) has no 3-separation. On the contrary, suppose that \((A, B)\) be a 3-separation of \(M_X^c \setminus \gamma\). Then, \(\min \{|A|, |B|\} \geq 3\) and

\[
r'(A) + r'(B) - r'(M_X^c \setminus \gamma) \leq 2.
\]

(5.4.7)

Without loss of generality assume that \(\{a\} \subset A\). Let \(A' = A \setminus a\) and \(B' = B\). Then, by Lemma 4.1.1, \(r'(A) = r(A') + 1\) and \(r'(B) \geq r(B')\). Now one of the following two cases occurs.

Case (I) \(|A| = 3\).

Let \(A = \{x, y, a\}\) where \(x, y \in E(M)\). Then, by Lemma 4.1.1 (2), we have \(r'(A) = r(A') + 1\). \(B'\) contains an odd circuit \(C\) of \(M\) such that \(\{x, y\} \cap C = \emptyset\). So, \(C \subset B\) and, by Lemma 4.1.1 (1), \(r'(B) = r(B') + 1\).
Hence, by inequality (5.4.7), we have \( r(A') + 1 + r(B') + 1 - r(M) - 1 \leq 2 \). That is, \( r(A') + r(B') - r(M) \leq 1 \) with \( |A'|, |B'| \geq 2 \). This gives a 2-separation of \( M \); a contradiction.

**Case (II) \(|A| > 3\).**

Applying Lemma 4.1.1 to \( A \) and \( B \) and using inequality (5.4.7), we get \( r(A') + r(B') - r(M) - 1 \leq 2 \). That is \( r(A') + r(B') - r(M) \leq 2 \) with \( |A'|, |B'| \geq 3 \). We conclude that \( M \) has a 3-separation; a contradiction.

Thus, \( M^c_X \setminus \gamma \) has no 3-separation. This implies that \( M^c_X \setminus \gamma \) is a 4-connected binary matroid.

In the following theorem, we give a sufficient condition for an \( n \)-connected binary matroid so that \( M^c_X \setminus \gamma \) is an \( n \)-connected minor of \( M^c_X \).

**Theorem 5.4.8.** Let \( M \) be an \( n \)-connected binary matroid with \( n \geq 4 \), \( |E(M)| \geq 2(n - 1) \) and let \( X \subset E(M) \) where \( |X| \geq n \). Suppose that for any \((n-2)\)-element subset \( S \) of \( E(M) \) there is an \( OX \)-circuit \( C \) of \( M \) such that \( S \cap C = \emptyset \). Then \( M^c_X \setminus \gamma \) is \( n \)-connected.

The proof follows by the arguments similar to one as given for the proofs of Theorems 5.4.2 and 5.4.7.

Thus, as a summery, we prove that given an \( n \)-connected binary matroid \( M \) of rank \( r \), \( M^c_X \setminus a \), \( M^c_X \setminus e \) and \( M^c_X \setminus \gamma \) are the only \( n \)-connected minors of rank \( (r + 1) \) of the es-splitting matroid \( M^c_X \). Alternatively, we provide a procedure to obtain \( n \)-connected matroids of rank \( (r + 1) \) from an \( n \)-connected matroid of rank \( r \). The matroids also have the property that each of them has one additional element than \( M \).