CHAPTER-VI

BIANCHI TYPE-II, VIII & IX MAGNETIZED COSMOLOGICAL MODELS IN SEN THEORY OF GRAVITATION BASED ON LYRA GEOMETRY*

6.1 INTRODUCTION:

The origin of the universe is one of the greatest cosmological mysteries even today. Astronomical observations have indicated that universe on the large scale can be treated as isotropic and homogeneous, but it is not so at small scales. In the early stage, the universe did not have the same property of isotropy as we have found today. In recent years there has been a lot of interest in alternative theories of gravitation. In chapter-I, we have conducted a systematic survey of the alternative theories of gravitation and the work done in these theories. Noteworthy among them is the theory of gravitation proposed by Sen (1957) based on Lyra (1951) geometry, a modified Riemannian geometry in which a gauge function has been introduced into the structure less manifold as a result of which the cosmological constant arises naturally from the geometry. This bears a remarkable resemblance to Weyl’s (1918) geometry. In subsequent investigations Sen (1957), Sen and Dunn (1971) formulated a new scalar-tensor theory of gravitation and constructed an analog of the Einstein’s field equations based on Lyra’s geometry. Halford (1972) has
shown that the scalar-tensor treatment based on Lyra’s geometry predicts the same effects as in general relativity. Several authors have studied cosmological models within the framework of Lyra geometry with a constant gauge vector in the time direction. Recently a lot of work has been done on string cosmological models and thick domain walls in Lyra geometry. Rao and Vinutha (2009) have studied axially symmetric cosmological models in a scalar tensor theory of gravitation based on Lyra (1951) geometry. Rao et al. (2012a) have obtained Bianchi types-II, VIII & IX string cosmological models with bulk viscosity in a theory of gravitation. Raj bali et al. (2012) have studied Bianchi type-IX barotropic fluid model with time-dependent displacement vector in Lyra geometry.

Magnetic field plays a vital role in the description of the energy distribution in the universe as it contains highly ionized matter. Also, electromagnetic field directly affects the expansion rate of the universe. The occurrence of magnetic fields on galactic scale is well-established fact today and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by
In this chapter we have discussed Bianchi type-II, VIII & IX magnetized cosmological models in Sen (1957) theory of gravitation which is based on Lyra (1951) geometry.

This chapter is organized as follows: In section 6.2, the field equations are obtained in the presence of magnetized cosmological models in Bianchi type- II, VIII and IX metrics. Section 6.3 deals with the solutions of field equations and in section 6.4 some important features of the models are discussed. Section 6.5 contains conclusions.

6.2 METRIC AND FIELD EQUATIONS:

We consider a spatially homogeneous Bianchi type-II, VIII & IX metrics of the form

\[ ds^2 = -dt^2 + R^2 [d\theta^2 + f^2(\theta)d\phi^2] + S^2 [d\phi + h(\theta)d\phi]^2 \]  \hspace{1cm} (6.2.1)

where \((\theta, \phi, \varphi)\) are the Eulerian angles, \(R\) and \(S\) are functions of ‘t’ only.

It represents

Bianchi type-II metric if \(f(\theta) = 1\) and \(h(\theta) = \theta\)

Bianchi type-VIII metric if \(f(\theta) = \cosh \theta\) and \(h(\theta) = \sinh \theta\)
Bianchi type- IX metric if $f(\theta) = \sin \theta$ and $h(\theta) = \cos \theta$

The field equations in normal gauge in Lyra manifold as obtained by Sen (1957) in the presence of electromagnetic field are

$$R_{ij} - \frac{1}{2} R g_{ij} + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -E_{ij}$$  \hspace{1cm} (6.2.2)

where $E_{ij}$ is the electromagnetic energy momentum tensor, $\phi_i$ is the displacement field and the other symbols have their usual meaning as in Riemannian geometry. The displacement field $\phi_i$ can be written as $\phi_i = (0,0,\beta(t))$.

We take the electromagnetic energy momentum tensor $E_{ij}$ as given by Lichnerowicz (1967) is

$$E_{ij} = \bar{\mu} \left[ |h|^2 \left( u_i u_j + \frac{1}{2} g_{ij} \right) - h_i h_j \right], \quad |h|^2 = h_i h^i$$  \hspace{1cm} (6.2.3)

where $\bar{\mu}$ is the magnetic permeability and $h_i$ is the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} E_{ijkl} F^k_{lj} u^j$$  \hspace{1cm} (6.2.4)
where $F^{kl}$ is the electromagnetic field tensor and $E_{ijkl}$ is the Levi Civita tensor density.

Let us assume the coordinates to be comoving, so that

$$u^1 = u^2 = u^3 = 0 \text{ and } u^4 = -1$$

Take the incident magnetic field to be in the direction of Z-axis, so that

$$h_1 = h_2 = h_4 = 0, \quad h_3 \neq 0$$

By virtue of the equation (6.2.4) the only non-vanishing component of $F_{ij}$ is $F_{12}$.

The set of Maxwell’s equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

(6.2.5)

leadsto $F^{12} R^2 S f(\theta) = \text{constant} = \zeta \text{(say)}$

Hence from (6.2.4), we get

$$h_3 = \frac{-\zeta}{2\mu}$$
For the metric (6.2.1), the components of electromagnetic field can be expressed as

\[ E_1^1 = E_2^2 = -E_3^3 = -E_4^4 = \frac{\zeta^2}{8\mu} \left[ \frac{1}{S^2} + \frac{h^2(\theta)}{R^2 f^2(\theta)} \right] = \eta \text{(say)} \quad (6.2.6) \]

where the quantity \( \eta \) is a function of \( 't' \) only.

The field equations (6.2.2) for the metric (6.2.1) with the help of equations (6.2.3) – (6.2.6) can be written as

\[ \frac{\ddot{R}}{R} + \frac{\dot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} + \frac{3}{4} \beta^2 = -\eta \quad (6.2.7) \]

\[ 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + \delta}{R^2} - \frac{3S^2}{4R^4} + \frac{3}{4} \beta^2 = \eta \quad (6.2.8) \]

\[ 2\frac{\dot{R}\dot{S}}{RS} + \frac{\dot{R}^2 + \delta}{R^2} - \frac{S^2}{4R^4} - \frac{3}{4} \beta^2 = \eta \quad (6.2.9) \]

Here the overhead dot denotes differentiation with respect to \( 't' \).

For \( \delta = 0, -1 & +1 \), the field equations (6.2.7) to (6.2.9) correspond to the Bianchi type II, VIII & IX universes respectively.
6.3 SOLUTIONS OF THE FIELD EQUATIONS:

Using the transformation \( dt = R^2 S \, dT \), the above field equations (6.2.7) to (6.2.9) can be written as

\[
\left( \frac{R'}{R} \right)' + \left( \frac{S'}{S} \right)' - \left( \frac{R'}{R} \right)^2 - 2 \left( \frac{R'S'}{RS} \right) + \frac{S^4}{4} + \frac{3}{4} \beta^2 R^4 S^2 = -\eta R^4 S^2 \tag{6.3.1}
\]

\[
2 \left( \frac{R'}{R} \right)' - \left( \frac{R'}{R} \right)^2 - 2 \left( \frac{R'S'}{RS} \right) - \frac{3S^4}{4} + \delta R^2 S^2 + \frac{3}{4} \beta^2 R^4 S^2 = \eta R^4 S^2 \tag{6.3.2}
\]

\[
\left( \frac{R'}{R} \right)^2 + 2 \left( \frac{R'S'}{RS} \right) - \frac{S^4}{4} + \delta R^2 S^2 - \frac{3}{4} \beta^2 R^4 S^2 = \eta R^4 S^2 \tag{6.3.3}
\]

Here the overhead dash denotes differentiation with respect to \( T \).

Since the field equations are highly non-linear in nature, in order to get a deterministic solution we take a relation between the metric potentials

\[
R = \alpha S \tag{6.3.4}
\]

where \( \alpha \) is an arbitrary constant.
From equations (6.3.1), (6.3.3) & (6.3.4), we get

\[ 2 \left( \frac{S'}{S} \right)' + \delta \alpha^2 S^4 = 0 \]  

(6.3.5)

**BIANCHI TYPE-II \((\delta = 0)\) MAGNETIZED COSMOLOGICAL MODEL:**

If \(\delta = 0\), the equation (6.3.5) can be written as

\[ 2 \left( \frac{S'}{S} \right)' = 0 \]  

(6.3.6)

From equation (6.3.6), we get

\[ S = \exp(aT + b) \]  

(6.3.7)

From equations (6.3.4) & (6.3.7), we get

\[ R = \alpha \exp(aT + b) \]  

(6.3.8)

From equations (6.2.6), (6.3.7) & (6.3.8), we get

\[ \eta = \left( \frac{\zeta^2}{8\bar{\mu}(\exp(aT + b))^2} \right) \left( \frac{\alpha^2 + \theta^2}{\alpha^2} \right) \]  

(6.3.9)
From equations (6.3.3), (6.3.7) & (6.3.9), we get the displacement vector

\[
\beta^2 = \frac{4a^2}{\alpha^4}(\exp(aT+b))^{-6} - \left[\left(\frac{\xi^2}{6\mu}\right)\left(\frac{\alpha^2 + \theta^2}{\alpha^2}\right) + \frac{1}{3\alpha^4}\right](\exp(aT+b))^{-2}
\]

(6.3.10)

The metric (6.2.1) can now be written as

\[
ds^2 = -[\alpha^4(\exp(aT+b))^6]dT^2 + [\alpha^2(\exp(aT+b))^2](d\theta^2 + d\phi^2) + [(\exp(aT+b))^2](d\varphi + \theta d\phi)^2
\]

(6.3.11)

Thus (6.3.11) together with (6.3.9) & (6.3.10) constitutes a Bianchi type-II magnetized cosmological model, in isotropic form, in Sen (1957) theory of gravitation.

**BIANCHI TYPE-VIII (\(\delta = -1\)) MAGNETIZED COSMOLOGICAL MODEL:**

If \(\delta = -1\), the equation (6.3.5) can be written as

\[
2\left(\frac{S'}{S}\right)' - \alpha^2 S^4 = 0
\]

(6.3.12)
From equation (6.3.12), with a suitable substitution, we get

\[ S''^2 = W^2 S^6 + \gamma^2 S^2 \]  \hspace{1cm} (6.3.13)

where \( \gamma^2 \) is a constant and \( W^2 = \frac{\alpha^2}{4} \).

From equation (6.3.13), we get

\[ S^2 = \left( \frac{\gamma}{W} \right) \left( \coth^2 (2\gamma T) - 1 \right)^{1/2} \]  \hspace{1cm} (6.3.14)

From equations (6.3.4) & (6.3.14), we get

\[ R^2 = \alpha^2 \left( \frac{\gamma}{W} \right) \left( \coth^2 (2\gamma T) - 1 \right)^{1/2} \]  \hspace{1cm} (6.3.15)

From equations (6.2.6), (6.3.14) & (6.3.15), we get

\[ \eta = \left( \frac{\zeta W}{8\tilde{\mu} \gamma} \right) \left( \frac{\alpha^2 \cosh^2 \theta + \sinh^2 \theta}{\alpha^2 \cosh^2 \theta} \right) \left( \coth^2 (2\gamma T) - 1 \right)^{-1/2} \]  \hspace{1cm} (6.3.16)
From equations (6.3.3), (6.3.14) & (6.3.16), we get the displacement vector

\[ \beta^2 = \left( \frac{W}{\alpha^2 \gamma} \right) 4W^2 - \left( \frac{1 + 4\alpha^2}{3} \right) - \left( \frac{\zeta^2 \alpha^2}{6\mu} \right) \left( \frac{\alpha^2 \cosh^2 \theta + \sinh^2 \theta}{\cosh^2 \theta} \right) \]

\[ \times \left( \coth^2(2\gamma T) - 1 \right)^{-1} + \frac{4W^3}{\alpha^4 \gamma} \left( \coth^2(2\gamma T) - 1 \right)^{\frac{3}{2}} \] (6.3.17)

The metric (6.2.1) can now be written as

\[ ds^2 = -\alpha^4 \left( \frac{\gamma}{W} \right)^3 (\coth^2(2\gamma T) - 1)^{\frac{3}{2}} dT^2 + \alpha^2 \left( \frac{\gamma}{W} \right) (\coth^2(2\gamma T) - 1)^{\frac{1}{2}} \]

\[ \times (d\theta^2 + \cosh^2 \theta d\phi^2) + \left( \frac{\gamma}{W} \right) (\coth^2(2\gamma T) - 1)^{\frac{1}{2}} (d\phi + \sinh \theta d\phi)^2 \]

(6.3.18)

Thus (6.3.18) together with (6.3.16) & (6.3.17) constitutes a Bianchi type-VIII magnetized cosmological model in Sen (1957) theory of gravitation.
BIANCHI TYPE-IX ($\delta = 1$) MAGNETIZED COSMOLOGICAL MODEL:

If $\delta = 1$, the equation (6.3.5) can be written as

$$2\left(\frac{S'}{S}\right) + \alpha^2 S^4 = 0$$

Equation (6.3.19), with a suitable substitution, we get

$$S'^2 = W^2 S^6 + \gamma^2 S^2$$

(6.3.20)

where $\gamma^2$ is a constant and $W^2 = -\frac{\alpha^2}{4}$.

From equation (6.3.20), we get

$$S^2 = \left(\frac{\gamma}{W}\right) \left(\coth^2 (2\gamma T) - 1\right)^{1/2}$$

(6.3.21)

From equations (6.3.4) & (6.3.21), we get

$$R^2 = \alpha^2 \left(\frac{\gamma}{W}\right) \left(\coth^2 (2\gamma T) - 1\right)^{1/2}$$

(6.3.22)
From equations (6.2.6), (6.3.21) & (6.3.22), we get

\[
\eta = \left( \frac{\xi^2 W}{8 \mu \gamma} \right) \left( \frac{\alpha^2 \sin^2 \theta + \cos^2 \theta}{\alpha^2 \sin^2 \theta} \right) \left( \coth^2 (2\gamma T) - 1 \right)^{-1/2} \tag{6.3.23}
\]

From equations (6.3.3), (6.3.21) & (6.3.23), we get the displacement vector

\[
\beta^2 = \left( \frac{W}{\alpha^4 \gamma} \right) 
\left[ 4W^2 + \left( \frac{4\alpha^2 - 1}{3} \right) - \left( \frac{\xi^2 \alpha^2}{6\mu} \right) \left( \frac{\alpha^2 \sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \right) \right] 
\times \left( \coth^2 (2\gamma T) - 1 \right)^{-1/2} + \left( \frac{4W^3}{\alpha^4 \gamma} \right) \left( \coth^2 (2\gamma T) - 1 \right)^{3/2} \tag{6.3.24}
\]

The metric (6.2.1) can now be written as

\[
\begin{align*}
\frac{\gamma}{W} & \left[ \alpha^4 \left( \frac{\gamma}{W} \right)^3 (\coth^2 (2\gamma T) - 1)^{3/2} \right] dT^2 + \alpha^2 \left( \frac{\gamma}{W} \right) (\coth^2 (2\gamma T) - 1)^{1/2} \\
\times \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) & + \left[ \left( \frac{\gamma}{W} \right) (\coth^2 (2\gamma T) - 1)^{1/2} \right] (d\phi + \cos \theta d\phi)^2
\end{align*} \tag{6.3.25}
\]

Thus (6.3.25) together with (6.3.23) & (6.3.24) constitutes a Bianchi type-IX magnetized cosmological model in Sen (1957) theory of gravitation.
VACUUM COSMOLOGICAL MODELS:

It is interesting to note that in the absence of magnetic field, i.e. if we assign zero to the constant $\zeta$ in equation (6.2.6), the components of the electromagnetic field $E^i_j$ will vanish and hence the metric (6.3.11) together with (6.3.10), the metric (6.3.18) together with (6.3.17) and the metric (6.3.25) together with (6.3.24) respectively constitutes Bianchi type-II ,VIII and IX vacuum cosmological models in Sen (1957) theory of gravitation with $\zeta = 0$.

6.4 SOME IMPORTANT FEATURES OF THE MODELS:

Bianchi type-II cosmological model ($\delta = 0$):

The spatial volume for the model is given by

$$V = \frac{1}{(-g)^{\frac{3}{2}}} = \left(\alpha^2 (\exp(aT + b))^3\right)$$

(6.4.1)
The expression for expansion scalar $\theta$, calculated for the flow vector $u^i$ is given by

$$\theta = u^i_{\cdot i} = \frac{-3a}{\alpha^2} \left( \exp(aT + b) \right)^{-3} \quad (6.4.2)$$

and the shear $\sigma$ is given by

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} = \frac{5a^2}{9\alpha^4} \left( \exp(aT + b) \right)^{-6} \quad (6.4.3)$$

The deceleration parameter is given by

$$q = (-3\theta^{-2})(\theta, u^i + \frac{1}{3} \theta^2) = 3\alpha^2 \left( \exp(aT + b) \right)^3 - 1 \quad (6.4.4)$$

The deceleration parameter $q$ is less than zero when $T$ approaches to $-\infty$. Hence it represents an accelerating universe.

The components of Hubble Parameter $H_1, H_2 \& H_3$ are given by

$$H_1 = \frac{R'}{R} = a, H_2 = \frac{R'}{R} = a \& H_3 = \frac{S'}{S} = a.$$
Therefore the generalized mean Hubble parameter \( H \) is

\[
H = \frac{1}{3} \left( H_1 + H_2 + H_3 \right) = a
\]  

(6.4.5)

The average anisotropy parameter is

\[
A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = 0, \text{ where } \Delta H_i = H_i - H \text{ (i=1,2,3)} \]  

(6.4.6)

**Bianchi type-VIII cosmological model \((\delta = -1)\):**

The spatial volume for the model is

\[
V = (-g)^{\frac{2}{3}} = \alpha^2 \left( \frac{\gamma}{W} \right)^3 \left( \text{coth}^2 (2\gamma T) - 1 \right)^{\frac{3}{4}} \text{cosh} \theta
\]  

(6.4.7)

The expression for expansion scalar \( \theta \), calculated for the flow vector \( u^i \) is given by

\[
\theta = u^i : i = -\frac{3W^{\frac{3}{2}}}{\alpha^2 \gamma^{\frac{1}{2}}} \left( \text{coth}^2 (2\gamma T) - 1 \right)^{\frac{3}{4}} \text{coth}(2\gamma T)
\]  

(6.4.8)

and the shear \( \sigma \) is given by

\[
\sigma^2 = \frac{1}{2} \sigma^{ij} \sigma_{ij} = \frac{5W^3}{2\alpha^4 \gamma} \left( \text{coth}^2 (2\gamma T) - 1 \right)^{\frac{3}{2}} \text{coth}^2 (2\gamma T)
\]  

(6.4.9)
The deceleration parameter

\[ q = \frac{\alpha^2 \gamma^2}{W^2} \left( \coth^2 (2\gamma T) - 1 \right)^{\frac{3}{2}} \cot^2 (2\gamma T)(3\cot^2 (2\gamma T) \right)
\]

\[ + 2 \cosech^2 (2\gamma T) - 1 \]

(6.4.10)

The deceleration parameter \( q \) is less than zero as \( T \) approaches to \( \pm \infty \). Hence it represents an accelerating universe.

The Hubble parameter (H) is

\[ H = \gamma \cot (2\gamma T) \]  

(6.4.11)

The average anisotropy parameter is

\[ A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = 0 \text{ , where } \Delta H_i = H_i - H \text{ (i = 1, 2, 3)} \]  

(6.4.12)

**Bianchi type-IX cosmological model (\( \delta = 1 \)):**

The spatial volume for the model is

\[ V = (-g)^2 \left( \alpha^2 \gamma^2 \cot^2 (2\gamma T) - 1 \right)^{\frac{3}{4}} \sin \theta \]  

(6.4.13)
The expression for expansion scalar $\theta$, calculated for the flow vector $u^i$, is given by

$$\theta = u^i : = \frac{-3W^{3/2}}{\alpha^2 \gamma^{\gamma/2}} (\coth^2 (2\gamma T) - 1)^{-\frac{3}{4}} \coth (2\gamma T)$$

(6.4.14)

and the shear $\sigma$ is given by

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} = \frac{5W^3}{2\alpha^4 \gamma} (\coth^2 (2\gamma T) - 1)^{-\frac{3}{2}} \coth^2 (2\gamma T)$$

(6.4.15)

The deceleration parameter

$$q = \frac{\alpha^2 \gamma^{\gamma/2}}{W^2} (\coth^2 (2\gamma T) - 1)^{\frac{3}{4}} \coth^{-2} (2\gamma T)(3 \coth^2 (2\gamma T))$$

$$+ 2 \cos \text{ech}^2 (2\gamma T) - 1$$

(6.4.16)

The deceleration parameter $q$ is less than zero as $T$ approaches to $\pm \infty$. Hence it represents an accelerating universe.

The Hubble parameter ($H$) is

$$H = \gamma \coth (2\gamma T)$$

(6.4.17)
The average anisotropy parameter is

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = 0$$, where $\Delta H_i = H_i - H$ $(i = 1, 2, 3)$ \hspace{1cm} (6.4.18)

Since for all the three models the average anisotropy parameter $A_m = 0$, all the three models, i.e., (6.4.6), (6.4.12) & (6.4.18) will always represent isotropic universes.

### 6.5 CONCLUSIONS:

In this chapter we have presented magnetized Bianchi type-II, VIII & IX cosmological models in Sen (1957) theory of gravitation based on Lyra (1951) geometry. We have noticed that the Bianchi type-II magnetized cosmological model has no initial singularity at $T = \frac{-\nu}{a}$ but the Bianchi type-VIII & IX magnetized cosmological models have initial singularity at $T = 0$. We can also observe that Bianchi type-II, VIII & IX magnetized cosmological models always represent accelerating universe. The spatial volume is decreasing as time $T$ increases except for Bianchi type-II magnetized cosmological model. In this case the spatial volume is increasing with time $T$. For Bianchi type-II magnetized
cosmological model, the expansion scalar $\theta$ and the shear scalar $\sigma$ will tend to unity as $T$ approaches to zero. For Bianchi type-VIII & IX magnetized cosmological models, the expansion scalar $\theta$ and the shear scalar $\sigma$ will tend to zero as $T$ approaches to zero. The models presented here will play a physically significant role in the studies of the interaction of electromagnetic and gravitation fields in Bianchi type-II, VIII & IX space times. It is heartening to note that closed form exact solutions are obtained when the fields are coupled non-linearly. Since our models are isotropic and accelerating, these models represent the present universe.