CHAPTER- 2
FUZZY PROGRAMMING TECHNIQUE
TO SOLVE MULTI-OBJECTIVE
TRANSPORTATION PROBLEM

2.1 Introduction

The multi-objective transportation problem is a vector minimum problem. In the case of the multi-objective fuzzy linear programming technique, only the objectives are fuzzy. The fuzzy linear programming technique for the multi-objective transportation problem gives an optimal compromise solution.

Diaz [25, 26] developed an algorithm for finding the solution of multi-objective transportation problem. Isermann [50] developed an algorithm for identifying all the non-dominated solution, for a linear multi-objective transportation problem. Ringuest and Rinks [72] developed two interactive algorithms for solving multi-objective transportation problem. Zimmermann [88] first applied suitable membership functions to solve linear programming problem with several objective functions. He showed that solutions obtained by fuzzy linear programming are always efficient. Bit et al. [7] applied the fuzzy programming technique with linear membership function to solve the multi-objective transportation problem.

Leberling [60] used a special-type nonlinear (hyperbolic) membership function for the vector maximum linear programming problem. He showed that solutions obtained by fuzzy linear programming with this type of non-linear membership function are always efficient. Dhingra and H. Moskowitz [27] defined other types of the non-linear (exponential, quadratic and
logarithmic) membership functions and applied them to an optimal design problem. Verma, Biswal and Biswas [84] used the fuzzy programming technique with some non-linear (hyperbolic and exponential) membership functions to solve a multi-objective transportation problem are always efficient.

2.2 Mathematical model

In a typical transportation problem, a homogeneous product is to be transported from each of m sources to n destinations. The sources are production facilities, warehouses, or supply point, characterized by available capacities $a_i$ (i = 1,2, …, m). The destinations are consumption facilities, warehouses, or demand points, characterized by required levels of demand $b_j$ (j = 1,2, …, n). A penalty $c_{ij}^p$ is associated with transportation of a unit of the product from sources $i$ to destination $j$ for the p-criteria. The penalty could represent transportation cost, delivery time, quantity of goods delivered, under used capacity, etc. A variable $X_{ij}$ represents the unknown quantity to be transported from origin $O_i$ to destination $D_j$. In the real world, however, transportation problems are not all-single objective type.

A Multi-objective transportation problem may be stated mathematically

\[
\text{Minimize } Z_p = \begin{cases} 
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^1 X_{ij} \\
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^2 X_{ij} \\
\vdots \\
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^p X_{ij} 
\end{cases} 
\] (2.1)
Subject to
\[ \sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1,2,...,m \]  
(2.2)
\[ \sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1,2,...,n \]  
(2.3)
\[ X_{ij} \geq 0 \quad \forall \quad i, j \]  
(2.4)

Where the subscript on \( Z_p \) and superscript on \( c^p_{ij} \) denote the \( P^{th} \) penalty criterion; \( a_i > 0 \ \forall \ \ i, \ b_j > 0 \ \forall \ \ j, \ c^p_{ij} \ \forall \ \ i, j \) and
\[ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \]  
(Balanced condition)

The balanced condition is treated as a necessary and sufficient condition for the existence of a feasible solution to the balanced linear transportation problem. A standard transportation problem has exactly \((m + n)\) constraints and \((m \times n)\) variables.

2.3 Fuzzy Algorithm to solve multi-objective transportation problem

Step 1:
Solve the Multi-objective multi-index transportation problem as a single objective transportation problem \( P \) times by taking one of the objectives at a time.

Step 2:
From the results of step 1, determine the corresponding values for every objective at each solution derived. According to each solution and value for every objective, we can find pay-off matrix as follows

\[
\begin{bmatrix}
Z_1(X) & Z_2(X) & \ldots & Z_p(X) \\
Z_1^{(1)} & Z_2^{(1)} & \ldots & Z_{1p}^{(1)} \\
Z_1^{(2)} & Z_2^{(2)} & \ldots & Z_{2p}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
Z_1^{(P)} & Z_2^{(P)} & \ldots & Z_{pp}^{(P)}
\end{bmatrix}
\]
Where, \( X^{(1)}, X^{(2)}, \ldots, X^{(p)} \) are the isolated optimal solutions of the \( P \) different transportation problems for \( P \) different objective functions

\[
Z_j = Z_j(X) \quad (i = 1, 2, \ldots, p \quad & \quad j = 1, 2, \ldots, p)
\]

be the \( i \)-th row and \( j \)-th column element of the pay-off matrix.

**Step 3:**

From step 2, we find for each objective the worst (\( U_p \)) and the best (\( L_p \)) values corresponding to the set of solutions, where,

\[
U_p = \max (Z_{1p}, Z_{2p}, \ldots, Z_{pp}) \quad \text{and} \quad L_p = Z_{pp} \quad p = 1, 2, \ldots, P
\]

An initial fuzzy model of the problem (2.1)-(2.4) can be stated as

Find \( X_{ij} \quad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n \)

so as to satisfy

\[
Z_p \leq L_p \quad p = 1, 2, \ldots, P
\]

Subject to

\[
\sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1, 2, \ldots, m \quad (2.6)
\]

\[
\sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1, 2, \ldots, n \quad (2.7)
\]

\[
X_{ij} \geq 0 \quad \forall \quad i, j \quad (2.8)
\]

**Step 4:** Case (i)

Define linear membership function for the \( p \)-th objective function as follows:

\[
\mu_p(X) = \begin{cases} 
1 & \text{if } Z_p(X) \leq L_p \\
\frac{U_p - Z_p(X)}{U_p - L_p} & \text{if } L_p < Z_p < U_p \\
0 & \text{if } Z_p \geq U_p
\end{cases}
\]

**Step 5:**

Find an equivalent crisp model by using a linear membership function for the initial fuzzy model.
Maximize \( \lambda \) 
subject to
\[
\lambda \leq \frac{U_p - Z_p(X)}{U_p - L_p}
\]  
(2.10)

\[
\sum_{j=1}^{n} X_{ij} = a_i, \quad i=1,2,...,m
\]  
(2.11)

\[
\sum_{i=1}^{m} X_{ij} = b_j, \quad j=1,2,...,n
\]  
(2.12)

\[
X_{ij} \geq 0 \quad \forall \ i, j \quad \text{and} \quad \lambda \geq 0
\]  
(2.13)

**Step 6:** Solve the crisp model by an appropriate mathematical programming algorithm.

Maximize \( \lambda \) 
Subject to
\[
C^p_i X^p_i + \lambda (U_p - L_p) \leq U_p \quad p = 1,2,...,P
\]  
(2.14)

subject to
\[
\sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1,2,...,m
\]

\[
\sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1,2,...,n
\]

\[
X_{ij} \geq 0 \quad \forall \ i, j
\]

Now, by using hyperbolic membership function for the P-th objective function

\[
\mu^p_{Z_p}(x) = \begin{cases} 
1 & \text{if } Z_p \leq L_p \\
\frac{1}{2} e^\left(\frac{(U_p + L_p) - Z_p(x)}{2} - \frac{(U_p + L_p) - Z_p(x)}{2}\right) + \frac{1}{2} & \text{if } L_p < Z_p < U_p \\
0 & \text{if } Z_p \geq U_p
\end{cases}
\]  
(2.15)

Where,
\[
\alpha_p = \frac{3}{U_p - L_p} = \frac{6}{U_p - L_p}
\]

Crisp model for the fuzzy model can be formulated as:

Maximize \( \lambda \) 
Subject to
\[
\begin{align*}
\lambda & \leq \frac{1}{2} \left( e^{\frac{2}{2} (U_p + L_p) Z_p(x)} \alpha_p - e^{\frac{2}{2} (U_p + L_p) Z_p(x)} \alpha_p \right) + \frac{1}{2} \left( e^{\frac{2}{2} (U_p + L_p) Z_p(x)} \alpha_p - e^{\frac{2}{2} (U_p + L_p) Z_p(x)} \alpha_p \right) \\
\sum_{j=1}^{n} X_{ij} &= a_i, \quad i = 1, 2, \ldots, m \\
\sum_{i=1}^{m} X_{ij} &= b_j, \quad j = 1, 2, \ldots, n \\
X_{ij} &\geq 0 \quad \forall \ i, j \quad \lambda \geq 0
\end{align*}
\]

(2.18)

(2.19)

(2.20)

(2.21)

Solve the crisp model as

Maximize \( X_{mn+1} \)

subject to

\[ \alpha_p Z_p(x) + X_{mn+1} \leq \alpha_p (U_p + L_p) / 2, \quad p = 1, 2, \ldots, P \]  

(2.22)

(2.23)

\[ \sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1, 2, \ldots, m \]

\[ \sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1, 2, \ldots, n \]

\[ X_{ij} \geq 0 \quad \forall \ i, j \quad \text{and} \quad X_{mn+1} \geq 0 \]

Where, \( X_{mn+1} = \tanh(2\lambda-1) \)

Now, by using exponential membership function for the \( p \)th objective function and is defined as

\[
\mu^{\epsilon}Z_p(x) = \begin{cases} 
1, & \text{if } Z_p \leq L_p \\
\frac{e^{-S\Psi_p(X)}}{1-e^{-S}}, & \text{if } L_p < Z_p < U_p \\
0, & \text{if } Z_p \geq U_p
\end{cases}
\]

(2.24)

Where, \( \Psi_p(X) = \frac{Z_p - L_p}{U_p - L_p} \)

\( \epsilon = 1, 2, \ldots, P \)

\( S \) is a non-zero parameter, prescribed by the decision maker.
4. Numerical Example

Minimize $Z_1 = 16X_{11} + 19X_{12} + 12X_{13} + 22 + 13X_{22} + 19X_{23}$
$\quad + 4X_{31} + 28X_{32} + 8X_{33}$ \hspace{1cm} (2.25)

Minimize $Z_2 = 9X_{11} + 14X_{12} + 12X_{13} + 16 + 10X_{22} + 14X_{23}$
$\quad + 8X_{31} + 20X_{32} + 6X_{33}$ \hspace{1cm} (2.26)

Subject to

\[
\sum_{j=1}^{3} X_{1j} = 14 \quad ; \quad \sum_{j=1}^{3} X_{2j} = 16 \quad ; \quad \sum_{j=1}^{3} X_{3j} = 12 \hspace{1cm} (2.27)
\]

\[
\sum_{i=1}^{3} X_{i1} = 10 \quad ; \quad \sum_{i=1}^{3} X_{i2} = 15 \quad ; \quad \sum_{i=1}^{3} X_{i3} = 17 \hspace{1cm} (2.28)
\]

$X_{ij} \geq 0 \quad i = 1,2,3. \quad j = 1,2,3.$ \hspace{1cm} (2.29)

For objective $Z_1$, we find the optimal solution as

\[
X^{(1)} = \begin{cases} 
X_{11} = 9 ; X_{12} = 5 ; X_{21} = 1, \\
X_{22} = 15 ; X_{32} = 12 \end{cases} 
\]

$Z_1 = 517$

For objective $Z_2$, we find the optimal solution as

\[
X^{(2)} = \begin{cases} 
X_{11} = 10 ; X_{13} = 4 ; X_{22} = 15, \\
X_{23} = 1 ; X_{33} = 12 \end{cases} 
\]

$Z_2 = 374$

Now for $X^{(1)}$ we can find out $Z_2$, \hspace{1cm} $Z_2(X^{(1)}) = 379$

Now for $X^{(2)}$ we can find out $Z_1$ \hspace{1cm} $Z_1(X^{(2)}) = 518$

Pay-off matrix is

\[
\begin{bmatrix}
Z_1 & Z_2 \\
X^{(1)} & 517 & 379 \\
X^{(2)} & 518 & 374
\end{bmatrix}
\]

From this matrix \hspace{1cm} $U_1 = 518, \hspace{1cm} U_2 = 379, \hspace{1cm} L_1 = 517, \hspace{1cm} L_2 = 374$

Find \{$X_{ij} \hspace{0.5cm}, \hspace{0.5cm} i=1,2,3, \hspace{0.5cm} j=1,2,3$\} \hspace{1cm} Soast to satisfy \hspace{1cm} $Z_1 \leq 517$ \hspace{1cm} and \hspace{1cm} $Z_2 \leq 374$,

Define membership function for the objective functions

$Z_1(X)$ and $Z_2(X)$ respectively
\[ \mu_1(X) = \begin{cases} 
1, & \text{if } Z_1(X) \leq 517 \\
\frac{518 - Z_1(X)}{518 - 517}, & \text{if } 517 < Z_1(X) < 518 \\
0, & \text{if } Z_1(X) \geq 518 
\end{cases} \]

\[ \mu_2(X) = \begin{cases} 
1, & \text{if } Z_2(X) \leq 374 \\
\frac{379 - Z_2(X)}{379 - 374}, & \text{if } 374 < Z_2(X) < 379 \\
0, & \text{if } Z_2(X) \geq 379 
\end{cases} \]

Find an equivalent crisp model

Maximize \( \lambda \),

\[ \lambda + Z_i(X) \leq 518 \quad \text{and} \quad 5\lambda + Z_j(X) \leq 379 \]

Solve the crisp model by using an appropriate mathematical algorithm.

\[
egin{align*}
16X_{11} + 19X_{12} + 12X_{13} + 22X_{21} + 13X_{22} + 19X_{23} + 14X_{31} \\
+ 28X_{32} + 8X_{33} + \lambda & \leq 518 \\
9X_{11} + 14X_{12} + 12X_{13} + 16X_{21} + 10X_{22} + 14X_{23} + 8X_{31} \\
+ 20X_{32} + 6X_{33} + 5\lambda & \leq 379 \\
X_{11} + X_{12} + X_{13} &= 14 \\
X_{21} + X_{22} + X_{23} &= 16 \\
X_{31} + X_{32} + X_{33} &= 12 \\
X_{11} + X_{21} + X_{31} &= 10 \\
X_{12} + X_{22} + X_{32} &= 15 \\
X_{13} + X_{23} + X_{33} &= 17 \\
X_i & \geq 0 \quad i = 1,2,3. \quad j = 1,2,3.
\end{align*}
\]

The optimal compromise solution of the problem is represented as

\[
X^* = \begin{cases} 
X_{11} = 9.5, & X_{13} = 4.5, \quad X_{21} = 0.5, \\
X_{22} = 15, & X_{23} = 0.5, \quad X_{33} = 12 , \\
\text{and rest all } x_{ij} \text{ are zeros}
\end{cases}
\]
\[ Z'_1 = 152+54+11+195+9.5+96=517.5 \]
and
\[ Z'_2 = 85.5+54+8+150+7+72=376.5 \]

\[ \lambda = 0.5 \]

If we use hyperbolic membership function with
\[ \alpha_1 = \frac{6}{518-517} = 6; \quad \alpha_2 = \frac{6}{379-374} = \frac{6}{5} \]
\[ \frac{U_1+L_1}{2} = \frac{1035}{2} = 517.5; \quad \frac{U_2+L_2}{2} = \frac{379+374}{2} = \frac{753}{2} = 376.5 \]

Then we get the membership functions \( \mu_1^H(Z_1) \) and \( \mu_2^H(Z_2) \)
for the objectives \( Z_1 \) & \( Z_2 \) respectively, are defined as follows:

\[
\mu^H_{Z_1}(x) = \begin{cases} 
1, & \text{if } Z_1(x) \leq 517 \\
\frac{1}{2} \left( \frac{e^{-\frac{(517.5-Z_1(x))6}{6}} + e^{-\frac{(517.5-Z_1(x))6}{6}}}{2} + 1 \right), & \text{if } 517 < Z_1(x) < 518 \\
0, & \text{if } Z_1(x) \geq 518
\end{cases}
\]

\[
\mu^H_{Z_2}(x) = \begin{cases} 
1, & \text{if } Z_2(x) \leq 374 \\
\frac{1}{2} \left( \frac{e^{-\frac{(376.5-Z_2(x))6}{6}} + e^{-\frac{(376.5-Z_2(x))6}{6}}}{2} + 1 \right), & \text{if } 374 < Z_2(x) < 379 \\
0, & \text{if } Z_2(x) \geq 379
\end{cases}
\]

\[
\mu^H_{Z_1}(x) = \begin{cases} 
1, & \text{if } Z_1(x) \leq 517 \\
\frac{1}{2} \tanh \left\{ \frac{(517.5-Z_1(x))6}{6} + \frac{1}{2} \right\}, & \text{if } 517 < Z_1(x) < 518 \\
0, & \text{if } Z_1(x) \geq 518
\end{cases}
\]
\[\mu^H Z_2(x) = \begin{cases} 
1, & \text{if } Z_2(X) \leq 374 \\
\frac{1}{2} \tanh \{376.5 - Z_2(X)\} \frac{6}{5} & \text{if } 374 < Z_2(X) < 379 \\
0, & \text{if } Z_2(X) \geq 379 
\end{cases} \]

We get an equivalent crisp model

Maximize \(X_{3x3+1}\)

Subject to

\[\alpha_i Z_i(X) + X_{10} \leq \frac{\alpha_i}{2} (U_i + L_i)\]

\[6(16X_{11} + 19X_{12} + 12X_{13} + 22X_{21} + 13X_{22} + 19X_{23} + 14X_{31} + 28X_{32} + 8X_{33}) + X_{10} \leq 6(517.5)\]

\[96X_{11} + 114X_{12} + 72X_{13} + 132X_{21} + 78X_{22} + 114X_{23} + 84X_{31} + 168X_{32} + 48X_{33}) + X_{10} \leq 3105\]

And

\[\alpha_j Z_j(X) + X \leq \frac{\alpha_j}{2} (U_j + L_j)\]

\[\frac{6}{5}(9X_{11} + 14X_{12} + 12X_{13} + 16X_{21} + 10X_{22} + 14X_{23} + 8X_{31} + 20X_{32} + 6X_{33}) + X_{10} \leq \frac{6}{5}(376.5)\]

\[X_{11} + X_{12} + X_{13} = 14\]

\[X_{21} + X_{22} + X_{23} = 16\]

\[X_{31} + X_{32} + X_{33} = 12\]

\[X_{11} + X_{12} + X_{13} = 10\]

\[X_{12} + X_{22} + X_{32} = 15\]

\[X_{13} + X_{23} + X_{33} = 17\]

\[X_{ij} \geq 0 \quad i = 1,2,3, \quad j = 1,2,3.\]

The problem is solved by using the linear interactive and discrete optimization (LINDO) software, the optimal compromise solution is
\[
X^* = \begin{cases} 
  x_{11} = 9.5, & x_{13} = 4.5, & x_{21} = 0.5, \\
  x_{22} = 15, & x_{23} = 0.5, & x_{33} = 12 \\
  \text{and rest all } x_i \text{ are zeros}
\end{cases}
\]

\[
Z_1^* = 517.5 \quad \text{and} \quad Z_2^* = 376.5
\]

\[
\lambda = 0.5
\]

\[
\mu^E Z_1(x) = \begin{cases} 
  1, & \text{if } Z_1 \leq 517 \\
  \frac{e^{-1} \Psi_1(x) - e^{-1}}{1 - e^{-S}}, & \text{if } 517 < Z_1 < 518 \\
  0, & \text{if } Z_1 \geq 518
\end{cases}
\]

\[
\mu^E Z_2(x) = \begin{cases} 
  1, & \text{if } Z_2 \leq 374 \\
  \frac{e^{-1} \Psi_2(x) - e^{-1}}{1 - e^{-S}}, & \text{if } 374 < Z_2 < 379 \\
  0, & \text{if } Z_2 \geq 379
\end{cases}
\]

Then an equivalent crisp model for fuzzy model can be formulated as

Maximize \( \lambda \)

Subject to

\[
\lambda \leq \frac{e^{-s_{\Psi p}(X)} - e^{-S}}{1 - e^{-S}}, \quad p = 1, 2, \ldots, P
\]

Constraints (2.27) - (2.29)

\[
\Psi_1(X) = \frac{Z_1 - L_1}{U_1 - L_1} = \frac{Z_1 - 517}{518 - 517} = Z_1 - 517 \quad \text{and}
\]

\[
\Psi_2(X) = \frac{Z_2 - L_2}{U_2 - L_2} = \frac{Z_2 - 374}{379 - 374} = \frac{Z_2 - 374}{5}
\]

\[
\Psi_1(X) = -16X_{11} - 19X_{12} - 12X_{13} - 22X_{21} - 13X_{22} - 19X_{23} - 14X_{31} - 28X_{32} - 8X_{33} + 517
\]

\[
\Psi_2(X) = -9X_{11} - 14X_{12} - 12X_{13} - 16X_{21} - 10X_{22} - 14X_{23} - 8X_{31} - 20X_{32} - 6X_{33} + 374
\]
Then the problem is \[ \lambda \leq \frac{e^{\Psi_1(x)} - e^{-1}}{1-e^{-1}}, \quad \text{and} \quad \lambda \leq \frac{e^{\Psi_2(x)} - e^{-1}}{1-e^{-1}}. \]

And subject to

\[
\begin{align*}
X_{11} + X_{12} + X_{13} &= 14 \\
X_{21} + X_{22} + X_{23} &= 16 \\
X_{31} + X_{32} + X_{33} &= 12 \\
X_{11} + X_{21} + X_{31} &= 10 \\
X_{12} + X_{22} + X_{32} &= 15 \\
X_{13} + X_{23} + X_{33} &= 17 \\
X_{ij} &\geq 0 \quad i = 1,2,3, \quad j = 1,2,3.
\end{align*}
\]

Then the problem can be simplified as

Maximize \( \lambda \)

Subject to

\[
\begin{align*}
e^{-S} &\Psi_p(X) - (1-e^{-S})\lambda \geq e^{-S} \quad p = 1,2,\ldots,P \\
(2.2) - (2.4) \quad &\forall \ i,j \quad &\lambda \geq 0
\end{align*}
\]

\[ \Rightarrow \text{Maximize } \lambda \]

\[
\begin{align*}
e^{\Psi_1(X)} - (1-e^{-1})\lambda &\geq e^{\Psi_1(X)} - (1-0.368)\lambda \geq 0.368 \\
&\Rightarrow e^{\Psi_1(X)} - (0.6321)\lambda \geq 0.368
\end{align*}
\]

\[
\begin{align*}
e^{\Psi_2(X)} - (1-e^{-1})\lambda &\geq e^{\Psi_2(X)} - (0.6321)\lambda \geq 0.368
\end{align*}
\]

The problem is solved by the (LINGO) software

\[
X^* = \begin{cases}
  x_{11} = 9.5, & x_{13} = 4.5, \quad x_{21} = 0.5. \\
  x_{22} = 15, & x_{23} = 0.5, \quad x_{33} = 12. \\
  \text{rest all } x_{ij} \text{ are zero's}
\end{cases}
\]

\[ Z_1 = 517.5 \quad \text{and} \quad Z_2 = 376.5 \quad \lambda = 0.3775 \]
Also, we get set of nondominated solutions, \(\{517, 379\}\) and \(\{518, 374\}\)

### 2.5 Conclusions

In this Chapter, three type of membership functions have been used to solve the multi-objective transportation problem. If we use the hyperbolic membership function then the crisp model becomes linear. The optimal compromise solution does not change. If we compare with the solution obtained by the linear membership function they are equal. However, if we use the exponential type membership function, with different values of \(S\) (parameter) then the crisp model becomes non-linear and the optimal compromise solution does not change significantly.