Chapter 3

Skew Polynomial Rings

One of the most active and important research areas in noncommutative algebra is the investigation of skew polynomial rings. In 1920 Noether and Schmeidler [57] were the first to consider this kind of ring, and they were later systematically studied by Ore [58] in 1933 both in the context of differential equations and as operators on finite fields. Skew polynomial rings were one of the earliest examples in noncommutative algebra. Since then, these rings have been extensively studied, for example, for characterizing various kinds of radicals (Jacobson and Baer) and Krull dimensions of such rings, for constructing finite-dimensional algebras, classifying all valuations of these algebras, etc. One can find a theoretical treatment of this material in many standard books on noncommutative algebras [18, 19, 27, 42, 43, 52].

Recently skew polynomials have been successfully applied in many areas, for example solving Ordinary differential equations [17, 20, 39, 64], Control theory [24, 33] and Coding theory [54].

In this Chapter we study polynomial rings over a ring $R$ in one variable $x$ which
does not commute with elements of $R$. We also give a construction of a skew polynomial ring and describe some basic results. This Chapter is divided into four Sections. In Section (3.1) we give some known definitions and results. In next Section (3.2) it is shown that if $R$ is a ring which is an order in an artinian ring, then $O(R)$ is an order in an artinian ring. In third Section (3.3) we give (without proof) some known relations between the prime ideals of a ring $R$ and that of skew polynomial rings. In the last Section (3.4) we find a relation between the minimal prime ideals of $R$ and those of the differential operator ring $R[x; \delta]$, where $R$ is a Noetherian $\mathbb{Q}$-algebra and $\delta$ is a derivation of $R$, [14]. We also prove that $U(S(R)) = U[x, \sigma]$ is a completely prime ideal of $S(R)$, where $R$ is a $\sigma(\ast)$-ring and $\sigma(U) = U$ for all $U \in MinSpec(R)$, [12].

3.1 Some basic results

Definition 3.1.1. Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $\delta$ a $\sigma$-derivation on $R$. The ring

$$R[x; \sigma, \delta] = \left\{ \sum_{i=0}^{n} x^i a_i, a_i \in R \right\}$$

in which addition is defined as usual and multiplication is defined subject to the relation

$$ax = x\sigma(a) + \delta(a)$$

for all $a \in R$. This ring $R[x; \sigma, \delta]$ is called a skew polynomial ring (or ore extension).

There are two archetypical cases of skew polynomial rings.

1. If $\sigma$ is the identity map, then $R[x; \delta]$ is called differential operator ring or skew polynomial ring with derivation type.
2. If \( \delta = 0 \), then \( R[x; \sigma] \) is called skew polynomial ring with endomorphism type.

We denote \( R[x; \sigma, \delta] \) by \( O(R) \); \( R[x; \delta] \) by \( D(R) \) and \( R[x; \sigma] \) by \( S(R) \).

This definition of noncommutative polynomial rings was first introduced by Ore [58]; who combined earlier ideas of Hilbert (in case \( \delta = 0 \)) and Schlessinger (in case \( \sigma = I \)).

**Definition 3.1.2.** Let \( R \) be a ring and \( \sigma \) be an endomorphism. Then \( R[[x; \sigma]] \) denotes the ring of power series \( \sum_{i=0}^{\infty} x^i a_i \), in which multiplication is subject to the relation \( ax = x \sigma(a) \). This is called a skew power series ring.

**Theorem 3.1.1.** (Hilbert Basis Theorem) If \( R \) is a right (left) Noetherian, then \( O(R) \) is right (left) Noetherian.

**Proof.** See Theorem (1.12) and Theorem (1.17) of [27].

**Definition 3.1.3.** An element \( c \) in a ring \( R \) is regular if it is neither a left nor a right zero divisor, i.e., if \( ac = 0 \) or \( ca = 0 \) implies \( a = 0 \), for all \( a \in R \).

The set of regular elements of \( R \) is denoted by \( C(0) \).

Note that if \( R \subseteq Q \) are rings and \( x \in R \) such that \( x \) is invertible in \( Q \), then \( x \) must be regular element of \( R \).

**Proposition 3.1.2.** Let \( R \) be a semiprime Noetherian ring. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \). If \( f \in O(R) \) is a regular element then there exists \( g \in O(R) \) such that \( gf \) has leading co-efficient regular in \( R \).

**Proof.** Let \( S = \{ a_m \in R \mid x^m a_m + \ldots + a_0 \in O(R) f \text{, some } m \} \cup \{ O \} \).

We will show that \( S \) is a left ideal of \( R \).
Let \( a_m \in S \) and \( c \in R \), then we have \( g = \sum_{i=0}^{n} x^i a_i \in O(R) \) and

\( f = \sum_{i=0}^{t} x^i d_i \) regular in \( O(R) \) such that leading coefficient of \( (x^n a_n)(x^t d_t) \)
is \( \sigma^t(a_n,d_t) = a_m \).

Now, \( \hat{g} = \sum_{i=0}^{n} x^i \sigma^{-t}(c) a_i \in O(R) \) and so, \( \hat{g} f \) has a leading coefficient in \( S \).

So, coefficient of \( x^n \sigma^{-t}(c) a_n x^t d_t \) is \( c \sigma^t(a_n) d_t \in S \).

We will show \( S \) is essential.

Let \( 0 \neq I \subseteq R \) be a left ideal of \( R \).

Then \( O(I) \) is a left ideal of \( O(R) \), because for \( \sum_{i=0}^{n} x^i a_i \in O(I) \) and \( \sum_{i=0}^{m} x^i b_i \in O(R) \),
we have \( \left( \sum_{i=0}^{m} x^i b_i \right) \left( \sum_{i=0}^{n} x^i a_i \right) \in O(I) \) as \( a_i \in I \).

Now, for \( f \in O(R) \) is regular, so \( O(R) f \) is an essential left ideal of \( O(R) \).

Therefore, \( O(I) \cap O(R) f \neq 0 \).

Let \( \sum_{j=0}^{k} x^j a_j \in O(I) \cap O(R) f, 0 \leq j \leq k \).

Then \( a_k \in I \cap S \) which implies that \( S \) is essential as a left ideal.

So, \( S \) contains a left regular element.

Now, \( R \) is semiprime, and Proposition (3.2.13) of Rowen [60] implies that \( S \) contains
a regular element.

Hence there exists \( g \in O(R) \) such that \( g f \) has leading coefficient regular in \( R \). \( \square \)

### 3.2 Quotient rings of skew polynomial rings

**Definition 3.2.1.** Let \( R \) be a ring and \( X \subseteq R \) be a multiplicative set. A right ring
of fractions (or right Ore quotient ring, or right ore localization) for \( R \) with respect
to \( X \) is a ring homomorphism \( \phi : R \rightarrow Q \) such that:
1. $\phi(x)$ is a unit of $Q$ for all $x \in X$.

2. Each element of $Q$ has the form $\phi(a)\phi(x)^{-1}$ for some $a \in R$ and $x \in X$.

3. $\text{ker}(\phi) = \{r \in R \mid rx = 0 \text{ for some } x \in X\}$.

**Definition 3.2.2.** Let $A$ be a multiplicative set in a ring $R$. Then $A$ satisfies the right Ore condition if $rA \cap aR \neq \phi$ for all $r \in R$ and $a \in A$ and $A$ is called a right Ore set. Furthermore let for any $r \in R$ and $a \in A$ such that $ar = 0$ we have $b \in A$ such that $rb = 0$, then $A$ is said to be right reversible. A multiplicative set $A$ which is a right Ore set and a right reversible set is called a right denominator set. A left denominator set is defined symmetrically. A denominator set is any right and left denominator set.

**Theorem 3.2.1.** (Goldie) Let $R$ be a right Noetherian ring and $N(R)$ its prime radical. Then $C(0) \subseteq C(N(R))$ and given any $c \in C(0)$ and $a \in R$ there exists $d \in C(N(R))$ such that $ad \in cR$.

*Proof.* See Goodearl and Warfield [27] (Theorem (10.8)). \qed

**Theorem 3.2.2.** (Small, Talintyre) A right Noetherian ring $R$ with prime radical $N(R)$ is a right order in a right Artinian ring if and only if $C(0) = C(N(R))$.

*Proof.* See Goodearl and Warfield [27] (Theorem (10.9)). \qed

**Definition 3.2.3.** Let $Q$ be a quotient ring and $R$ be its subring, not necessarily containing 1. $R$ is called a right order in $Q$ if each $q \in Q$ has the form $ab^{-1}$ for some $a, b \in R$, $b$ regular. A left order is defined symmetrically.
Proposition 3.2.3. Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. If $R$ is an order in an artinian ring $S$, then $\sigma$ can be extended to an automorphism $\alpha$ of $S$ and $\delta$ can be extended to an $\alpha$-derivation $\partial$ of $S$.

Proof. Define $\alpha$ on $S$ by $\alpha(as^{-1}) = \sigma(a)\sigma(s)^{-1}$.

$\alpha$ is well defined.

Let $as^{-1} = bt^{-1} \in S$ then there exists $c, d \in R$ such that $ac = bd$ and $sc = td$.

Now, $\alpha(as^{-1}) = \alpha(bt^{-1})$ if $\sigma(a)\sigma(s)^{-1} = \sigma(b)\sigma(t)^{-1}$

i.e, if $\sigma(a)\sigma(c)\sigma(c)^{-1}\sigma(s)^{-1} = \sigma(b)\sigma(d)\sigma(d)^{-1}\sigma(t)^{-1}$

i.e, if $\sigma(ac)\sigma(sc)^{-1} = \sigma(bd)\sigma(td)^{-1}$ which is true.

In $S$ define addition as:

For any $as^{-1}, bt^{-1} \in S$ there exists $c, d \in R$ such that $sc = td$ and

$as^{-1} + bt^{-1} = (ac + bd)(td)^{-1}$

Now, $\alpha(as^{-1} + bt^{-1}) = \alpha((ac + bd)(td)^{-1}) = \sigma(ac + bd)\sigma(td)^{-1}$

and $\alpha(as^{-1}) + \alpha(bt^{-1}) = \sigma(a)\sigma(s)^{-1} + \sigma(b)\sigma(t)^{-1}$

$= \sigma(a)\sigma(c)\sigma(c)^{-1}\sigma(s)^{-1} + \sigma(b)\sigma(d)\sigma(d)^{-1}\sigma(t)^{-1}$

$= \sigma(ac)\sigma(sc)^{-1} + \sigma(bd)\sigma(td)^{-1}$

$= (\sigma(ac) + \sigma(bd))\sigma(td)^{-1}$ as $sc = td$.

$= \sigma(ac + bd))\cdot \sigma(td)^{-1}$.

Now, multiplication in $S$ define addition as:

For any $as^{-1}, bt^{-1} \in S$ there exists $u, w \in R$ such that $su = bw$ and define

$as^{-1}bt^{-1} = au(tw)^{-1}$

Now, $\alpha(as^{-1}bt^{-1}) = \alpha(au.(tw)^{-1}) = \sigma(au).\sigma(tw)^{-1}$

Now, $\alpha(as^{-1}).\alpha(bt^{-1}) = (\sigma(a)\sigma(s)^{-1})(\sigma(b)\sigma(t)^{-1})$

$= (\sigma(a)\sigma(u)\sigma(u)^{-1}\sigma(s)^{-1})(\sigma(b)\sigma(w)\sigma(w)^{-1}\sigma(t)^{-1})$
\[ (\sigma(au) - \sigma(su)^{-1})(\sigma(bw)\sigma(tw)^{-1}) \]
\[ = (\sigma(au)\sigma(tw)^{-1}) \text{ as } su = bw. \]

\( \alpha \) is onto:

Let \( as^{-1} \in S \) then \( a = \sigma(b) \) and \( s = \sigma(u) \) for some \( b \in R \) and \( u \in C(0) \).

Now, \( \alpha(bu^{-1}) = \sigma(b)\sigma(u)^{-1} = as^{-1} \).

\( \alpha \) is one-one:

Let \( \alpha(as^{-1}) = 0.t^{-1} \), the zero element of \( S \), then \( \sigma(a)\sigma(s)^{-1} = 0.t^{-1} \) which implies that \( \sigma(a) = 0 \) i.e., \( a = 0 \).

So, \( as^{-1} = 0s^{-1} \) the zero element of \( S \). So, \( \alpha \) is an automorphism of \( S \).

Now we define \( \partial \) on \( S \) by \( \partial(as^{-1}) = (\delta(a) - as^{-1}\delta(s))\sigma(s)^{-1} \) for any \( as^{-1} \in S \).

\( \delta \) is well defined:

Let \( as^{-1} = bt^{-1} \in S \) then as above there exists \( c, d \in R \) such that \( ac = bd \) and \( sc = td \).

Now, \( \partial(as^{-1}) = \partial(bt^{-1}) \)

if \( (\delta(a) - as^{-1}\delta(s))\sigma(s)^{-1} = (\delta(b) - bt^{-1}\delta(t))\sigma(t)^{-1} \)
i.e., \( (\delta(a) - as^{-1}\delta(s))\sigma(c)\sigma(c)^{-1}\sigma(s)^{-1} = (\delta(b) - bt^{-1}\delta(t))\sigma(d)\sigma(d)^{-1}\sigma(t)^{-1} \)
i.e., \( (\delta(a)\sigma(c) - as^{-1}\delta(s)\sigma(c))\sigma(sc)^{-1} = (\delta(b)\sigma(d) - bt^{-1}\delta(t)\sigma(d))\sigma(td)^{-1} \)
i.e., \( \delta(a)\sigma(c) - as^{-1}\delta(s)\sigma(c) = \delta(b)\sigma(d) - bt^{-1}\delta(t)\sigma(d) \) as \( sc = td \).
i.e., \( \delta(a)\sigma(c) - as^{-1}(\delta(s)\sigma(c) + s\delta(c) - s\delta(c)) \)
\[ = \delta(b)\sigma(d) - bt^{-1}(\delta(t)\sigma(d) + t\delta(d) - t\delta(d)) \]
i.e., \( \delta(a)\sigma(c) - as^{-1}\delta(sc) + as^{-1}s\delta(c) = \delta(b)\sigma(d) - bt^{-1}\delta(td) - bt^{-1}t\delta(d) \)
i.e., \( \delta(ac) - as^{-1}\delta(sc) = \delta(bd) - bt^{-1}\delta(td) \) which is true.

We now show that \( \partial \) is an \( \alpha \)-derivation.

For any \( as^{-1}, bt^{-1} \in S \)
\[ \partial(as^{-1} + bt^{-1}) = \partial(ac + bd)(td)^{-1}, \text{ as before} \]
\[ = [\delta(ac + bd) - (ac + bd)(td)^{-1}\delta(td)]\sigma(td)^{-1} \]
\[ = [\delta(ac + bd) - (ac^{-1}s^{-1} + bbd^{-1}t^{-1})(\delta(t)\sigma(d) + t\delta(d))]\sigma(td)^{-1} \]
as \(sc = td\).
\[ = [\delta(ac + bd) - as^{-1}(\delta(s)\sigma(c) + s\delta(c)) - bt^{-1}(\delta(t)\sigma(d) + t\delta(d))]\sigma(td)^{-1} \]
\[ = [\delta(a)\sigma(c) + a\delta(c) + \delta(b)\sigma(d) + b\delta(b) - as^{-1}\delta(s)\sigma(c) - a\delta(c) \]
\[ - bt^{-1}\delta(t)\sigma(d) - b\delta(d)]\sigma(td)^{-1} \]
\[ = [\delta(a)\sigma(c) + \delta(b)\sigma(d) - as^{-1}\delta(c)\sigma(c) - bt^{-1}\delta(t)\sigma(d)]\sigma(td)^{-1} \]
\[ \ldots (\ast) \]

Now, \(\alpha(as^{-1}) + \alpha(bt^{-1}) = (\delta(a) - as^{-1}\delta(s))\sigma(s^{-1}) + (\delta(b) - bt^{-1}\delta(t))\sigma(t^{-1}) \]
\[ = (\delta(a)\sigma(c) - as^{-1}\delta(s)\sigma(c))\sigma(c)^{-1}\sigma(s^{-1}) + (\delta(b)\sigma(d) \]
\[ - bt^{-1}\delta(t)\sigma(d)\sigma(d^{-1})\sigma(t^{-1}) \]
\[ = [\delta(a)\sigma(c) - as^{-1}\delta(s)\sigma(c) + \delta(b)\sigma(d) - bt^{-1}\delta(t)\sigma(d)]\sigma(td)^{-1} \]
as \(sc = td\).
\[ = \sigma(as^{-1} + bt^{-1}) \text{ by (\ast)} \]

Now, \(\alpha(as^{-1} \cdot bt^{-1}) = \alpha(au \cdot (tv)^{-1}) \text{ as before for some } u, v \in R \)
\[ = [\delta(au) - au(tv)^{-1}\delta(tv)]\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u) + a\delta(u) - au^{-1}tv^{-1}\delta(t)\sigma(v) - au^{-1}tv^{-1}t \]
\[ + au^{-1}tv^{-1}t\delta(v)]\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u)+a\delta(s^{-1}bv) - a(s)^{-1}bt^{-1}\delta(t)\sigma(v) - as^{-1}b\delta(v)]\sigma(tv)^{-1} \]
as \(su = bv\).
\[ = [\delta(a)\sigma(u) + a(\delta(s^{-1})\sigma(bv) + s^{-1}\delta(bv)) - as^{-1}bt^{-1}\delta(t)\sigma(v) \]
\[ - as^{-1}b\delta(v)]\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u) + a(-s^{-1}\delta(s)\sigma(s)^{-1}\sigma(s)\sigma(u) + s^{-1}\delta(b)\sigma(v) \]
\[ - as^{-1}b\delta(v)]\sigma(tv)^{-1} \]
\[ +s^{-1}b\delta(v) - as^{-1}bt^{-1}\delta(t)\sigma(v) - as^{-1}b\delta(v)]\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u) - as^{-1}\delta(s)\sigma(u) + as^{-1}\delta(b)\sigma(v) - as^{-1}bt^{-1}\delta(t)\sigma(v)]\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u) - as^{-1}\delta(s)\sigma(u)]\sigma(tv)^{-1} + as^{-1}[\delta(v)\sigma(v) - bt^{-1}\delta(t)\sigma(v)]\sigma(tv)^{-1} \]
\[ = \partial(as^{-1})\alpha(bt^{-1}) + as^{-1}\partial(bt^{-1}) \]

Since, \( \delta(as^{-1})\alpha(bt^{-1}) = (\delta(a) - as^{-1})\delta(s)\sigma(s)^{-1}\sigma(b)\sigma(t^{-1}) \)
\[ = (\delta(a) - as^{-1}\delta(s))\sigma(u)\sigma(v)^{-1}\sigma(s)^{-1}\sigma(b)\sigma(v)^{-1}\sigma(t^{-1}) \]
\[ = [\delta(a)\sigma(u) - as^{-1}\delta(s)\sigma(u)]\sigma(su)^{-1}\sigma(bv)\sigma(tv)^{-1} \]
\[ = [\delta(a)\sigma(u) - as^{-1}\delta(s)\sigma(u)]\sigma(tv)^{-1} \]

Hence, \( \alpha \) is an automorphism of \( S \) and \( \partial \) is an \( \alpha \)-derivation of \( S \). \( \square \)

**Definition 3.2.4.** Let \( R \) be a ring and \( U \) be a right Ore-set in \( R \). Let \( M \) be a right \( R \)-module. The set \( T_U(M) = \{ m \in M \mid mu = 0, \text{ for some } u \in U \} \) is called the \( U \)-torsion submodule of \( M \). \( M \) is said to be \( U \)-torsion if and only if \( T_U(M) = M \) and is said to be torsion free if \( T_U(M) = 0 \).

**Lemma 3.2.4.** Let \( R \) be a right Noetherian ring, \( \sigma \) is an automorphism and \( \delta \) a \( \sigma \)-derivation of \( R \). If \( M \) is the set of monic polynomials of \( O(R) \), then \( M \) is a right denominator set in \( O(R) \).

**Proof.** It is sufficient to show that for any \( g \in O(R) \) and \( f \in M \) we have,

\[ \frac{O(R)}{f \cdot O(R)} \]

is right \( M \)-torsion.

Let \( J = \{ h \in O(R) \text{ such that } gh \in f \cdot (O(R)) \} \).

Now, \( \varphi : \frac{O(R)}{J} \to \frac{O(R)}{f \cdot O(R)} \) defined by

\[ \varphi(s) = gs + f \cdot O(R) \]

is an embedding.

Now, \( \frac{O(R)}{f \cdot O(R)} \) is finitely generated over \( R \) and so is \( M \)-torsion as in [5]. \( \square \)
Theorem 3.2.5. Let $R$ be a right Noetherian ring. If $M$ is the set of monic polynomials of $O(R)$, then $|O(R)_M|_r = |R|_r$.

Proof. Let $L(A)$ denote the lattice of right ideals in a ring $A$.

As outlined in [5], we construct a strictly increasing function

$\varphi : L(R) \rightarrow L(O(R)_M)$ and $\psi : L(O(R)_M) \rightarrow L(R)$ to prove the theorem.

Now, $\varphi : L(R) \rightarrow L(O(R)_M)$ defined by $\varphi(I) = I(O(R)_M)$ for any ideal $I \subseteq R$, is the extension map.

Now, as in [5] consider $\eta : L(O(R)) \rightarrow L(R)$ the leading coefficient map by $\eta_i(J)$ to be the right ideal of $R$ generated by the leading coefficient elements of $J$ with degree $i$ for any right ideal $J$ of $O(R)$.

Let $J \subseteq K$ be a right ideal of $O(R)$, then $\eta(J) = \eta(K)$ if and only if, $K/J$ is finitely generated as an $R$-module.

So, for any right ideal $T$ of $O(R)_M$ if we put $\psi(T) = \eta(O(R) \cap T)$, then by Lemma (3.2.4) $\psi$ is strictly increasing. \hfill \Box

Lemma 3.2.6. Let $R$ be a ring which is an order in a right artinian ring $S$. Let $\sigma$ be an automorphism and $\delta$ a $\sigma$-derivation of $R$. Then:

1. Every regular element of $R$ is regular in $O(R)$.

2. Set of regular elements of $R$ satisfies the right Ore-condition in $O(R)$.

3. Any element of $O(S)$ has the form $f(x)c^{-1}$ for some $f(x) \in O(R)$ and some $c$ regular in $R$.

4. If $g(x) = f(x)c^{-1}$ is regular in $O(S)$, then $f(x)$ is regular in $O(R)$. 

5. Let $M$ be the set of monic polynomials of $O(S)$, then every regular element of $O(R)$ is right regular element of $O(S)_{M}$.

Proof. 1. Let $f = x^n a_n + \ldots + a_0 \in O(R)$ and $c \in C(O)$ of $R$.

Let $fc = 0$, then $a_i c = 0$, $0 \leq i \leq n$ so that $a_i = 0$ i.e., $f = 0$.

Let $cf = 0$ i.e., $c(x^n a_n + \ldots + a_0) = 0$ i.e., $x^n \sigma^n(c) a_n = 0$ which implies that $a_n = 0$ as $\sigma^n(c) \in C(0)$ with the same process we get that $a_i = 0$, $0 \leq i \leq n$.

So, $c$ is regular in $O(R)$.

2. Let $c \in C(0)$ and $f = \sum_{i=0}^{n} x^i a_i \in O(R)$.

Then, $c^{-1} f = c^{-1} \sum_{i=0}^{n} x^i a_i$

$= x^n \sigma^n(c)^{-1} a_n + \ldots + c^{-1} a_0$

$= \sum_{i=0}^{n} x^i \sigma^i(c_i)^{-1} a_i, c_i \in R$

$= \sum_{i=0}^{n} x^i b_i d_i^{-1} = \sum_{i=0}^{n} x^i u_i d^{-1}$ by Lemma (5.1) of [27],

where $d \in C(0), u_i \in R$.

$= gd^{-1}$, where $g = \sum_{i=0}^{n} x^i u_i \in O(R)$.

So, $fd = cg$. So, $C(0)$ satisfies right ore condition in $O(R)$.

3. Let $g(x) = x^n a_n + \ldots + a_0 \in O(S), a_i \in S$.

Now, $a_i = b_i c_i^{-1} = d_i c^{-1}$ by Lemma (5.1) of [27], where $d_i \in R$ and $c \in C(0)$.

So, $g(x) = (x^n d_n + \ldots + d_0)c^{-1} = f(x).c^{-1}$.

4. Let $h(x) \in O(R)$ be such that $f(x).g(x) = 0$

Then, $f(x).c^{-1}.h(x) = 0$ in $O(S)$.

So that $c.h(x) = 0$, which implies that $h(x) = 0$ as $c$ is regular in $O(R)$. 
5. Let $f$ be a regular element of $O(R)$.

   Let $f.gh^{-1} = 0$, the zero element of $O(S)_M$, then there exists $u, t \in O(S)$ such that $1.u = gt$ and $f.(gh^{-1}) = (fu)(ht)^{-1}$.

   Therefore, $fu = 0$. So, $fkd^{-1} = 0, k \in O(R), d \in C(0)$.

   So, $fk = 0$, which implies that $k = 0$ so that $u = 0$ so $gt = 0$.

   Hence $f$ is right regular.

\[\square\]

**Theorem 3.2.7.** If $R$ is an order in an artinian ring $S$, then $O(R)$ is an order in $O(S)_M$, where $M$ is the set of monic polynomials of $S$. Also $O(S)_M$ is artinian.

*Proof.* We can extend $\sigma$ to an automorphism $\alpha$ of $S$ and $\delta$ an $\alpha$-derivation $\partial$ of $S$ by Proposition (3.2.3).

Let $M$ be the set of monic polynomials of $O(S)$.

Then, since $O(S)$ is a right Noetherian ring, so by Lemma (3.2.4) $M$ is a right denominator set in $O(S)$.

So, by Theorem (3.2.5) $|O(S)_M| = |S| = 0$. So, $O(S)_M$ is right artinian ring.

Now, any regular element of $R$ is regular in $O(R)$ by Lemma (3.2.6) and the set of regular elements of $R$ forms a right denominator set in $O(R)$ by Lemma (3.2.6).

Also any element of $O(S)$ is of the form $f(x)c^{-1}$ for some $f(x) \in R$ and some $C \in C(0)$ by Lemma (3.2.6).

Moreover, if $f(x)c^{-1}$ is regular element of $O(S)$, then $f(x)$ is a regular element of $O(R)$.

So, any element of $O(S)_M$ is of the form $g(x)h(x)^{-1}$, where $g(x) \in O(R)$ and $h(x)$ regular in $O(R)$.

Also, every regular element of $O(R)$ is right regular as an element of $O(S)_M$ by
Lemma (3.2.6) and so, is invertible in $O(S)_M$.

Thus, $O(R)$ is a right order in $O(S)_M$. □

### 3.3 Prime ideals of skew polynomial rings

**Lemma 3.3.1.** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$.

1. If $P$ is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of $R$ and $\sigma(P \cap R) = P \cap R$.

2. If $Q$ is a prime ideal of $R$ such that $\sigma(Q) = Q$, then $S(Q)$ is a prime ideal of $S(R)$ and $S(Q) \cap R = Q$.

**Proof.** The proof follows on the same lines as in McConnell and Robson [53] (Lemma (10.6.4)). □

**Lemma 3.3.2.** Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra. Let $\delta$ be a derivation of $R$. Then:

1. If $P$ is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of $R$ and $\delta(P \cap R) \subseteq P \cap R$.

2. If $U$ is a prime ideal of $R$ such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.

**Proof.** See Goodearl and Warfield [27] (Theorem (2.22)). □
Lemma 3.3.3. Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then:

1. If $P$ is a prime ideal of $O(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of $R$ with $\delta(P \cap R) \subseteq P \cap R$ and $\sigma(P \cap R) = P \cap R$.

2. If $U$ is a prime ideal of $R$ such that $\sigma(U) = U$ and $\delta(U) \subseteq U$, then $O(U)$ is a prime ideal of $O(R)$ and $O(U) \cap R = U$.

Proof. See Goodearl and Warfield [27] (Theorem (2.22)) and McConnell and Robson [53] (Lemma (10.6.4)).

3.4 Differential operator rings

Theorem 3.4.1. Let $R$ be Noetherian $\mathbb{Q}$-algebra and $\delta$ be a derivation of $R$. Then $P \in MinSpec(D(R))$ if and only if $P = (P \cap R)[x; \delta]$ and $P \cap R \in MinSpec(R)$.

Proof. Let $P_1 \in MinSpec(R)$.

Then by Lemma(3.4) of Gabriel [26] implies that $\delta(P_1) \subseteq P_1$.

Now Proposition (14.2.5)(ii) of McConnel and Robson [52] implies that $P_1[x; \delta] \in Spec(S(R))$.

If possible suppose $P_1[x; \delta] \notin MinSpec(D(R))$ and $P_2 \subset P_1[x; \delta]$ be a minimal prime ideal of $D(R)$.

Then $P_2 = (P_2 \cap R)[x; \delta] \subset P_1[x; \delta] \in Spec(D(R))$.

So, $P_2 \cap R \subset P_1$, which is not possible as Lemma (2.21) of Goodearl and Warfield
[27] implies that $P_2 \cap R \in \text{Spec}(R)$.

Hence, $P_1[x; \delta] \in \text{MinSpec}(D(R))$.

Conversely let $P \in \text{MinSpec}(D(R))$.

Then Lemma (2.21) of Goodearl and Warfield [27] implies that $P \cap R \in \text{Spec}(R)$.

Suppose $P \cap R \notin \text{MinSpec}(R)$.

Let $P_1 \subset P \cap R$ be a minimal prime ideal of $R$.

Then, $P_1[x; \delta] \subset (P \cap R)[x; \delta] \subseteq P$.

Now as in first paragraph above $P_1[x; \delta] \in \text{Spec}(D(R))$ and we get a contradiction.

Hence, $P \cap R \in \text{MinSpec}(R)$.

We now prove the following (Proposition (2) of [12]).

**Proposition 3.4.2.** Let $R$ be a $\sigma(\ast)$-ring and $U \in \text{MinSpec}(R)$ be such that $\sigma(U) = U$. Then $U(S(R)) = U[x; \sigma]$ is a completely prime ideal of $S(R)$.

*Proof.* Proposition (2.3.1) implies that $P(R)$ is completely semiprime ideal of $R$ and further more $U$ is completely prime by Proposition (1.11) of [63].

Now, we note that $\sigma$ can be extended to an automorphism $\varphi$ of $R/U$.

Now, it is well known that $S(R)/U(S(R)) \simeq (R/U)[x; \varphi]$.

Hence $U(S(R))$ is a completely prime ideal of $S(R)$. \qed

**Corollary 3.4.3.** Let $R$ be a Noetherian $\sigma(\ast)$-ring and $U \in \text{MinSpec}(R)$. Then $U(S(R)) = U[x; \sigma]$ is a completely prime ideal of $S(R)$.

*Proof.* By Theorem (2.3.3), $\sigma(U) = U$. Now use the above Proposition (3.1.2). \qed