Chapter I

REGULAR RINGS

Continuous geometry is a complete complemented irreducible modular lattice in which join and meet are continuous. Von Neumann invented Regular Rings in order to “Coordinise” these lattices. In this chapter we present the basic properties of general regular rings, the nature and use of idempotents, the class of all principal right ideals (left ideals) as a complemented modular lattice, and other general properties of regular rings which are useful in the remaining chapters.

1.1 Definition: Suppose R is a ring with 1. A right ideal (left ideal) I of R is a set
   $I \subseteq R$ such that (i) $x, y \in I \Rightarrow x+y \in I$, (ii) $x \in I, z \in R \Rightarrow xz \in I \ (zx \in I)$.

1.2 Note: Suppose $a \in R$ then there exist minimal right ideal (left ideal) exist containing a which is called the principal right (left) ideal denoted by $(a)_r \ (a)_l$ is the set of all $az \ (za), \ z \in R$.
   i.e, $(a)_r = \{az / z \in R\}$ and $(a)_l = \{za / z \in R\}$.

1.3 Note: The set of all right ideals form a partially ordered set with respect to set theoretical inclusion $I \subseteq J$. This set has a minimum element:
   $(0) = (0)_r$ and a maximum one : $R = (1)_r$.

1.4 Note (1): For any set of right ideals $I_1, I_2, \ldots \exists$ a maximal right ideal $I \ni I \subseteq I_1, I_2, \ldots \ And \ I_1 \cap I_2 \cap \ldots$ is the maximal right ideal contained in every right ideal $I_1, I_2, \ldots \ and \ it \ is \ denoted \ by \ glb \ \{I_1, I_2, \ldots\}$. 
For any set of left ideals \( I_1, I_2, \ldots \), \( \exists \) a minimal right ideal \( I \ni I \supseteq I_1, I_2, \ldots \) and it is denoted by \( \text{lub} \{I_1, I_2, \ldots \} \).

1.5 Note: For the right ideals \( I_1, I_2 \), \( \text{glb} \{I_1, I_2\} \) is denoted by \( I_1 \land I_2 \).

And \( \text{lub} \{I_1, I_2\} \) is denoted by \( I_1 \lor I_2 \).

Thus the set of right ideals form a lattice with \( \land, \lor \) Zero (0), unit \( R \).

Two right ideals \( I, J \) are inverses if \( I \land J = (0) \) and \( I \lor J = R \).

Clearly \( I \lor J = \{ x + y / x \in I, y \in J \} \).

1.6 Definition: Suppose \( R \) is a ring with unit 1. An element \( e \in R \) is said to be an idempotent if \( e^2 = e \).

1.7 Lemma: \( e \) is an idempotent iff \( 1 - e \) is an idempotent.

Proof: Suppose \( e \) is an idempotent \( \Rightarrow e^2 = e \)

Now consider \( 1 - e \).

\[
e (1 - e) = e.1 - e.e = e - e^2 = e - e = 0.
\]

\[
(1 - e)^2 = (1 - e) (1 - e) = 1 (1 - e) - e(1 - e)
\]

\[
= 1 - e - 0 = 1 - e
\]

\[
\therefore (1 - e)^2 = 1 - e. \therefore 1 - e \text{ is an idempotent.}
\]

Suppose that \( 1 - e \) is an idempotent

put \( f = 1 - e \Rightarrow f \) is an idempotent \( \Rightarrow 1 - f \) is an idempotent

\( \Rightarrow 1 - (1 - e) \) is an idempotent \( \Rightarrow e \) is an idempotent.

1.8 Lemma: For any idempotent \( e, x \in (e) \), \( \Leftrightarrow ex = x \).

Proof Suppose \( e \) is an idempotent and let \( x \in (e) \).

\( \Rightarrow x = ey \) for some \( y \in R \).
Consider $e^x = e(e^y) = e^2y = ey = x$. \therefore ex = x.

Suppose $x = ex \rightarrow x \in (e)$. 

**1.9 Lemma**: I, J are inverse right ideals iff $I = (e)_r$, $J = (1 - e)_r$ for a suitably chosen idempotent e.

**Proof**: Suppose I, J are inverse right ideals

$\Rightarrow I \cap J = (0)$ and $I \setminus J = R$

$\therefore 1 \in R \Rightarrow 1 \in I \setminus J$

$\Rightarrow 1 = x + y$ where $x \in I$, $y \in J$

Let $z \in I$. $z = 1z = (x + y)z = xz + yz$

$\therefore x \in I$, $xz \in I$

$\therefore z \in I$ and $xz \in I \Rightarrow yz \in I$.

But $yz \in J$. \therefore $yz \in I \cap J \Rightarrow yz = 0$

So $z = xz \in (x)_r$. \therefore $I \subseteq (x)_r$.

Clearly $(x)_r \subseteq I$. \therefore $I = (x)_r$.

Similarly $J = (y)_r = (1 - x)_r$.

$x(1 - x) = (1 - x) \in I$

Similarly $x(1 - x) = (1 - x)x \in J$

$\therefore x(1 - x) \in I \cap J = (0) \Rightarrow x(1 - x) = 0 \Rightarrow x.1 - x.x = 0$

$\Rightarrow x - x^2 = 0 \Rightarrow x = x^2$.

$\therefore x$ is an idempotent.

let $x = e$. Similarly $y = 1 - e$.

$\therefore I = (e)_r$ and $J = (1 - e)_r$. 

Conversely suppose \( I = (e)_r \) and \( J = (1 - e)_r \), \( e \) is an idempotent

\[
x \in I \cap J \Rightarrow x \in I, x \in J
\]

\[
\Rightarrow x \in (e)_r, x \in (1 - e)_r
\]

\[
\Rightarrow ex = x, (1 - e)x = x.
\]

Consider \((1 - e)x = x \Rightarrow 1.x - e.x = x \Rightarrow x - ex = x \Rightarrow x - x = x \Rightarrow x = 0
\]

\[
\therefore I \cap J = (0).
\]

\[
\because 1 = e + (1 - e) \in I \lor J.
\]

\[
\therefore I \lor J = R
\]

\[
\therefore I \text{ and } J \text{ are inverses.}
\]

1.10 Lemma: If \( I \) and \( J \) are inverses and \( I = (e)_r, J = (1 - e)_r, e \) is an idempotent.

Then \( e \) is determined uniquely.

Proof: Suppose \( \exists \) another idempotent \( f \) such that \( I = (f)_r, J = (1 - f)_r \).

\[
\therefore e \in I = (f)_r \Rightarrow fe = e \text{ and } (1 - f)e = 0.
\]

Replace \( e, f \) by \((1 - e), (1 - f)\) then we have \( f(1 - e) = 0 \) and \( fe = f \)

Hence \( e = f \).

1.11 Lemma: The following are equivalent.

1. \((a)_r = (e)_r\), where \( e \) is an idempotent.

2. \( \exists x \cdot \exists a = a\), and \( ax = e \), \( e \) is an idempotent

Proof: \( 1 \Rightarrow 2 \).

Suppose \((a)_r = (e)_r\), \( e \) is an idempotent.

\[
\therefore a \in (e)_r \Rightarrow ea = a.
\]

\[
\therefore e \in (a)_r \Rightarrow e = ax \text{ for some } x \in R.
\]
Consider \( axa = ea = a \). \( \therefore axa = a \) and \( e = ax \)

\[ 1 \Rightarrow 2. \]

\[ 2 \Rightarrow 1 : \]

Suppose \( \exists x \ni axa = a, \) and \( ax = e, \) \( e \) is an idempotent.

Let \( y \in (a)_r \Rightarrow y = a z \) for some \( z \in R. \)

\( \Rightarrow y = axaz = eaz \in (e)_r. \) \( \therefore (a)_r \subseteq (e)_r. \)

Let \( y \in (e)_r \Rightarrow y = e z \) for some \( z \in R. \) \( \Rightarrow y = axz \in (a)_r \)

\( \therefore (e)_r \subseteq (a)_r. \) \( \therefore (a)_r = (e)_r. \) \( \therefore 2 \Rightarrow 1. \)

\( \therefore (1) \) and \( (2) \) are equivalent

1.12 Lemma : The three following properties are equivalent for an element \( a. \)

1. \( \exists x \) with \( axa = a \) exists.

2. \( \exists \) an idempotent \( e \) such that \( (a)_r = (e)_r. \)

3. \( \exists \) a right ideal \( J \) which is inverse to \( (a)_r \) exists.

Proof : By 1.11 lemma, \( (1) \) and \( (2) \) are equivalent

By 1.9 lemma, \( (2) \) and \( (3) \) are equivalent.

1.13 Definition : A ring \( R \) with unit 1 is called a regular ring if for every

\( a \in R \exists x \in R \) such that \( a x a = a. \) Here after \( R \) stands for a regular ring.

1.14 Note : For any regular ring \( R \) and \( a \in R, \) the following conditions are equivalent.

1. \( \exists \) an idempotent \( e \) such that \( (a)_r = (e)_r. \) \( [(a)_r = (e)_r]. \)

2. \( \exists \) a right ideal (left ideal) \( J \) which is inverse of \( (a)_r \) [\( (a)_l \)].

1.15 Note : Every regular ring is a semi simple ring.

[A ring is called left semi simple (right semi simple) if it is a direct sum of
minimal left (right) ideals. A ring $R$ is a left (right) semi simple if every left
(right) ideal in $R$ is a direct summand. If $R$ is regular ring with unit 1, forevery
$a \in R$, $(a \neq 0) \exists$ idempotent $e \cdot (a) = (e)$, [or $(a) = (e)$.] such that $I \oplus J = R.$
Where $I = (e)$, and $J = (1 - e)$. $[I = (e), J = (1 - e)].$

1.16 **Note :** A ring $R$ with unit 1 is left semi simple iff right semi simple.

1.17 **Notations :** $R$ is a regular ring. The set of all principal right ideals is denoted
by $R_R$. The set of all principal left ideals of $R$ is denoted by $L_R$.

1.18 **Definition :** Suppose $I$ is a right ideal of $R$, then $I^r = \{x \in R/ xy = 0 \ \forall \ y \in I\}$
Suppose $I$ is a left ideal of $R$, $I^l = \{x \in R/ yx = 0 \ \forall \ y \in I\}$.

1.19 **Lemma :** $I^l$ is a left ideal.

**Proof :** let $x, y, \in I^l$.

$\Rightarrow xz = 0, yz = 0 \ \forall \ z \in I$.

$\Rightarrow (x+y)z = xz + yz = 0 + 0 = 0 \ \forall \ z \in I$

$\therefore x + y \in I^l$.

let $x \in I^l$ and $r \in R \Rightarrow xy = 0 \ \forall \ y \in I$ and $r \in R$.

$rx.y = r(xy) = r(0) = 0. \ \forall \ y \in I$.

$\therefore rx \in I^l. \ \therefore I^l$ is a left ideal.

1.20 **Lemma :** $I \subseteq J \Rightarrow I^l \supseteq J^r$.

**Proof :** Suppose $I \subseteq J$. let $x \in J^r \Rightarrow xy = 0 \ \forall y \in J$.

$\Rightarrow xy = 0 \ \forall y \in I \Rightarrow x \in I^l (\because I \subseteq J). \ \therefore I^l \supseteq J^r$.

1.21 **Lemma :** $I \subseteq I^r$.

**Proof :** Let $x \in I \Rightarrow yx = 0 \ \forall y \in I^l \Rightarrow x \in I^r. \ \therefore I \subseteq I^r$. 
1.22 Note : \( I \subseteq I'. \)

1.23 Lemma : \( I' = I'^{rl}. \)

Proof: \( I \subseteq I^{rl} \) (By 1.21 lemma)

\[ \Rightarrow I' \supseteq I'^{rl} \) (By 1.20 lemma). \]

By the above note, replacing \( I \) by \( Il \), we have \( I' \subseteq I'^{rl}. \)

\[ \therefore I' = I'^{rl}. \]

1.24 Note : \( I' = I'^{lr} \) for any principal left ideal.

1.25 Lemma : For every principal right ideal \( I \) of \( R \), there exists a principal left ideal \( J \) such that \( I = J^r. \)

Proof: Suppose \( I \) is a principal right ideal. Then \( \exists \) an idempotent \( e \) such that \( I = (e)_r. \) Let \( J = (1 - e)_r. \) Then \( J \) is a principal left ideal.

Claim : \( I = J^r. \)

\[ x \in I \iff x \in (e)_r \iff ex = x, (1 - e)x = 0 \iff z(1 - e)x = 0 \forall z \in R. \]

\[ \iff yx = 0 \forall y \in (1 - e)_l = J \iff x \in J^r. \]

1.26 Note : For any principal left ideal \( I \), \( \exists \) a principal right ideal \( J \) such that \( I = J^r \).

1.27 Lemma : If \( I \) is a principal right ideal, then \( I = I'^{lr}. \)

Proof : Suppose \( I \) is a principal left ideal.

\[ \Rightarrow \exists \text{ a principal left ideal } J \ni I = J^r \) (By 1.25 lemma)

\[ I^{lr} = J^{lr} = J^r \) (from the note after 1.23 lemma) \( = I. \)

1.28 Note : If \( I \) is a principal left ideal, then \( I = I'^{rl}. \)

1.29 Lemma : If \( I \) is a principal right ideal. Then \( I' \) is a principal left ideal.

Proof : Suppose \( I \) is a principal right ideal.
\[ \Rightarrow I = J' \text{ for some principal left ideal } J. \]

\[ I^l = J'^l = J \text{ (from note after 1.27 lemma)} \]

\[ \therefore I^l \text{ is a principal left ideal} \]

**1.30 Theorem:** \( I \to I^l \) is a one to one mapping of \( R_R \to L_R \), \( I \to I' \) is a one to one mapping from \( L_R \) to \( R_R \). They are inverse to each other, and anti monotonic.

**Proof:**

Define \( f : R_R \to L_R \) by \( f(I) = I^l \).

Suppose \( f(I_1) = f(I_2) \Rightarrow I_1^l = I_2^l \Rightarrow I_1'^l = I_2'^l \Rightarrow I_1 = I_2. \)

\[ \therefore f \text{ is one - one}. \]

Similarly \( g : L_R \to R_R \) by \( g(I) = I' \) is one-one

and \( gof = \text{Identity on } R_R \) and \( fog = \text{Identity on } L_R \).

For \( (gof) (I) = g(f(I)) = g(I^l) = I'^l = I \) (By 1.27 lemma).

Similarly \( (fog) (I) = I \).

Suppose \( I, J \in R_R \) and \( I \subseteq J \Rightarrow I^l \supseteq J' \Rightarrow f(I) \subseteq f(J). \)

\[ \therefore \text{These mappings are inverses to each other and anti-monotonic}. \]

**1.31 Lemma:** If \( I, J \) are principal right ideals, then \( \exists \) two idepotents \( e, f \) such that \( ef = fe = 0 \) and \( I \lor J = (e)_r \lor (f)_r \).

**Proof:**

Suppose \( I, J \) are principal right ideals.

\[ \Rightarrow \exists \text{ an idempotent } e \text{ such that} \]

\[ I = (e)_r \text{ and } J = (b)_r. \text{ Put } J_1 = ((1 - e)b)_r. \]

**Claim:** \( I \lor J = I \lor J_1. \)

\[ I \lor J = \{eu + bv / u, v \in R\}. \]

\[ I \lor J_1 = \{eu^1 + (1-e)bv / u^1, v \in R\} = \{e(u^1-bv)+bv / u^1, v \in R\} \]
eu' + (1−e)bv = e (u'−bv) + bv.

u = u' − bv,  u' = u + bv.  Let x ∈ I ∨ J.

⇒ x = eu + bv = eu + bv + ebv − ebv = e(u+bv)+(1−e)bv

∈ I ∨ J₁.  ∴ I ∨ J ⊆ I ∨ J₁

Similarly I ∨ J₁ ⊆ I ∨ J.  ∴ I ∨ J = I ∨ J₁

∴ J₁ is principal right ideal  ∃ an idempotent f₁  ∃ J₁ = (f₁)r.

∵ f₁ ∈ J₁ = ((1−e)b)r  ⇒  f₁ = (1−e) bw.

ef₁ = e(1−e)bw = (e−e²) bw = (e−e) bw = 0.  ∴ ef₁ = 0.

put f = f₁(1−e).  ff₁ = f₁(1−e)f₁ = f₁(f₁−ef₁) = f₁f₁ = f₁² = f₁.

∴ ff₁ = f₁.

f² = f.f = ff₁(1−e) = f₁(1−e) = f  (∴  f = f₁(1−e))

∴ f² = f.

∴ f is an idempotent.

f = f₁(1−e) ∈ (f₁)r

f₁ = ff₁ ∈ (f₁)r.  ∴ (f₁)r = (f₁)r

∴ J₁ = (f₁)r.  (∴ J₁ = (f₁)r)

∴ I ∨ J = I ∨ J₁ = (e)r  ∨ (f₁)r.

ef = ef₁ (1−e) = 0(1−e) = 0  (∵ ef₁ = 0).  ∴ ef = 0

fe = f₁ (1−e)e = f₁ (e−e²) = f₁ (e−e) = f₁ (0) = 0.  fe = 0

∴ I ∨ J = (e)r  ∨ (f₁)r, where ef = fe = 0 and e² = e, f² = f.

1.32 Lemma : If I, J are principal right ideals, then I ∨ J is a principal right ideal.

Proof: Suppose I, J are principal right ideals. By 1.31 Lemma, ∃ two idempotents
e, f with \( ef = 0, fe = 0 \) and \( I \lor J = (e) \lor (f) \).

\[
e + f \in (e) \lor (f) \Rightarrow (e+f) \subseteq (e) \lor (f)
\]

\[
\therefore (e+f)e = e^2 + fe = e^2 = e.
\]

\[
\therefore e \in (e+f).
\]

Similarly \( (f) \subseteq (e+f) \).

\[
\therefore (e) \lor (f) \subseteq (e+f)
\]

From (1) & (2) \( (e) \lor (f) = (e+f) \).

\[
\therefore I \lor J = (e+f).
\]

\[
\therefore I \lor J \text{ is a principal right ideal and } e + f \text{ is idempotent}
\]

**1.33 Note:** If \( I, J \) are principal left ideals, then \( I \lor J \) is a principal left ideal.

**1.34 Note:** \( (I \lor J)^r = I^l \land J^l \) \( (I \lor J)^l = I^l \land J^l \)

**1.35 Lemma:** For every principal right ideals \( I, J \), \( I \lor J \) and \( I \land J \) are also principal right ideals.

**Proof:** Suppose \( I, J \) are principal right ideals, By 1.32 Lemma, \( I \lor J \) is a principal right ideal.

\[
\therefore I, J \text{ are principal right ideals, then } I^l, J^l \text{ are principal left ideals.}
\]

By above note \( I^l \lor J^l \) is also principal left ideal.

\[
\Rightarrow (I^l \lor J^l)^r \text{ is a principal right ideal.} \therefore (I^l \lor J^l)^r = I^l \land J^l
\]

\[
= I \land J \text{ is a principal right ideal (By 1.27 lemma)}
\]

**1.36 Note:** By 1.35 lemma, \( R_R^l \) is a lattice. Clearly \( 0 = (0) \), \( R = (1) \), the unit with the meet. \( I \lor J, I \land J \). Since for every principal right ideal \( I \), \( \exists \) a principal right ideal \( J \) \( \ni I \land J = (0), I \lor J = R \).

\[
\therefore R_R^l \text{ is a complemented lattice.}
\]

**1.37 Theorem:** \( R_R^l \) is a modular complemented lattice. For suppose \( I, J, K \) are principal right ideals and \( I \subseteq K \), then \( (I \lor J) \land K = I \lor (J \land K) \)
Proof: \[\therefore I \subseteq I \lor J \Rightarrow I \land K \subseteq (I \lor J) \land K\]
\[\Rightarrow I \subseteq (I \lor J) \land K (\therefore I \subseteq K).\]
\[J \land K \subseteq (I \lor J) \land K.\]
\[\therefore I \lor (J \land K) \subseteq (I \lor J) \land K (\therefore I \subseteq I \lor J, J \land K \subseteq K) \quad (1)\]
\[x \in (I \lor J) \land K \Rightarrow x \in I \lor J \quad \text{and} \quad x \in K \Rightarrow x = y + z, \quad y \in I, z \in J\]
\[\therefore x \in K \quad \text{and} \quad x \in I \lor J.\]
\[\text{i.e.,} \quad x = y + z, \quad y \in I, z \in J.\]
\[\text{Now } x \in K, \quad y \in I \subseteq K.\]

So, \(z \in K\), hence \(y \in J \land K \Rightarrow x \in I \lor (J \land K).\)
\[\therefore (I \lor J) \land K \subseteq I \lor (J \land K) \quad (2)\]
\[\therefore \text{From (1) and (2), } (I \lor J) \land K = I \lor (J \land K).\]

1.38 Note: By the above theorem, in any regular ring, the collection of principal right ideals forms a complemented modular lattice. Von Neumann proved a nearly complete converse: With certain exceptions, every complemented modular lattice is isomorphic to the lattice of principal right ideals of a regular ring, and the regular ring is unique upto isomorphism [38].

1.39 Note: By interchanging right and left, \(L_R\) has the same facts as \(R_R\) has.

1.40 Theorem: \(R_R\) and \(L_R\) are both complemented modular lattices with meet \(I \land J\), join \(I \lor J\), zero (0), unit \(R\). They are anti isomorphic.

Proof: \(R_R\) is complemented, modular lattice: Suppose \(I, J \in R_R\), then by above 1.35 lemma, \(I \lor J\) and \(I \land J \in R_R\).
By 1.36 note, 1.37 Theorem $R_R$ is a modular complemented lattice.

By interchanging right and left, $L_R$ is a modular complemented lattice.

And $R_R$, $L_R$ are anti isomorphic.

1.41 Definition: Suppose $R$ is a regular ring.

$$B = \{a \in R/ax = xa \forall x \in R\}$$ is called the center of $R$.

Clearly $B$ is a commutative ring with 1.

1.42 Theorem: The center $B$ of $R$ is a regular ring.

Proof: Suppose $R$ is a regular ring.

Claim: $B$ is a regular ring.

Let $a \in B \Rightarrow a \in R \Rightarrow \exists x \in R \exists axa = a$ (\because R is regular)

$\Rightarrow a^2x = a$ (\because $a \in B$). Let $x^3 = a^2x^3$.

Let $z \in R$. $xa^2z = za^2x$ (\because $a^2x = a$, $a^2x \in B$)

$$= a^2zx$$

\therefore $a^2z$ commutes with $x$. \therefore $a^2z$ commutes with $x^3$.

$a^2x^3z = x^3a^2z = a^2zx^3 = za^2x^3$. \therefore $a^2x^3 \in B$ and

$a \cdot a^2x^3. a = ax ax ax a = axa = a$

\therefore $a \in B$, $a^2x^3 \in B$ and $a \cdot a^2x^3. a = a$.

\therefore $B$ is a regular ring.

1.43 Lemma: If $a \in B$, then $(a)_r = (a)_l$ and $(a)_r^l = (a)_l^r$

Proof: Since $a \in B$, $ax = xa \forall x \in R$, so $(a)_r = (a)_l$.

Suppose $x \in (a)_r^l \Rightarrow xy = 0 \forall x \in (a)_r^l$.

$$\Rightarrow xaz = 0 \forall z \in R \Rightarrow xa = 0$$
\[ x \in (a)_l^r \Rightarrow yx = 0 \quad \forall \ y \in (a)_l \Rightarrow zax = 0 \quad \forall \ z \in R \]

i.e. \( ax = 0. \)

\[ \therefore \ xa = 0 \text{ and } ax = 0 \text{ are equivalent.} \quad \therefore \ (a)_l^r = (a)_r^l. \]

**1.44 Note:** For \( a \in R, (a)_l = (a)_r \) is denoted by \( (a)_r^l \). \( (a)_l^r \) is denoted by \( (a)_r^l \).

**1.45 Lemma:** 1) A principal right ideal \( I \) is at the same time a left ideal if \( I = (a)_r \) for some \( a \in B \).

2) If \( I \) is left ideal and \( I = (e)_r \) for some idempotent \( e \in R \) then this \( e \) is uniquely determined by \( I \) and \( e \in B \). Hence \( I = (e)_r \).  

**Proof:** 1) Suppose \( I \) is principal right ideal at the same time left ideal

\[ I = (e)_r \text{ and } I = (f)_l \text{ where } e, f \text{ are idempotents.} \]

\[ e \in I \Rightarrow e \in (f)_l \Rightarrow ef = e. \quad f \in I \Rightarrow f \in (e)_r \Rightarrow ef = f. \quad \therefore \ e = f \]

\[ \therefore \ I = (e)_r = (f)_l \]

Conversely suppose that \( I = (e)_r = (e)_l \) for some idempotent \( e \).

Then \( I \) is a principal right ideal at the same time a principal left ideal.

2) Suppose \( I = (e)_r \) is a left (right) ideal and \( e \) is an idempotent.

**Claim:** \( I = (e)_r \).

Let \( x \in I \Rightarrow x, e \in I \text{ and } I \text{ is a left ideal.} \)

So \( (x)_r, (e)_r \) and \( (x)_l \lor (e)_l \subseteq I. \)

\( (x)_l \lor (e)_l \) is a principal left ideal.

\[ \therefore (x)_l \lor (e)_l = (f)_l \text{ where } f \text{ is an idempotent.} \]

\[ \Rightarrow f \in (e)_r \Rightarrow ef = f \text{ and } e \in (e)_l \subseteq (x)_l \lor (e)_l = (f)_l \]

\[ \Rightarrow e \in (f)_l \Rightarrow ef = e. \quad \therefore \ e = f. \]
\[ x \in (x)_l \subseteq (x)_l \lor (e)_l = (f)_l = (e)_l \Rightarrow x \in (e)_l. \quad \therefore I \subseteq (e)_l \]

But \( (e)_l \subseteq I. \quad \therefore I = (e)_l. \]

Now \( I = (e)_l = (e)_l. \]

For every \( x : ex \in (e)_l = I = (e)_l \Rightarrow exe = ex \]
and \( xe \in (e)_l = I = (e)_l \Rightarrow exe = xe. \]
\[ \therefore ex = xe \forall x \in B. \quad \text{So } e \in B, I = (e)_l. \]

Suppose \( I = (g)_l \) for another idempotent \( g \)

\[ g \in (g)_l = I = (e)_l \Rightarrow ge = g. \quad e \in (e)_l = I = (g)_l \Rightarrow ge = e. \quad \therefore e = g. \]

**1.46 Lemma**: The only reductions of \( R \) (i.e., its decompositions into two right & left ideal direct summands) are: \( R = (e)_* + (1-e)_* \), \( e \) is an idempotent and \( e \in B \). [The only pairs of inverse sets \( I \) and \( J \) which are both right and left ideals are \( I = (e)_* \), \( J = (1-e)_* \), \( e \) is an idempotent and \( e \in B \)]

**Proof**: Suppose \( I = (e)_* \), \( J = (1-e)_* \), \( e \) is an idempotent and \( e \in B \).

Then \( R = (e)_* + (1-e)_* \), \( I \) and \( J \) are right and left ideals.

Conversely suppose \( I \) and \( J \) are right and left inverse ideals such that \( R = I + J, I \cap J = (0) \). By 1.45 lemma, \( I \) is principal right ideal at the same time left ideal then \( I = (e)_l = (e)_r, e \) is idempotent and \( e \in B \).

Let \( J = (1-e)_r \) clearly \( (1-e)_l = (1-e)_r \).
\[ \therefore R = (e)_* + (1-e)_*. \]

**1.47 Definition**: \( R \) is a regular ring. Then for every right and left ideal \( I \) has an inverse \( J \) which is right and left ideal. Then this is called total reducibility of \( R \).
1.48 Theorem: R is totally reducible. That is for every right and left ideal I has inverse which is right and left ideal J can be found.

Proof: By 1.46 lemma, the only reduction of R is its decomposition into two right and left ideal direct summands.

i.e., the only pairs of inverse sets I and J which are both right and left ideal.

\[ I = (e)_r, \quad J = (1-e)_r, \quad e \text{ is an idempotent and } e \in B. \]

1.49 Definition: R is said to be irreducible if it has only two ideals. i.e., (0) and R.

1.50 Theorem: R is irreducible iff B is a division algebra.

Proof: Suppose R is irreducible.

⇒ R has no ideals except (0) and R.

⇒ 0 and 1 are the only idempotents belong to B.

Suppose B is division algebra.

Then for any idempotent \( e \in B \) and \( e(1-e) = 0 \)

⇒ \( e = 0, \ 1-e = 0 \) ⇒ \( e = 0 \) or 1

⇒ the only idempotents of B are 0 and 1. So, R is irreducible.

Conversely suppose R is irreducible. Then B has 0, 1 as the only idempotents.

Let \( a \in B \) and \( a \neq 0 \).

Since B is regular ring, \( (a)_r = (e)_r \) in B, and \( e \) is idempotent & \( e \in B \).

⇒ \( e = 0 \) or 1 \( \therefore a \neq 0, \ \therefore e \neq 0. \ \therefore e = 1. \)

\( \therefore (a)_r = (1)_r \)

\( 1 \in (1)_r \Rightarrow 1 \in (a)_r \ \therefore 1 = ax \) for some \( x \in B \ = \ xa \ (\therefore a \in B). \)

\( \therefore 1 = ax = xa. \ \therefore \exists a^{-1} \in B. \)

\( \therefore B \) is division algebra.
1.51 Definition: Let \( Z_R = R_R \cap L_R \)

(\( Z_R \) is the set of all right and left principal ideals).

1.52 Note: There is one to one correspondence between \( Z_R \) and the set of all idempotents \( e \in B \). For \( I \in Z_R \Rightarrow I \) is principal right and left ideal.

\[ \Rightarrow I = (e)_r \text{ and } e \text{ is idempotent and } e \in B. \]

Suppose \( e \in B \Rightarrow e \) is central idempotent, then \( (e)_r = (e)_l = (e)_e \) is principal right at the same time principal left ideal.

\[ \therefore \text{There is one-one correspondance between } Z_R \text{ and central idempotents}. \]

1.53 Lemma: If \( e \) is an idempotent, then \((1-e)xe = 0 \ \forall \ x \in R\), is equivalent to \( e \in B \).

Proof: Suppose \( e \) is central idempotent.

let \( x \in R \), consider \((1-e)xe = (1-e)ex = (e-e^2)x = (e-e)x = 0.x = 0.\)

Conversely suppose \( e \) is idempotent and \((1-e)xe = 0 \ \forall \ x \in R \).

Suppose \((1-e)xe = 0 \Rightarrow exe = xe, xe \in (e)_r \)

As \((e)_r \) is a right ideal. \( xey \in (e)_r \) and \( ey \) is the general element of \((e)_r \)

So \( u \in (e)_r \Rightarrow u = ey \text{ for some } y \in R. \)

\[ \Rightarrow xu = xey \in (e)_r. \quad \therefore (e)_r \text{ is a left ideal} \]

\[ \therefore e \text{ is central idempotent.} \]

\[ \text{i.e. } e \in B. \text{ (By 1.45 lemma)}. \]

1.54 Theorem: \( Z_R \) is the set of all \( I \in R_R \) which has unique inverse \( J \in R_R \).

Proof: Suppose \( I \in Z_R \).

\[ \Rightarrow I \text{ is principal right ideal at the same time left ideal}. \]
\[ I = (e)^r \text{ for unique idempotent } e \in R. \]

let \( J = (1-e)^r \Rightarrow J \) is principal right ideal at the same time left ideal and \( (1-e) \) is unique. \( \Rightarrow J \) is the unique inverse of \( I \).

\[ \therefore \text{ Every } I \in Z_R \text{ has unique inverse } J \in Z_R. \]

Conversely suppose \( I = (e)^r \) for a unique idempotent \( e \in R \).

Put \( e_1 = e + ex \)(1–e).
\[ e_1e = [e + ex (1–e)]e = e^2 + ex (1–e)e = e^2 + ex (0) = e^2 = e. \]
\[ e_1 \in (e_1)^r. \]
\[ ee_1 = e[e + ex (1–e)] = e^2 + e^2x \( (1–e) \)
\[ = e + ex (1–e) = e_1. \quad \therefore e_1 \in (e_1)^r \]
\[ \therefore I = (e)^r = (e_1)^r \]
\[ e_1^2 = e_1e_1 = e_1ee_1 = ee_1 = e_1 \]
\[ \therefore e_1^2 = e_1. \quad \therefore e_1 \text{ is idempotent. } \therefore e = e_1. \]
\[ ex (1–e) = 0 \{ \text{by 1.53 lemma, } (1–e) \in B[By taking (1–e) in the place of e] \}
\[ \therefore e \in B. \quad \therefore \text{I has unique inverse } J = (1–e)^r. \]

1.55 Lemma: The unique inverse of \( (e)^* \), \( e \) is an idempotent and \( e \in B \).

Proof: Suppose \( e \) is an idempotent
\[ \Rightarrow (e)^* \text{ is a principal right ideal and at the same time left ideal.} \]

So by 1.9 lemma, \( (1–e)^* \) is inverse of \( (e)^* \), and is unique by 1.54 theorem.

Claim: \[ (e)^* = (1–e)^* \quad (e)^* = (e)^r \Rightarrow (1–e)^r. \]
\[ x \in (e)^r \Rightarrow xy = 0 \quad \forall y \in (e)^r \Rightarrow xez = 0 \quad \forall z \Rightarrow xe = 0. \]
\( x \in (e)_* \implies yx = 0 \quad \forall \ y \in (e)_* \implies zex = 0 \quad \forall \ z \implies ex = 0. \)

\[
\therefore x \in (e)_* \iff xe = 0 = ex
\]

\((e)_* = \{x / ex = 0\} = \{x / (1-e)x = x\} = (1-e)_*.
\]

\[
\therefore (e)_* = (1-e)_*.
\]

\[
\therefore (e)_* \text{ is unique inverse of (e)_* for } e \in B, \text{ and e is an idempotent.}
\]

**1.56 Lemma**: If \(e, f\) are idempotents and \(e, f \in B\). Then \((e)_* \land (f)_* = (ef)_*\) and \(ef \in B\) and idempotent.

**Proof**: Clearly \(ef\) is an idempotent

For \((ef)^2 = (ef)(ef) = ee ff \implies e, f \in B\)

\[
= e^2f^2 = ef.
\]

\[
\therefore ef \in (e)_*.
\]

\[
\therefore (ef)_* \subseteq (e)_*.
\]

Similarly \((ef)_* \subseteq (f)_* \implies (ef)_* \subseteq (e)_* \land (f)_*\)

\[
x \in (e)_* \land (f)_* \implies x \in (e)_* \text{ and } x \in (f)_*.
\]

\[
\implies ex = x \text{ and } fx = x.
\]

\[
\implies (ef)x = x \implies x \in (ef)_*.
\]

\[
\therefore (e)_* \land (f)_* \subseteq (ef)_*.
\]

\[
\therefore (e)_* \land (f)_* = (ef)_*.
\]

**1.57 Lemma**: If \(e, f\) are idempotents and \(e, f \in B\). Then \(e + f - ef\) is an idempotent and \(e + f - ef \in B\) and \((e)_* \lor (f)_* = (e + f - ef)_*\).

**Proof**: \((e + f - ef) = 1 - (1-e)(1-f) \in B\).
and $e + f - ef$ is also an idempotent.

By 1.40 theorem,

$$ (e)_* \lor (f)_* = ((e)_* \land (f)_*)_* = ((1-e)_* \lor (1-f)_*)_* = (e + f - ef)_* $$

1.58 Note: $\land$ and $\lor$ satisfy distributive laws.

i.e. $e, f, g \in B$ and

$$ ((e)_* \lor (f)_*) \land (g)_* = ((e)_* \land (g)_*) \lor ((f)_* \land (g)_*). $$

For the above LHSs and RHSs are equal to $(eg + fg - efg)_*$. 

1.59 Theorem: $Z_R$ is a complemented boolean algebra.

Proof: From above 1.53 lemma, 1.55 lemma, 1.56 lemma and 1.57 lemma, we have the result.

1.60 Definition: Given an idempotent $e$,

$$ R(e) = \{x / xe = ex = x\} $$

Then $R(e)$ is regular ring with unit $e$.

For : Clearly $R(e)$ is a ring with unit $e$.

$R(e)$ is regular:

Suppose $a \in R(e) \Rightarrow a \in R \Rightarrow \exists x \in R \exists a x a = a$

$e exe = exe = exe e$

$$ x^1 = exe \in R(e). $$

Now consider $ax^1 a = a exe a = ae. x. ea$

$$ = a x a \therefore a \in R(e) \text{ so } ea = ae = a \] = a$$

$$ \therefore R(e) \text{ is regular. } $$