Fuzzy statements usually take truth values in the interval \([0,1]\) of real numbers, while the ordinary (or conventional or crisp) statements take truth values in the two element set \(\{F,T\}\) or \(\{0,1\}\), where \(F\) or 0 stands for ‘false’ and \(T\) or 1 stands for ‘True’. However, \([0,1]\) is found to be insufficient to have the truth values of certain fuzzy statements. For example, let us consider the statement ‘Guntur is a good city’. The truth value of this statement may not be a real number in \([0,1]\). Being good city may have several components: good in environment, good in cleanliness, good in educational facilities, good in health facilities, good in political awareness, good in literacy among the public, good in public transport systems etc. The truth value corresponding to each component may be a real number in \([0,1]\). If \(n\) is the number of such components under consideration, then the truth value of the statement ‘Guntur is a good city’ is a \(n\)-tuple of real numbers in \([0,1]\); that is, it is an element in \([0,1]^n\).

\([0,1]^n\) is not a totally ordered set when \(n > 1\), under the usual coordinate-wise ordering, where \([0,1]\) is considered with the usual ordering of real numbers. However, \([0,1]^n\) satisfies certain rich lattice theoretic properties; for example \([0,1]^n\) is a complete lattice satisfying the infinite meet distributivity; that is, for any element \(a\) and any subset \(X\) of \([0,1]^n\),

\[
a \land \left( \bigvee_{x \in X} x \right) = \bigvee_{x \in X} (a \land x).
\]

For this reason, we consider a complete lattice \(L\) satisfying the infinite meet distributivity to have the truth values of fuzzy statements.
If X denotes the set of all cities in India, then the set A of good cities in X is actually not a subset of X, but it is a fuzzy subset of X, since being good is fuzzy. That is, A can be considered as a function of X into L, a lattice of the type discussed above. Such a fuzzy subset is called an L-fuzzy subset of X. In this chapter, we first discuss in section 1.1 about the frames, which are the complete lattices satisfying the infinite meet distributivity. In section 1.2, we deal with L-fuzzy subsets of a given set and several lattice theoretic properties of these. In section 1.3, we consider a lattice X = (X, ∧, ∨) and define the notion of an L-fuzzy ideal of X and prove several structural properties of these. Finally, in section 1.4, we define the notion of an L-fuzzy filter of a lattice X as an L-fuzzy ideal of the dual of X and extend all the results on L-fuzzy ideals to L-fuzzy filters.

1.1 FRAMES

In this section we discuss the lattices in which all the fuzzy objects we consider later in the thesis have truth values. First of all, let us recall that a partially ordered set is called a lattice if every non-empty finite subset of it has greatest lower bound (glb) and least upper bound (lub). If (L, ≤) is a lattice and a and b ∈ L, then the glb and lub of \{a, b\} is denoted by a ∧ b and a ∨ b respectively and, in this case, the binary operations ∧ and ∨, which are respectively called ‘meet’ and ‘join’, satisfy the idempotent, associative, commutative and the absorption laws. Also, recall that if \((L, ∧, ∨)\) is an algebraic structure in which ∧ and ∨ are idempotent, associative and commutative binary operations on L satisfying the absorption laws, then \((L, ≤)\) is a lattice in which a ∧ b and a ∨ b are precisely the glb and lub of \{a, b\} respectively, for any a and b ∈ L, where ≤ is defined by
'\( a \leq b \iff a = a \land b (\iff a \lor b = b) \)'.

Let us begin with the following.

1.1.1 Definition

A lattice \( (L, \land, \lor) \) is called a Brouwerian lattice (or implicative lattice) if, for any elements \( a \) and \( b \) in \( L \), there exists a largest element \( x \) in \( L \) such that \( a \land x \leq b \) and this largest element \( x \) in \( L \) will be denoted by \( a \rightarrow b \).

The symbol \( \rightarrow \) will be read as ‘arrow’. If \( L=(L, \land, \lor) \) is a Brouwerian lattice, then \( \rightarrow \) can be treated as another binary operation on \( L \) satisfying the following.

For any \( a, b \) and \( x \in L \),

\[
\begin{align*}
a \land x \leq b & \iff x \leq a \rightarrow b.
\end{align*}
\]

1.1.2 Examples

(1) Consider the interval \((0,1]\) of real numbers \( x \) such that \( 0 < x \leq 1 \). Then \((0,1]\) is a lattice in which, for any \( a \) and \( b \),

\[
\begin{align*}a \land b &= \min\{a,b\} \quad \text{and} \quad a \lor b = \max\{a,b\}.
\end{align*}
\]

Also, for any \( a \) and \( b \in (0,1] \), define

\[
\begin{align*}a \rightarrow b &= \begin{cases}
1 & \text{if } a \leq b \\
b & \text{if } b < a
\end{cases}
\end{align*}
\]

Then, it can be easily verified that, for any \( x \in (0,1] \),

\[
\begin{align*}a \land x \leq b & \iff x \leq a \rightarrow b.
\end{align*}
\]

Therefore \((0,1]\) is a Brouwerian lattice. In fact, for any real numbers \( r \) and \( s \) such that \( r < s \), the interval \((r,s]\) is a Brouwerian lattice where \( \land, \lor \) and \( \rightarrow \) are defined as given above, in which \( 1 \) is replaced by \( s \) (note that \( a \rightarrow b \) is the largest element in the lattice for any \( a \leq b \)).
(2) In general, any chain $C$ with the largest element $1$ is a Brouwerian lattice, where $\land, \lor$ and $\rightarrow$ are defined as given in (1) above.

(3) If $\{L_i\}_{i \in I}$ is a non-empty class of Brouwerian lattices, then the product $L = \prod_{i \in I} L_i$ is again a Brouwerian lattice, where, for any $a = (a_i)_{i \in I}$ and $b = (b_j)_{j \in I} \in L$,

\[
\begin{align*}
    a \land b &= (a_i \land b_j)_{i \in I} \\
    a \lor b &= (a_i \lor b_j)_{i \in I} \\
    \text{and } a \rightarrow b &= (a_i \rightarrow b_j)_{i \in I}.
\end{align*}
\]

(4) In particular, $[0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and $[0,1]^n$, for any positive integer $n$, are Brouwerian lattices.

(5) Let $X$ be any topological space and $\mathcal{O}(X)$ be the set of all open subsets of $X$. Then $(\mathcal{O}(X), \cap, \cup)$ is a lattice, where $\cap$ and $\cup$ are the usual set intersection and union. For any open sets $A$ and $B$ in $X$, define

\[
A \rightarrow B = \text{the interior of } (X-A) \cup B.
\]

Then, for any open set $G$ in $X$, it can be easily verified that,

\[
A \land G \subseteq B \iff G \subseteq ((X-A) \cup B)^\circ = A \rightarrow B.
\]

Therefore $\mathcal{O}(X)$ forms a Brouwerian lattice.

(6) Recall that a complemented distributive lattice is called Boolean algebra. If $(B, \land, \lor, ')$ is a Boolean algebra, then it is a Brouwerian lattice in which $a \rightarrow b = a' \lor b$ for any $a$ and $b$ in $B$, where $a'$ is the complement of $a$ in $B$; for, $a \land (a' \lor b) = (a \land a') \lor (a \land b) = a \land b \leq b$ and for any $x \in B, a \land x \leq b$

\[
\Rightarrow a' \lor (a \land x) \leq a' \lor b \Rightarrow (a' \lor a) \land (a' \lor x) \leq a' \lor b \Rightarrow x \leq a' \lor x \leq a' \lor b.
\]

Note that for any elements $a$ and $b$ in a Brouwerian lattice $L$, $a \land b \leq a$ and hence $b \leq a \rightarrow a$. In other words, $a \rightarrow a$ is the largest element in any Brouwerian
lattice $L$, for any $a \in L$. Thus every Brouwerian lattice is bounded above. In example (6) above, we have proved that every Boolean algebra is a Brouwerian lattice. Note that the converse of this is not true; for consider the lattice $\emptyset (X)$ of open subsets of a topological space (given in example (5) above). Then $\emptyset (X)$ is a Brouwerian lattice which is not a Boolean algebra, except in the case when each open set in $X$ is closed also. However, a Brouwerian lattice must be necessarily distributive, as proved below.

1.1.3 Theorem

Every Brouwerian lattice is distributive.

**Proof:** Let $(L, \wedge, \vee)$ be a Brouwerian lattice, in which, for any $a$ and $b \in L$, $a \rightarrow b$ is the element in $L$ such that

$$a \wedge x \leq b \iff x \leq a \rightarrow b$$

for any $x \in L$.

Let $a$, $b$ and $c \in L$ and $s = (a \wedge b) \vee (a \wedge c)$.

Then clearly $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$. Also, since $a \wedge b \leq s$, we have $b \leq a \rightarrow s$.

Similarly $c \leq a \rightarrow s$ and hence $b \vee c \leq a \rightarrow s$, which implies that

$$a \wedge (b \vee c) \leq s = (a \wedge b) \vee (a \wedge c).$$

Therefore $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Thus $(L, \wedge, \vee)$ is a distributive lattice.

The converse of the above result is not true; for, consider the example given below.

1.1.4 Example

Consider the set $\mathbb{N}$ of non-negative integers and define

$$a \leq b \text{ if and only if } a \text{ divides } b \text{ for any } a, b \in \mathbb{N}.$$

Then $\left(\mathbb{N}, \leq\right)$ becomes a lattice in which

$$a \wedge b \text{ = greatest common divisor of } a \text{ and } b$$

and $a \vee b = \text{least common multiple of } a \text{ and } b$. 

31
which are usually denoted by \((a, b)\) and \([a, b]\) respectively for any \(a\) and \(b\) in \(\mathbb{N}\). We observe that \(a \land (b \lor c) = (a \land b) \lor (a \land c)\) for any \(a\) and \(b\) in \(\mathbb{N}\). This is trivial if any of \(a, b\) and \(c\) is 0 or 1. Therefore, we can assume that \(a, b\) and \(c\) are positive integers greater than 1 and hence we can write

\[
a = \prod_{i=1}^{n} p_i^{\alpha_i}, \quad b = \prod_{i=1}^{n} p_i^{\beta_i} \quad \text{and} \quad c = \prod_{i=1}^{n} p_i^{\gamma_i}
\]

where \(p_1, \ldots, p_n\) are distinct primes and \(\alpha_i, \beta_i, \gamma_i\) are non-negative integers.

Then

\[
a \land (b \lor c) = \left( \prod_{i=1}^{n} p_i^{\alpha_i} \right) \land \left( \prod_{i=1}^{n} p_i^{\max(\beta_i, \gamma_i)} \right)
\]

\[
= \prod_{i=1}^{n} p_i^{\min(\alpha_i, \max(\beta_i, \gamma_i))}
\]

\[
= \prod_{i=1}^{n} p_i^{\max(\min(\alpha_i, \beta_i), \min(\alpha_i, \gamma_i))}
\]

\[
= \left( \prod_{i=1}^{n} p_i^{\min(\alpha_i, \beta_i)} \right) \lor \left( \prod_{i=1}^{n} p_i^{\min(\alpha_i, \gamma_i)} \right)
\]

\[
= (a \land b) \lor (a \land c).
\]

Thus \(\left(\mathbb{N}, \land, \lor\right)\) is a distributive lattice, which is bounded above (since \(a\) divides 0 and hence \(a \leq 0\) for any \(a \in \mathbb{N}\), so that 0 is the largest element in \(\mathbb{N}\)). However, this is not a Brouwerian lattice. For, consider the elements 2 and 1 in \(\mathbb{N}\). Then

\[
2 \land p = 1 \quad \text{for all odd primes} \ p
\]

and 0 is the only element bigger than all odd primes and \(2 \land 0 = 2 \land 1\). That is, there is no largest element \(x\) such that \(2 \land x \leq 1\). In fact, for any positive integer a greater than 1, there is no largest \(x\) in \(\mathbb{N}\) such that \(a \land x \leq 1\) (equivalently, \(a \land x = 1\), since 1 is the least element in \(\mathbb{N}\)).
In 1.1.3 and 1.1.4 above, we have observed that the distributivity of a lattice is not equivalent to the Brouwerian property. However, in the case of complete lattices, the Brouwerian property is proved to be equivalent to an infinite distributivity, namely the distributivity of meet over infinite joins. First of all let us recall that a partially ordered set \((L, \leq)\) is called a complete lattice if every subset of \(L\) has glb and lub in \(L\) and, in this case, for any \(A \subseteq L\), glb of \(A\) will be denoted by glb \(A\) or inf \(A\) or \(\bigwedge_{a \in A} a\) or simply \(\bigwedge A\) and, similarly, lub of \(A\) will be denoted by lub \(A\) or sup \(A\) or \(\bigvee_{a \in A} a\) or simply \(\bigvee A\).

1.1.5 Definition

A complete lattice \((L, \leq)\) is said to satisfy the infinite meet distributivity if

\[
a \bigwedge \left( \bigvee_{b \in B} b \right) = \bigvee_{b \in B} (a \land b)
\]

for any \(a \in L\) and \(B \subseteq L\). Also, \(L\) is said to satisfy the infinite join distributivity if

\[
a \bigvee \left( \bigwedge_{b \in B} b \right) = \bigwedge_{b \in B} (a \lor b)
\]

for any \(a \in L\) and \(B \subseteq L\).

Clearly, any complete lattice satisfying the infinite meet distributivity is necessarily distributive. However, a complete distributive lattice need not satisfy the infinite meet distributivity; for, consider the following.

1.1.6 Example

The lattice \(\mathbb{N}\) given in Example 1.1.4 is a complete lattice and also distributive. However, this does not satisfy the infinite meet distributivity; for, let \(B = \) The set of all odd positive integers.
Then
\[ 2 \land \left( \lor_{b \in B} b \right) = 2 \land 0 = 2 \]
and
\[ \lor_{b \in B} (2 \land b) = 1, \text{ since } 2 \land b = 1 \text{ for all } b \in B \]
and hence
\[ 2 \land \left( \lor_{b \in B} b \right) \neq \lor_{b \in B} (2 \land b). \]

It is known that the distributivity of \( \land \) over \( \lor \) in any lattice is equivalent to the distributivity of \( \lor \) over \( \land \); that is, \( a \land (b \lor c) = (a \land b) \lor (a \land c) \) for all \( a, b \) and \( c \) if and only if \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \) for all \( a, b \) and \( c \).

However, this cannot be extended to infinite distributivities of \( \land \) over \( \lor \) and \( \lor \) over \( \land \) in complete lattices. Consider the following example given in 1.1.8. First we prove the following elementary result in number theory.

1.1.7 Theorem

Let \( B \) be a nonempty set of positive integers, then there exists a finite subset \( F \) of \( B \) such that
\[ \gcd B = \gcd F. \]

**Proof:** Let
\[ I = \{ a_1 b_1 + a_2 b_2 + \ldots + a_n b_n : b_i \in B \text{ and } a_i \in \mathbb{Z} \} \]

Then, it can be easily verified that \( I \) is an ideal of the ring \( \mathbb{Z} \) of integers. It is well known that \( \mathbb{Z} \) is a principal ideal domain; that is, \( \mathbb{Z} \) is an integral domain in which every ideal is a principal ideal. In particular, \( I \) is a principal ideal of \( \mathbb{Z} \) and hence there exists a non-negative integer \( d \) such that
\[ I = d\mathbb{Z} = \{ da : a \in \mathbb{Z} \}. \]

Now, \( d \in I \) and hence \( d = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \) for some \( b_1, b_2, \ldots, b_n \in B \) and
Take 
\[ F = \{b_1, b_2, \ldots, b_n\} \]
which is a finite subset of \( B \). We prove that 
\[ \gcd B = d = \gcd F. \]

Put \( x = \gcd B \) and \( y = \gcd F \). Since \( x \) divides for all \( b \in B \) it follows that \( x \) divides \( b_i \) for all \( 1 \leq i \leq n \) and hence \( x \) divides \( d \). Also, since \( F \subseteq 1 = d\mathbb{Z} \), \( d \) divides every \( b_i \) and hence \( d \) divides \( \gcd F = y \).

Also, since \( y \) divides each \( b_i \), we have that \( y \) divides \( d \) and hence \( y \) divides \( x \) (since \( B \subseteq 1 = d\mathbb{Z} \)). All these imply that \( \gcd B = \gcd F \).

### 1.1.8 Example

Let \( \mathbb{N} \) be the set of non-negative integers. Then we have seen in Example 1.1.4 that 
\( (\mathbb{N}, \leq) \) forms a distributive lattice, in which 
\[ a \wedge b = \gcd \{a, b\} \]
and 
\[ a \vee b = \lcm \{a, b\} \]
for any elements \( a \) and \( b \) in \( \mathbb{N} \). Also, \( (\mathbb{N}, \leq) \) is a complete lattice which does not satisfy the infinite meet distributivity (refer 1.1.6). However, this complete lattice \( (\mathbb{N}, \leq) \) satisfies the infinite join distributivity; that is, for any \( a \in \mathbb{N} \) and \( B \subseteq \mathbb{N} \),
\[ a \vee \bigwedge_{b \in B} = a \vee [\gcd B] \]
\[ = \vee (\gcd F) \text{ for some finite } F \subseteq B \]
\[ = \vee (b_1 \wedge b_2 \wedge \ldots \wedge b_n) \text{ for some } b_i \in B \]
\[ = (a \vee b_1) \wedge (a \vee b_2) \wedge \ldots \wedge (a \vee b_n), \ b_i \in B \]
\[ \geq \bigwedge_{b \in B} (a \vee b) \geq a \vee (\bigwedge_{b \in B} b) \]
and hence \( a \lor (\bigwedge_{b \in B} b) = \bigwedge_{b \in B} (a \lor b) \).

1.1.9 Definition

A non-trivial complete lattice satisfying the infinite meet distributivity is called a frame.

1.1.10 Examples

(1) The interval \([0,1]\) in the real number system \(\mathbb{R}\) (with the usual order) is a frame; for, let \(a \in [0,1]\) and \(B \subseteq [0,1]\). If \(b \leq a\) for all \(b \in B\), then \(\bigvee_{b \in B} b \leq a\) and

\[
a \land \left( \bigvee_{b \in B} b \right) = \bigvee_{b \in B} b = \bigvee_{b \in B} (a \land b).
\]

If \(a < b\) for some \(b \in B\), then \(a < \bigvee_{b \in B} b\) and

\[
a \land \left( \bigvee_{b \in B} b \right) = a = \bigvee_{b \in B} (a \land b) \text{ (by the absorption law)}
\]

Thus \([0,1]\) satisfies the infinite meet distributivity and hence \([0,1]\) is a frame. In fact, any closed and bounded interval in the real number system \(\mathbb{R}\) is a frame.

(2) More generally, any complete chain (totally ordered set) is a frame (by the same argument given in (1) above).

(3) For any positive integer \(n\), \([0,1]^n\) together with coordinate wise ordering is a frame; here, for any \(B \subseteq [0,1]^n\) and \(a = (a_1, \ldots, a_n) \in [0,1]^n\),

\[
a \land \left( \bigvee_{b \in B} b \right) = (a_1, \ldots, a_n) \land (\bigvee_{b \in B} b_1, \ldots, \bigvee_{b \in B} b_n)
\]

\[
= (a_1 \land (\bigvee_{b \in B} b_1), \ldots, a_n \land (\bigvee_{b \in B} b_n))
\]

\[
= (\bigvee_{b \in B} (a_1 \land b_1), \ldots, \bigvee_{b \in B} (a_n \land b_n))
\]

\[
= \bigvee_{b \in B} (a \land b).
\]

Note that, for \(n > 1\), \([0,1]^n\) is not a totally ordered set (since \((0,1,0,\ldots,0)\) and \((1,0,0,\ldots,0)\) are not comparable).
This provides an example of a frame which is not a totally ordered set.

(4) Let \( \{L_i\}_{i \in I} \) be a non empty class of frames and \( L = \prod_{i \in I} L_i \). Then \( L \) is a frame, where the ordering on \( L \) is coordinate wise.

1.1.11 Theorem

A complete lattice is a frame if and only if it is a Brouwerian lattice. In particular, any finite Brouwerian lattice is a frame.

**Proof:** Let \( (L, \leq) \) be a complete lattice. Suppose that \( L \) is a frame. Let \( a \) and \( b \in L \) and \( X = \{ x \in L : a \wedge x \leq b \} \).

Put \( x_0 = \bigvee_{x \in X} x \). Since \( L \) satisfies the infinite meet distributivity, we have

\[
a \wedge x_0 = a \wedge \left( \bigvee_{x \in X} x \right) = \bigvee_{x \in X} (a \wedge x) \leq b \quad \text{(since } a \wedge x \leq b \text{ for all } x \in X \text{)}.
\]

Therefore \( x_0 \) is the largest element in \( L \) such that \( a \wedge x \leq b \). That is, \( x_0 = a \rightarrow b \). Thus \( L \) is a Brouwerian lattice. Conversely, suppose that \( L \) is a Brouwerian lattice. To prove the infinite meet distributivity in \( L \), let us take \( a \in L \) and \( B \subseteq L \). Put \( s = a \wedge \left( \bigvee_{b \in B} b \right) \) and \( t = \bigvee_{b \in B} (a \wedge b) \). Then \( a \wedge b \leq s \) for all \( b \in B \) and hence \( t \leq s \).

Also, since \( a \wedge b \leq t \), we have \( b \leq a \rightarrow t \) for all \( b \in B \) and hence \( \bigvee_{b \in B} b \leq a \rightarrow t \) which implies that

\[
s = a \wedge \left( \bigvee_{b \in B} b \right) \leq t.
\]

Thus \( s = t \) and hence \( a \wedge \left( \bigvee_{b \in B} b \right) = \bigvee_{b \in B} (a \wedge b) \).

That is, \( L \) satisfies the infinite meet distributivity and therefore \( L \) is a frame.
Note that a frame need not satisfy the infinite join distributivity, even though it satisfies the infinite meet distributivity. For, consider the following.

1.1.12 Example

Consider the set $\mathbb{N}$ of non-negative integers with the partial order dual to the one given in 1.1.4;
that is, for any $a$ and $b \in \mathbb{N}$, let us define
\[ a \leq b \text{ if and only if } a \text{ is a multiple of } b. \]
Then $\langle \mathbb{N}, \leq \rangle$ is a complete lattice in which, for any subset $A$,
\[ \text{lub } A = \text{greatest common divisor of all } a \in A. \]
and \[ \text{glb } A = \text{least common multiple of all } a \in A. \]
In 1.1.8, we have proved that $\langle \mathbb{N}, \leq \rangle$ satisfies the infinite meet distributivity and hence this $\langle \mathbb{N}, \leq \rangle$ is a frame, not satisfying the infinite join distributivity.

1.2 FUZZY SETS

It is well known that, if $A$ is any algebraic structure and $X$ is any non-empty set, then the set $A^X$ of all mappings of $X$ into $A$ can be made as an algebraic structure of type same as that of $A$ by defining the fundamental operations point-wise. In fact, if $A$ is an equationally definable algebraic structure (like a group or a ring or a module or a vector space), then $A^X$ is an algebra belonging to the variety generated by $A$.

In particular, if $A$ is the two-element lattice $\{0,1\}$ with $0 < 1$, then $A^X$ is a Boolean algebra for any non-empty set $X$, since $\{0,1\}$ is a Boolean algebra. In this context, let us recall the following well known result.
1.2.1 Theorem

Let \( X \) be any non-empty set and \( \mathbb{2} = \{0, 1\} \) the two element Boolean algebra. For any subset \( S \) of \( X \), define \( \chi_S : X \rightarrow \mathbb{2} \) by

\[
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \notin S 
\end{cases}
\]

for any \( x \in X \). Then \( S \mapsto \chi_S \) is an isomorphism of the Boolean algebra \( \mathcal{P}(X) \) of all subsets of \( X \) onto \( \mathbb{2}^X \).

**Proof:** The following can be easily verified for any subsets \( S \) and \( T \) of \( X \).

\[
\chi_{S \cap T} = \chi_S \cap \chi_T \\
\chi_{S \cup T} = \chi_S \cup \chi_T \\
\chi_{X - S} = \left( \chi_S \right)' \\
\chi_S = \chi_T \iff S = T \\
\chi_{A^{-1}(\emptyset)} = A \text{ for any } A \in \mathbb{2}^X.
\]

Here the operations \( \cap, \cup \) and \( ' \) mentioned on the right sides of the above equations are defined point-wise, as these are defined in \( \mathbb{2} \) as given in the following tables.

<table>
<thead>
<tr>
<th>( \wedge )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \vee )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Since $\mathbb{2}$ is a Boolean algebra, $\mathbb{2}^X$ and hence $\mathcal{P}(X)$ is a Boolean algebra. Thus the usual subsets of $X$ can be identified with the mappings of $X$ into $\mathbb{2}$. We generalise this situation by replacing $\mathbb{2}$ with general frame $L$. The idea behind choosing a frame to replace $\mathbb{2}$ is that most of the fuzzy statements take truth values in $[0, 1]^n$ for a suitable positive integer $n$ and that $[0,1]^n$ is a frame, as discussed in 1.1.10(3). Note that, being a Boolean algebra, $\mathbb{2}$ is also a frame where the operation $\rightarrow$ is as given below.

\[
\begin{array}{c|cc}
\rightarrow & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{array}
\]

**1.2.2 Definition**

Let $X$ be any non-empty set and $L = (L, \wedge, \vee)$ be a frame. Any mapping of $X$ into $L$ is called an $L$-fuzzy subset of $X$ or, simply, a fuzzy subset of $X$ when there is no ambiguity about $L$.

Any complete lattice $L$ should possess the smallest element 0 and the largest element 1 and hence $\mathbb{2} = \{0, 1\}$ can be treated as sublattice of $L$. Therefore any subset $S$ of $X$ can be treated as an $L$-fuzzy subset of $X$ by identifying $S$ with $\chi_S$, as given in 1.2.1. Let us recall that $\chi_S$ is called the characteristic map of $S$. For the sake of distinguishing the subsets of $X$ from the fuzzy subsets of $X$, the subsets of $X$ are usually called the crisp subsets of $X$. 

The set of all $L$-fuzzy subsets of $X$ will be denoted by $\mathcal{F} S (X)$. Note that $\mathcal{F} S (X)$ is precisely equal to $L^X$. The lattice structure on $L$ can be extended to $\mathcal{F} S (X)$, as given below.

1.2.3 Definition

Let $L$ be a frame and $X$ a non-empty set. For any $L$-fuzzy subsets $A$ and $B$ of $X$, define

$$A \leq B \; \text{if and only if} \; A(x) \leq B(x) \; \text{for all} \; x \in X.$$ 

It can be easily verified that $\leq$ is a partial order on the set $\mathcal{F} S (X)$ of $L$-fuzzy subsets of $X$ and is called the point wise ordering. The following is a straightforward verification.

1.2.4 Theorem

For any crisp subsets $S$ and $T$ of $X$,

$$\chi_S \leq \chi_T \; \text{if and only if} \; S \subseteq T.$$ 

1.2.5 Definition

Let $X$ be a non-empty set and $L$ a frame. For any $L$-fuzzy subset $A$ of $X$ and $\alpha \in L$, define

$$A_\alpha = A^{-1}(\{\alpha, 1\}) = \{x \in X : \alpha \leq A(x)\}.$$ 

Then $A_\alpha$ is called the $\alpha$-cut of $A$.

Clearly the $\alpha$-cut of $A$ is a crisp subset of $X$. In the following we describe the point-wise ordering on $L$-fuzzy subsets in terms of the inclusion ordering on the crisp subsets of $X$.

1.2.6 Theorem

Let $A$ and $B$ be $L$-fuzzy subsets of $X$. Then

$$A \leq B \iff A_\alpha \subseteq B_\alpha \; \text{for all} \; \alpha \in L.$$
Proof: Suppose that \( A \leq B \). Then, for any \( \alpha \in L \),

\[
x \in A_\alpha \Rightarrow \alpha \leq A(x) \leq B(x) \Rightarrow \alpha \leq B(x) \Rightarrow x \in B_\alpha \quad \text{and hence} \quad A_\alpha \subseteq B_\alpha.
\]

Conversely, if \( A_\alpha \subseteq B_\alpha \) for all \( \alpha \in L \), then for any \( x \in X \),

\[
x \in A_{\alpha(x)} \subseteq B_{\alpha(x)} \quad \text{and hence} \quad A(x) \leq B(x) \quad \text{and, this implies that} \quad A \leq B.
\]

The following is a straightforward verification using the definition that \( A \leq B \) if and only if \( A(x) \leq B(x) \) for all \( x \in X \).

**1.2.7 Theorem**

Let \( X \) be a non empty set, \( L \) a frame and \( F_S(X) \) be the set of all \( L \)-fuzzy subsets of \( X \).

Then \( F_S(X) \) is a complete lattice with respect to the point-wise ordering in which, for any \( \{A_i\}_{i \in I} \subseteq F_S(X) \)

\[
\left( \text{glb} \{A_i\}_{i \in I} \right)(x) = \text{glb} \{A_i(x)\}_{i \in I} \quad \text{and}
\]

\[
\left( \text{lub} \{A_i\}_{i \in I} \right)(x) = \text{lub} \{A_i(x)\}_{i \in I} .
\]

**1.2.8 Theorem**

\( F_S(X) \) is a frame in which, for any \( A \) and \( B \),

\[
(A \rightarrow B)(x) = A(x) \rightarrow B(x) \quad \text{for all} \quad x \in X.
\]

Proof: Let \( A, B, \) and \( C \in F_S(X) \), then

\[
C \leq A \rightarrow B \iff C(x) \leq (A \rightarrow B)(x) = A(x) \rightarrow B(x) \quad \text{for all} \quad x \in X
\]

\[
\iff A(x) \land C(x) \leq B(x) \quad \text{for all} \quad x \in X
\]

\[
\iff A \land C \leq B.
\]

The lattice operations on the set \( F_S(X) \) of fuzzy subsets of \( X \) can be described in terms of the \( \alpha \)-cuts as given in the following.
1.2.9 Theorem

Let \( \{A_i\}_{i \in I} \) be a non-empty set of L-fuzzy subsets of \( X \) and \( A = \bigwedge_{i \in I} A_i \). Then for each \( \alpha \in L \), \( A_{\alpha} = \bigcap_{i \in I} A_{i \alpha} \).

**Proof:** For any \( x \in X \) and \( \alpha \in L \),

\[
x \in A_{\alpha} \iff \alpha \leq A(x) = \bigwedge_{i \in I} A_i(x)
\]

\[
\iff \alpha \leq A_i(x) \text{ for all } i \in I
\]

\[
\iff x \leq A_{i \alpha} \text{ for all } i \in I
\]

\[
\iff x \in \bigcap_{i \in I} A_{i \alpha}
\]

Even though the \( \alpha \)-cut of the glb of \( A_i \)'s is simply the set intersection of the \( \alpha \)-cuts of \( A_i \)'s, the \( \alpha \)-cut of the lub of \( A_i \)'s need not be the set union of the \( \alpha \)-cuts of \( A_i \)'s. For, consider the following.

1.2.10 Example

Let \( L = [0,1] \times [0,1] \) and \( S \) and \( T \) proper subsets of a set \( X \) such that \( S \subseteq T \). Define \( A \) and \( B : X \to L \) as follows:

\[
A(x) = (0,0) = B(x) \text{ for all } x \not\in S
\]

\[
A(x) = (1,1) = B(x) \text{ for all } x \in S \cap T
\]

\[
A(x) = (1,0) \text{ if } x \in S \setminus T
\]

and \( B(x) = (0,1) \text{ if } x \in S \setminus T \).

Let \( \alpha = (1,1) \). Then \( (A \lor B)_{\alpha} = S \) and

\[
A_{\alpha} \cup B_{\alpha} = S \cap T \neq S.
\]

Therefore

\[
(A \lor B)_{\alpha} \neq A_{\alpha} \cup B_{\alpha}.
\]
However, the \( \alpha \)–cut of the lub of \( A_i \)'s can be expressed in terms of the \( \alpha \)–cuts of \( A_i \)'s in the form given in 1.2.12 below. First let us have the following.

1.2.11 Definition
Let \( L \) be a frame and \( \alpha \in L \). A subset \( M \) of \( L \) is said to be a cover of \( \alpha \) (or \( \alpha \) is said to be covered by \( M \)) if \( \alpha \leq \bigvee_{\beta \in M} \beta \).

1.2.12 Theorem
Let \( \{ A_i \}_{i \in I} \) be a class of \( L \)-fuzzy subsets of \( X \) and \( A = \bigvee_{i \in I} A_i \) and \( \alpha \in L \). Then the \( \alpha \)–cut of \( A \) can be expressed as

\[
A_{\alpha} = \bigcup \left\{ \bigcap_{\beta \in M} \left( \bigcup_{i \in I} A_{\beta} \right) : M \text{ is a cover of } \alpha \right\}
\]

**Proof:** Let us consider the set

\[
Y = \bigcup \left\{ \bigcap_{\beta \in M} \left( \bigcup_{i \in I} A_{\beta} \right) : M \text{ is a cover of } \alpha \right\}.
\]

If \( y \in Y \), then there is a cover \( M \) of \( \alpha \) such that

\[
y \in \bigcap_{\beta \in M} \left( \bigcup_{i \in I} A_{\beta} \right)
\]

and hence, for each \( \beta \in M \), there exists \( i \in I \) such that \( y \in A_{\beta} \); that is, \( \beta \leq A_i(y) \leq A(y) \) which implies that \( \bigvee_{\beta \in M} \beta \leq A(y) \) and therefore

\[
\alpha \leq \bigvee_{\beta \in M} \beta \leq A(y) \text{ and hence } y \in A_{\alpha}.
\]

Thus \( Y \subseteq A_{\alpha} \). On the other hand,

\[
y \in A_{\alpha} \Rightarrow \alpha \leq A(y) = \bigvee_{i \in I} A_i(y)
\]

\[
\Rightarrow \left\{ A_i(y) \right\}_{i \in I} \text{ is a cover of } \alpha \text{ and}
\]

\[
y \in A_i(y) \text{ Where } \beta_i = A_i(y) \Rightarrow y \in Y
\]
and hence $A_{\alpha} \subseteq Y$. Thus $A_{\alpha} = Y$.

Next, we prove that any $L$-fuzzy subset of $X$ is completely determined by its $\alpha$-cuts, $\alpha \in L$. First of all, it is clear from 1.2.6 that, for any $L$–fuzzy subsets $A$ and $B$ of $X$,

$$A = B \Leftrightarrow A_{\alpha} = B_{\alpha} \text{ for all } \alpha \in L.$$ 

In the next two results we prove that any $L$-fuzzy subset $A$ can be expressed in terms of its $\alpha$–cuts.

**1.2.13 Theorem**

Let $A$ be an $L$-fuzzy subset of $X$. Then $\{A_{\alpha} : \alpha \in L\}$ is a class of crisp subsets of $X$ such that

$$\bigcap_{\alpha \in M} A_{\alpha} = A \bigvee_{\alpha \in M} \alpha \text{ for all } M \subseteq L.$$ 

**Proof:** If $M$ is the empty subset of $L$, then $\bigvee_{\alpha \in M} \alpha = 0$ and $A_{\alpha} = X = \bigcap_{\alpha \in M} A_{\alpha}$. Now, let $M$ be any non-empty subset of $L$. Then, for any $x \in X$, we have

$$x \in \bigcap_{\alpha \in M} A_{\alpha} \Leftrightarrow x \in A_{\alpha} \text{ for all } \alpha \in M$$

$$\Leftrightarrow \alpha \leq A(x) \text{ for all } \alpha \in M$$

$$\Leftrightarrow \bigvee_{\alpha \in M} \alpha \leq A(x)$$

$$\Leftrightarrow x \in M \bigvee_{\alpha \in M} \alpha.$$ 

Thus $\bigcap_{\alpha \in M} A_{\alpha} = A \bigvee_{\alpha \in M} \alpha$ for all subsets $M$ of $L$. 

45
1.2.14 Theorem

Let X be a nonempty set and L a frame. Suppose that $\left\{ S_\alpha \right\}_{\alpha \in L}$ is a class of crisp subsets of X such that $\bigcap_{\alpha \in M} S_\alpha = \bigvee_{\alpha \in M} S_\alpha$ for any subset M of L. Then there exists a unique L-fuzzy subset A of X such that $A_\alpha = S_\alpha$ for each $\alpha \in L$.

Proof: First of all observe that, for any $\alpha$ and $\beta \in L$,

$$\alpha \leq \beta \Rightarrow S_\alpha \cap S_\beta = S_{\alpha \land \beta} = S_\beta$$

$$\Rightarrow S_\beta \subseteq S_\alpha.$$ 

Define $A : X \rightarrow L$ by

$$A(x) = \begin{cases} 0 & \text{if } x \not\in S_\beta \text{ for any } \beta \in L \\ \vee \{ \beta \in L : x \in S_\beta \} & \text{otherwise} \end{cases}$$

Then A is an L-fuzzy subset of X. For any $\alpha \in L$ and $x \in X$, we have

$$x \in S_\alpha \Rightarrow \alpha \leq A(x) \text{ (by the definition of A(x))}$$

$$\Rightarrow x \in A_\alpha,$$

and hence $S_\alpha \subseteq A_\alpha$.

On the other hand, suppose that $x \in A_\alpha$. Then

$$\alpha \leq A(x) = \vee \{ \beta \in L : x \in S_\beta \}.$$ 

Since L is a frame, it satisfies the infinite meet distributivity and hence

$$\alpha = \alpha \land \left( \vee \{ \beta \in L : x \in S_\beta \} \right)$$

$$= \vee \{ \alpha \land \beta : \beta \in L \text{ and } x \in S_\beta \}.$$ 

Now, since $S_\beta \subseteq S_{\alpha \land \beta}$ and hence $x \in S_{\alpha \land \beta}$ when ever $x \in S_\beta$.

We get that

$$x \in \bigcap \left\{ S_{\alpha \land \beta} : \beta \in L \text{ and } x \in S_\beta \right\} = S_\alpha.$$
Thus $A_\alpha \subseteq S_\alpha$. Therefore $A_\alpha = S_\alpha$ for all $\alpha \in L$.

The uniqueness of $A$ follows from theorem 1.2.6.

1.2.15 Corollary

Let $Y_1 \subseteq Y_2 \subseteq \ldots$ be an increasing sequence of subsets of a set $X$ such that $\bigcup_{n=1}^{\infty} Y_n = X$.

Let $L$ be a frame and $1 = \alpha_1 \geq \alpha_2 \geq \ldots$ be a decreasing sequence in $L$. Define $A : X \rightarrow L$ by

$$A(x) = \alpha_n \text{ where } n \text{ is the least positive integer such that } x \in Y_n.$$ 

Then $A$ is an $L$-fuzzy subset of $X$ and, for any $\alpha \in L$, the $\alpha$-cut $A_\alpha$ is given by

$$A_\alpha = \begin{cases} \emptyset & \text{if } \alpha \nleq \alpha_n \text{ for all } n \\ Y_n & \text{if } n \text{ is the least such that } \alpha \geq \alpha_n \end{cases}$$

Proof: For each $\alpha \in L$, define

$$S_\alpha = \begin{cases} \emptyset & \text{if } \alpha \nleq \alpha_n \text{ for all } n \\ Y_n & \text{if } n \text{ is the least such that } \alpha \geq \alpha_n \end{cases}$$

Then $\{S_\alpha\}_{\alpha \in L}$ satisfies the property given in theorem 1.2.14 and hence there is a unique $L$-fuzzy subset $A$ of $X$ such that $A_\alpha = S_\alpha$ for all $\alpha \in L$.

The following is an useful tool in working with the $L$-fuzzy subsets (or crisp subsets) of a set $X$. A class $\{A_i\}_{i \in I}$ of $L$-fuzzy subsets of $X$ is called directed above if, for any $A_i$ and $A_j$ ($i$ and $j \in I$), there exists $A_k$, $k \in I$, such that $A_i \leq A_k$ and $A_j \leq A_k$.

1.2.16 Theorem

Let $X$ be a nonempty set and $L$ a frame. Suppose that $\{A_i\}_{i \in I}$ is a directed above class of $L$-fuzzy subsets of $X$ and $x_1, x_2, \ldots, x_n \in X$. Then

$$\bigwedge_{r=1}^{n} \left( \bigvee_{i \in I} A_i(x_r) \right) = \bigvee_{i \in I} \left( \bigwedge_{r=1}^{n} A_i(x_r) \right).$$

47
Proof: Let $\alpha$ and $\beta$ denote respectively the left hand side and right hand side of the above equation. For each $j \in I$, $A_j(x_r) \leq \bigvee_{i \in I} A_i(x_r)$ for all $1 \leq r \leq n$ and hence

$$\bigwedge_{i=1}^n A_j(x_r) \leq \bigwedge_{i=1}^n \left( \bigvee_{i \in I} A_i(x_r) \right) = \alpha$$

which implies that

$$\beta = \bigvee_{j \in I \setminus r=1} \left( \bigwedge_{i=1}^n A_j(x_r) \right) \leq \alpha.$$  

Also, by the infinite meet distributivity in $L$, we have

$$\alpha = \bigvee_{i_1, i_2, \ldots, i_n \in I} \left( A_{i_1}(x_1) \wedge A_{i_2}(x_2) \wedge \ldots \wedge A_{i_n}(x_n) \right) \ldots . (*)$$

Now, for any $i_1, i_2, \ldots, i_n \in I$, there exists $j \in I$ such that $A_{i_r} \leq A_j$ for all $1 \leq r \leq n$ and hence

$$A_{i_1}(x_1) \wedge A_{i_2}(x_2) \wedge \ldots \wedge A_{i_n}(x_n) \leq A_j(x_1) \wedge A_j(x_2) \wedge \ldots A_j(x_n) \leq \beta.$$  

From this and (*) above, we get that $\alpha \leq \beta$. Thus

$$\alpha = \beta.$$  

we close this section with an observation that the characteristic map of the empty set $\chi_{\emptyset}$ is the constant map $\bar{0}$ which maps all elements of $X$ onto 0 and hence $\chi_{\emptyset}$ is the smallest element in $\mathcal{F} \mathcal{S}(X)$. Similarly $\chi_X$ is the constant map $\bar{1}$ and is the largest element in $\mathcal{F} \mathcal{S}(X)$. In other words, $\emptyset$ and $\chi_{\emptyset}$ are the smallest elements in $\mathcal{P}(X)$ and $\mathcal{F} \mathcal{S}(X)$ respectively, while $X$ and $\chi_X$ are the greatest elements in $\mathcal{P}(X)$ and $\mathcal{F} \mathcal{S}(X)$ respectively.

For any $\alpha \in L$, the $L$-fuzzy subset $A$ of $X$ defined by $A(x) = \alpha$ for all $x \in X$ is called a constant $L$-fuzzy subset of $X$ and is denoted simply by $\bar{\alpha}$. Therefore $\alpha_{\emptyset} = \bar{0}$ and $\chi_X = \bar{1}$.
1.3 FUZZY IDEALS

In this section we introduce the concept of a fuzzy ideal of a lattice and prove certain elementary properties of these. In particular, we prove that the fuzzy ideals of a given lattice form an algebraic lattice. To begin with, let us recall that a nonempty subset I of a lattice \( X = (X, \wedge, \vee) \) is called an \emph{ideal} of X if

\[
a \text{ and } b \in I \Rightarrow a \lor b \in I \text{ and } a \land x \in I \text{ for all } x \in X.
\]

In other words, I is an ideal of X if and only if I is closed under the operation \( \lor \) and I is an initial segment (that is, \( y \leq a \in I \Rightarrow y \in I \)). First we prove the following.

1.3.1 Theorem

The following are equivalent to each other for any lattice \( X = (X, \wedge, \vee) \)

1. \( X \) has smallest element 0
2. \( X \) has a smallest ideal
3. The class \( \mathcal{I}(X) \) of ideals of X is a Moore class.

\textbf{Proof:} (1) \( \Rightarrow \) (2): If I is an ideal, then I is non-empty and hence there exists \( a \in I \) and
\( 0 = a \land 0 \in I \).

Thus \( \{0\} \subseteq I \) for all ideals I of X and \( \{0\} \) is an ideal of X.

(2) \( \Rightarrow \) (3): Let \( I_o \) be the smallest ideal. If \( \{I_\alpha\}_{\alpha \in \Delta} \) is a class of ideals of X, then
\( I_o \subseteq I_\alpha \) for all \( \alpha \in \Delta \) and hence \( \phi \neq I_o \subseteq \bigcap_{\alpha \in \Delta} I_\alpha \). Thus \( \bigcap_{\alpha \in \Delta} I_\alpha \neq \phi \). It can be easily verified that \( \bigcap_{\alpha \in \Delta} I_\alpha \) is an ideal of X.

(3) \( \Rightarrow \) (1): For any \( x \in X \), \( (x) = \{ y \in X : y \leq x \} \) is an ideal of X and hence \( \bigcap_{x \in X} (x) \) is an ideal; in particular, it is not empty and hence, there exists \( a \in X \) such that \( a \in \bigcap_{x \in X} (x) \);
that is, \( a \leq x \) for all \( x \in X \).
Since we intend to make fuzzy ideals to form an algebraic lattice and in particular, a complete lattice, the above theorem compels us to consider lattices with smallest elements. Throughout the thesis, we consider lattices $X = (X, \land, \lor)$ with zero, unless otherwise stated. The smallest element in a lattice, if it exists, is usually denoted by 0 and call it the zero element. For any set $X$ and for any element $\alpha$ in a frame $L$, we write $\alpha$ to denote the constant map of $X$ into $L$ which maps every element of $X$ onto $\alpha$. In particular $\mathbf{0}$ is called the zero map, since $\mathbf{0} \leq A$ for any $A : X \to L$.

1.3.2 Definition

Let $X = (X, \land, \lor)$ be a lattice with zero and $L$ a frame. An $L$-fuzzy subset $A$ of $X$ is called an $L$-fuzzy ideal of $X$ (or simply, fuzzy ideal of $X$, when there is no ambiguity about $L$) if

$$A(0) = 1$$

and

$$A(x \lor y) = A(x) \land A(y)$$

for all $x$ and $y \in X$.

First of all, note that any fuzzy ideal $A$ of $X$ is not the constant map $\mathbf{0}$ (since $A(0) = 1$) and is an antitone, in the sense that,

$$x \leq y \Rightarrow A(y) = A(x \lor y) = A(x) \land A(y) \Rightarrow A(x) \geq A(y),$$

for any $x$ and $y \in X$.

We prove the following result, which facilitates to identify any (crisp) ideal of $X$ with a fuzzy ideal of $X$.
1.3.3 Theorem

Let \( X \) be a lattice with zero, \( S \subseteq X \) and \( L \) a frame. Then \( S \) is a (crisp) ideal of \( X \) if and only if the characteristic map \( \chi_S \) is an \( L \)-fuzzy ideal of \( X \).

**Proof:** Recall that \( \chi_S : X \rightarrow L \) is defined by

\[
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \notin S 
\end{cases}
\]

for any \( x \in X \). Therefore \( S \) is not empty if and only if \( \chi_S \) is not the zero in \( F_S(X) \) (that is \( \chi_S \neq 0 \)).

Now, suppose that \( S \) is an ideal of \( X \) and \( x, y \in X \). Then \( S \neq \emptyset \) and hence \( 0 \in S \) so that \( \chi_S(0) = 1 \). Also,

\[
x \lor y \in S \iff x \text{ and } y \in S \text{ (since } x \leq x \lor y \text{ and } y \leq x \lor y) \\
\iff \chi_S(x \lor y) = 1 = \chi_S(x) \land \chi_S(y)
\]

\[
x \lor y \notin S \iff x \notin S \text{ or } y \notin S \\
\iff \chi_S(x) = 0 \text{ or } \chi_S(y) = 0 \\
\iff \chi_S(x \lor y) = 0 = \chi_S(x) \land \chi_S(y).
\]

Thus, in either case, \( \chi_S(x \lor y) = \chi_S(x) \land \chi_S(y) \) for all \( x \) and \( y \in X \) and hence \( \chi_S \) is an \( L \)-fuzzy ideal of \( X \).

Conversely suppose that \( \chi_S \) is an \( L \)-fuzzy ideal of \( X \).

Then \( \chi_S(0) = 1 \) and hence \( 0 \in S \) and \( S \neq \emptyset \). Now,

\[
x \text{ and } y \in S \Rightarrow \chi_S(x) = 1 = \chi_S(y) \\
\Rightarrow \chi_S(x \lor y) = \chi_S(x) \land \chi_S(y) = 1 \land 1 = 1
\]
\[ \Rightarrow x \vee y \in S \]

and \( x \in S, z \in X \Rightarrow \chi_S(x \wedge z) \geq \chi_S(x) = 1 \)

\[ \Rightarrow \chi_S(x \wedge z) = 1 \]

\[ \Rightarrow x \wedge z \in S \]

Thus \( S \) is an ideal of \( X \).

Note that, for any \( S \subseteq X \) and \( \alpha \in L \), the \( \alpha \)-cut of the characteristic map \( \chi_S \) is given by

\[
(\chi_S)_\alpha = \begin{cases} 
X & \text{if } \alpha = 0 \\
S & \text{if } \alpha \neq 0 
\end{cases}
\]

Since \( X \) is always an ideal of \( X \), it follows from the above theorem that \( \chi_S \) is a fuzzy ideal of \( X \) if and only if the \( \alpha \)-cut of \( \chi_S \) is an ideal of \( X \) for each \( \alpha \in L \). The following is a generalization of Theorem 1.3.3. First of all, note that, for any fuzzy subset \( A \) of \( X \),

\[ A_\beta \subseteq A_\alpha \text{ whenever } \beta \leq \alpha \]

and hence \( A_\beta \subseteq A_\alpha \) for all \( \beta \in L \), which implies that

\[ A_\beta \neq \phi \text{ for all } \beta \in L \Leftrightarrow A_\alpha \neq \phi \]

\[ \Leftrightarrow A(x) = 1 \text{ for some } x \in X. \]

Now, we prove the following, which generalises Theorem 1.3.3.

1.3.4 Theorem

Let \( L \) be a frame and \( A \) be an \( L \)-fuzzy subset of a lattice \( X = (X, \wedge, \vee) \) with zero. Then \( A \) is an \( L \)-fuzzy ideal of \( X \) if and only if the \( \alpha \)-cut \( A_\alpha \) is a crisp ideal of \( X \) for each \( \alpha \in L \).
Proof: Suppose that \( A \) is an \( L \)-fuzzy ideal of \( X \). Then \( A(0)=1 \) and, by the above discussion, \( A_\alpha \neq \phi \) for all \( \alpha \in L \).

Now let \( \alpha \in L \) be fixed, then

\[
x \text{ and } y \in A_\alpha \Rightarrow \alpha \leq A(x) \text{ and } \alpha \leq A(y)
\]

\[
\Rightarrow \alpha \leq A(x) \wedge A(y) = A(x \vee y)
\]

\[
\Rightarrow x \vee y \in A_\alpha
\]

and \( x \in A_\alpha \) and \( z \in X \) \( \Rightarrow \alpha \leq A(x) \leq A(x \wedge z) \)

\[
\Rightarrow x \wedge z \in A_\alpha
\]

Thus \( A_\alpha \) is a crisp ideal of \( X \) for all \( \alpha \in L \).

Conversely suppose that \( A_\alpha \) is a crisp ideal of \( X \) for all \( \alpha \in L \). In particular \( A_1 \) is an ideal of \( X \) and hence \( 0 \in A_1 \), so that \( A(0) \geq 1 \) which is equivalent to saying that \( A(0) = 1 \).

Now, let \( x \) and \( y \in X \). Then

\[
x \leq y \Rightarrow x \leq y \in A_{A(y)}
\]

\[
\Rightarrow x \in A_{A(y)} \text{ (since } A_{A(y)} \text{ is an ideal)}
\]

\[
\Rightarrow A(y) \leq A(x).
\]

That is, \( A \) is an antitone,

Since \( x \leq x \vee y \) and \( y \leq x \vee y \), we get that

\[
A(x \vee y) \leq A(x) \text{ and } A(y)
\]

and hence \( A(x \vee y) \leq A(x) \wedge A(y) \).

To prove the other inequality, put \( \beta = A(x) \wedge A(y) \). Then \( \beta \leq A(x) \) and \( \beta \leq A(y) \) and hence \( x \) and \( y \in A_j \). Since \( A_j \) is an ideal of \( X \), it follows that \( x \vee y \in A_j \) and hence
Thus $A(x \vee y) = A(x) \wedge A(y)$ for all $x$ and $y \in X$. Therefore $A$ is an $L$-fuzzy ideal of $X$.

1.3.5 Definition

For any lattice $(X, \wedge, \vee)$ with zero and any frame $L$, let $\mathfrak{F} | L(X)$ denote the set of all $L$-fuzzy ideals of $X$.

1.3.6 Theorem

$\mathfrak{F} | L(X)$ is closed under point-wise infimums.

**Proof:** Let $\{A_i\}_{i \in \Delta}$ be a class of $L$-fuzzy ideals of $X$ and $A$ be the point-wise infimum of $\{A_i\}_{i \in \Delta}$. That is, $A$ is defined by

$$A(x) = \wedge_{i \in \Delta} A_i(x) = \operatorname{glb} \{A_i(x)\}_{i \in \Delta} \text{ in } L$$

for any $x \in X$. Then $A(0) = 1$ (since $A_i(0)=1$ for all $i \in \Delta$).

Also, for any $x$ and $y \in X$,

$$A(x \vee y) = \wedge_{i \in \Delta} A_i(x \vee y)$$

$$= \wedge_{i \in \Delta} (A_i(x) \wedge A_i(y))$$

$$= \left( \wedge_{i \in \Delta} A_i(x) \right) \wedge \left( \wedge_{i \in \Delta} A_i(y) \right)$$

$$= A(x) \wedge A(y).$$

Thus $A$ is an $L$-fuzzy ideal of $X$ and $A \in \mathfrak{F} | L(X)$.

1.3.7 Theorem

$\mathfrak{F} | L(X)$ is a complete lattice under point-wise ordering.

**Proof:** Recall that the point-wise ordering on $\mathfrak{F} | L(X)$ is defined by
\[ A \leq B \iff A(x) \leq B(x) \text{ for all } x \in X, \]

for any \( A \) and \( B \in \mathcal{F} \downarrow \mathcal{L}(X) \). Also, \( \chi_{[0]} \) and \( \chi_{x} \) are clearly the smallest element and greatest element respectively in \( \mathcal{F} \downarrow \mathcal{L}(X) \). Note that

\[
\chi_{[0]}(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0
\end{cases}
\]

and \( \chi_{x} \) = the constant map \( \bar{1} \)

and \( \chi_{[0]} \leq A \leq \chi_{x} \) for all \( A \in \mathcal{F} \downarrow \mathcal{L}(X) \). Therefore \((\mathcal{F} \downarrow \mathcal{L}(X), \leq)\) is a bounded partially ordered set. Also, by the above theorem (1.3.6), every subset of \( \mathcal{F} \downarrow \mathcal{L}(X) \) has greatest lower bound (namely, its point-wise infimum). Thus, \( \mathcal{F} \downarrow \mathcal{L}(X) \) is a complete lattice in which the least upper bound of \( \{A_{i}\}_{i \in \Delta} \) is given by the point-wise infimum of

\( \{ A \in \mathcal{F} \downarrow \mathcal{L}(X) : A_{i} \subseteq A \text{ for all } i \in \Delta \} \).

1.3.8 Note

For any \( \mathcal{L} \)- fuzzy ideals \( \{A_{i}\}_{i \in \Delta} \) of \( X \),

\[
\text{glb}\{A_{i}\}_{i \in \Delta} = \bigwedge_{i \in \Delta} A_{i}, \text{ the pointwise infimum}
\]

\[
\text{lub}\{A_{i}\}_{i \in \Delta} = \bigvee_{i \in \Delta} A_{i} = \bigwedge_{i \in \Delta} \{ A \in \mathcal{F} \downarrow \mathcal{L}(X) : A_{i} \subseteq A \text{ for all } i \in \Delta \}
\]

Note that, even though \( \bigwedge_{i \in \Delta} A_{i} \) is the point-wise infimum, \( \bigvee_{i \in \Delta} A_{i} \) is not the point-wise supremum, that is,

\[
\left( \bigwedge_{i \in \Delta} A_{i} \right)(x) = \bigwedge_{i \in \Delta} A_{i}(x)
\]

and \( \left( \bigvee_{i \in \Delta} A_{i} \right)(x) \) is not necessarily equal to \( \bigvee_{i \in \Delta} A_{i}(x) \) for any \( x \in X \).
The following is an important observation which supports the assumption that $A(0) = 1$ for any $L$-fuzzy ideal $A$, in the sense that the assumption $A(0) = 1$ is not a burden, but only a technical one.

**1.3.9 Theorem**

Let $I$ be an $L$-fuzzy subset of a lattice $X$ such that

$$I(x \lor y) = I(x) \land I(y)$$

for all $x$ and $y \in X$.

Define $\overline{I}: X \to L$ by

$$\overline{I}(x) = \begin{cases} 1 & \text{if } x = 0 \\ I(x) & \text{if } x \neq 0 \end{cases}$$

Then $\overline{I}$ is an $L$-fuzzy ideal of $X$.

**Proof:** Clearly $\overline{I}(0) = 1$. Let $x$ and $y \in X$. If $x \lor y = 0$ then $x = 0 = y$ and hence

$$\overline{I}(x \lor y) = 1 = 1 \land 1 = \overline{I}(x) \land \overline{I}(y)$$

If $x \lor y \neq 0$, then $x \neq 0$ or $y \neq 0$; say $x \neq 0$, then

$$\overline{I}(x \lor y) = I(x \lor y) = I(x) \land I(y) = \overline{I}(x) \land \overline{I}(y).$$

Thus $\overline{I}$ is an $L$-fuzzy ideal of $X$.

Actually, $\overline{I}$ defined above is the smallest $L$-fuzzy ideal of $X$ containing $I$, in the sense that, for any $L$-fuzzy ideal $J$ of $X$,

$$I \leq J \iff \overline{I} \leq J.$$  

In the next result, we describe the smallest $L$-fuzzy ideal of $X$ containing a given $L$-fuzzy subset of $X$ and deduce the point-wise description of the supremum of a given class of $L$-fuzzy ideals in terms of their point-wise supremum. First, we have the following.

**1.3.10 Definition**

For any $L$-fuzzy subset $A$ of a lattice $X$, define $\overline{A}: X \to L$ by
\( \overline{A}(x) = \bigwedge \{I(x): I \text{ is an } L\text{-fuzzy ideal of } X \text{ and } A \leq I \} \) for any \( x \in X \). By theorem 1.3.6, \( \overline{A} \) is an L-fuzzy ideal of X and, in fact \( \overline{A} \) is the smallest (with respect to the point-wise ordering in \( F \vert _L (X) \)) L-fuzzy ideal of X containing \( A \); in the sense that, for any \( I \in F \vert _L (X) \),

\[ A \leq I \iff \overline{A} \leq I \]

\( \overline{A} \) is called the \textit{L-fuzzy ideal of } X \textit{ generated by } A.

In the following result we give a point-wise description of \( \overline{A} \) for any given L-fuzzy subset \( A \) of \( X \).

**1.3.11 Theorem**

Let \( L \) be a frame and \( A \) an L-fuzzy subset of a lattice \( X = (X, \wedge, \vee) \). Then the L-fuzzy ideal \( \overline{A} \) generated by \( A \) is given by

\[ \overline{A}(0) = 1 \]

and

\[ \overline{A}(x) = \bigvee \{ \bigwedge \{ A(a_i): a_i, \ldots, a_n \in X \text{ and } x \leq \bigvee a_i \} \} \text{, for any } 0 \neq x \in X . \]

**Proof:** Let \( \overline{A} \) be defined as given above. Clearly \( A(x) \leq \overline{A}(x) \) for all \( x \in X \) and hence \( A \leq \overline{A} \).

Also, if \( I \) is an L-fuzzy ideal of \( X \) such that \( A \leq I \), then, for any \( x \in X \) and \( a_1, a_2, \ldots, a_n \in X \) with \( x \leq \bigvee a_i \),

\[ \bigwedge \{ A(a_i): a_i, \ldots, a_n \in X \text{ and } x \leq \bigvee a_i \} \leq \bigwedge \{ I(a_i): a_i, \ldots, a_n \in X \text{ and } x \leq \bigvee a_i \} \leq I(x) \]

and therefore \( \overline{A}(x) \leq I(x) \) for all \( x \in X \), so that \( \overline{A} \leq I \). Finally we prove that \( \overline{A} \) is an L-fuzzy ideal of \( X \). By the very definition of \( \overline{A} \), we have

\[ \overline{A}(0) = 1. \]
If $x \leq y$ are elements in $X$, then, for any $a_1, a_2, \ldots, a_n \in X$

$$y \leq \bigvee_{i=1}^{n} a_i \Rightarrow x \leq \bigvee_{i=1}^{n} a_i \Rightarrow \bigwedge_{i=1}^{n} A(a_i) \leq \overline{A}(x)$$

which implies that $\overline{A}(y) \leq \overline{A}(x)$. Therefore $\overline{A}$ is an antitone. From this it follows that

$$\overline{A}(x \vee y) \leq \overline{A}(x) \wedge \overline{A}(y)$$

for all $x$ and $y$ in $X$. On the other hand, by the infinite meet distributivity in $L$, we have

$$\overline{A}(x) \wedge \overline{A}(y)$$

$$= \left( \bigvee \left\{ \bigwedge_{i=1}^{n} A(a_i) : a_i \in L \text{ and } x \leq \bigvee_{i=1}^{n} a_i \right\} \right) \wedge \left( \bigvee \left\{ \bigwedge_{j=1}^{m} A(b_j) : b_j \in L \text{ and } y \leq \bigvee_{j=1}^{m} b_j \right\} \right)$$

$$= \bigvee \left\{ \bigwedge_{i=1}^{n} A(a_i) \wedge \bigwedge_{j=1}^{m} A(b_j) : x \leq \bigvee_{i=1}^{n} a_i \text{ and } y \leq \bigvee_{j=1}^{m} b_j \right\}$$

$$= \bigvee \left\{ \bigwedge_{i=1}^{n} A(a_i) \wedge \bigwedge_{j=1}^{m} A(b_j) : x \vee y \leq \bigvee_{i=1}^{n} a_i \vee \bigvee_{j=1}^{m} b_j \right\}$$

$$\leq \overline{A}(x \vee y) \leq \overline{A}(x) \wedge \overline{A}(y).$$

Thus $\overline{A}(x \vee y) = \overline{A}(x) \wedge \overline{A}(y)$ for all $x$ and $y \in X$. Thus $\overline{A}$ is the $L$-fuzzy ideal of $X$ generated by $A$.

**1.3.12 Corollary**

Let $\{A_i\}_{i \in \Delta}$ be a class of $L$-fuzzy ideals of a lattice $X$. Then the supremum $\bigvee_{i \in \Delta} A_i$ of $\{A_i\}_{i \in \Delta}$ in $F \cap L(X)$ is given by

$$\left( \bigvee_{i \in \Delta} A_i \right)(x) = \bigvee_{i \in \Delta} \left( \bigwedge_{j=1}^{n} A_i(a_j) : a_1, \ldots, a_n \in X \text{ and } x \leq \bigvee_{j=1}^{n} a_j \right)$$

**1.3.13 Corollary**

For any $L$-fuzzy ideals $A$ and $B$ of $X$, the supremum $A \vee B$ of $A$ and $B$ in $F \cap L(X)$ is given by
\[(A \lor B)(x) = \lor \left\{ \land_{i=1}^{n} \left( A(a_i) \lor B(a_i) \right) : x \leq \lor_{i=1}^{n} a_i, a_i \in X \right\}.\]

Note that \( \lor_{i \in \Delta} A_i \) is precisely the L- fuzzy ideal generated by the point-wise supremum of A_i’s.

1.3.14 Definition

Let \( X = (X, \land, \lor) \) be a lattice, L a frame, \( x \in X \) and \( \alpha \in L \).

Define \( \alpha_x : X \to L \) by

\[
\alpha_x(y) = \begin{cases} 1 & \text{if } y \leq x \\ \alpha & \text{if } y \not\leq x \end{cases}
\]

for any \( y \in X \).

1.3.15 Theorem

\( \alpha_x \) defined above is an L- fuzzy ideal of \( X \) for any \( \alpha \in L \) and \( x \in X \).

Proof: By the definition of \( \alpha_x \), we clearly have \( \alpha_x(0) = 1 \).

Let \( y \) and \( z \in X \). If \( y \lor z \leq x \), then \( y \leq x \) and \( z \leq x \) and hence

\[
\alpha_x(y \lor z) = 1 = 1 \land 1 = \alpha_x(y) \land \alpha_x(z).
\]

If \( y \lor z \not\leq x \) then either \( y \not\leq x \) or \( z \not\leq x \) and hence

\[
\alpha_x(y) = \alpha \quad \text{or} \quad \alpha_x(z) = \alpha, \quad \text{so that}
\]

\[
\alpha_x(y \lor z) = \alpha = 1 \land \alpha = \alpha \land \alpha = \alpha_x(y) \land \alpha_x(z)
\]

Thus \( \alpha_x \) is an L- fuzzy ideal of \( X \).

1.3.16 Definition

\( \alpha_x \) is called the \( \alpha \)-level principal fuzzy ideal corresponding to \( x \). Note that \( \alpha_0 = \chi_{[0]} \)

and, if \( X \) has largest element 1, then \( \alpha_1 = \chi_X \).
In definition 1.3.14, we can replace the principal crisp ideal by a general crisp ideal as follows.

### 1.3.17 Definition

Let $I$ be a crisp ideal of a lattice $X$ and $L$ a frame. For any $\alpha \in L$, define $\alpha_I : X \rightarrow L$ by

$$
\alpha_I(y) = \begin{cases} 
1 & \text{if } y \in I \\
\alpha & \text{if } y \notin I
\end{cases}
$$

For any $\alpha \in L$, the interval $[\alpha,1]$ is a frame and $\alpha_I$ can be considered as the characteristic map $\chi_I$ of $X$ into $[\alpha,1]$ and hence $\alpha_I$ is an $[\alpha,1]$-fuzzy ideal of $X$ and hence an $L$-fuzzy ideal of $X$.

### 1.3.18 Definition

$\alpha_I$ defined above is called the $\alpha$–level fuzzy ideal corresponding to $I$.

For any $x \in X$, if $[x] = \{ y \in X : y \leq x \}$, then clearly $\alpha_x = \alpha_{[x]}$. Also, note that $\alpha_{[0]} = \alpha_0 = \chi_{[0]}$ and $\alpha_x = \alpha_1 = \chi_x$. Further, $0_I = \chi_I$ for any ideal $I$ of $X$.

If we define $\alpha_I$ as in 1.3.17 for any $\alpha \in L$ and any subset $I$ of $X$, then one can easily verify that $\alpha_I$ is an $L$-fuzzy ideal of $X$ if and only if $I$ is a crisp ideal of $X$. Now, we prove in the following that, for a given $\alpha \in L$, the $\alpha$-level $L$–fuzzy ideals of $X$ form a sublattice of $\mathcal{F} \mid L(X)$, isomorphic to the lattice $\mathcal{I}(X)$ of ideals of $X$.

### 1.3.19 Theorem

Let $L$ be a frame, $X$ a lattice and $0 \neq \alpha \in L$. Then $I \mapsto \alpha_I$ is an isomorphism of the lattice $\mathcal{I}(X)$ of ideals of $X$ onto the lattice of all $\alpha$–level $L$- fuzzy ideals of $X$ corresponding to crisp ideals of $X$.

**Proof:** Let $I$ and $J$ be any crisp ideals of $X$. Then it can be easily verified that

$$
\alpha_{I \cap J} = \alpha_I \wedge \alpha_J
$$
and $\alpha_i \leq \alpha_j \iff I \subseteq J$.

Recall that $I \lor J = \{ x \in X : x \leq a \lor b, a \in I, b \in J \}$

and $(\alpha_i \lor \alpha_j)(x) = \lor \{ \bigwedge_{i=1}^{n} (\alpha_i \lor \alpha_j) : x \leq \bigvee_{i=1}^{n} a_i, a_i \in X \}$. Clearly we have $\alpha_i \leq \alpha_{1 \lor j}$ and $\alpha_j \leq \alpha_{1 \lor i}$ hence $\alpha_i \lor \alpha_j \leq \alpha_{1 \lor i}$. Also, $\alpha_i(x) = 1$ or $\alpha$ and $\alpha_j(x) = 1$ or $\alpha$ for any $x \in X$ and hence

$(\alpha_i \lor \alpha_j)(x) \geq \alpha = \alpha_{1 \lor i}(x)$, if $x \not\in I \lor J$.

On the other hand, suppose $x \in I \lor J$. Then there exist $a \in I$ and $b \in J$ such that $x \leq a \lor b$

$$(\alpha_i \lor \alpha_j)(x) \geq (\alpha_i(a) \lor \alpha_j(a)) \land (\alpha_i(b) \lor \alpha_j(b))$$

$$\geq \alpha_i(a) \land \alpha_j(b)$$

$$= 1 \land 1$$

$$= 1 = \alpha_{1 \lor i}(x).$$

Therefore $(\alpha_i \lor \alpha_j)(x) \geq \alpha_{1 \lor i}(x)$ for all $x \in X$. Thus $\alpha_i \lor \alpha_j = \alpha_{1 \lor i}$. Thus $I \mapsto \alpha_j$ is an embedding of $| \mathcal{L} | (X)$ into $\mathcal{L} | (X)$.

In theorem 1.3.4, we have proved that the $\alpha$ - cuts of any $\mathcal{L}$ – fuzzy ideal $A$ of a lattice $X$ are crisp ideals of $X$. Infact these $\alpha$ - cuts completely determine the fuzzy ideal in the sense of the following.

1.3.20 Theorem

Let $L$ be a frame and $X = (X, \land, \lor)$ be a lattice with zero. Let $\{I_\alpha\}_{\alpha \in L}$ be a class of crisp ideals of $X$ such that

$$\bigcap_{\alpha \in M} I_\alpha = I_{\lor M}$$

for any $M \subseteq L$. 

61
For any \( x \in X \) define \( A(x) = \bigvee \{ \alpha \in L : x \in I_\alpha \} \). Then A is an \( L \) – fuzzy ideal of X such that \( I_\alpha \) is precisely the \( \alpha \) – cut of A, for any \( \alpha \in L \). Conversely, every \( L \) – fuzzy ideal of X can be obtained as above.

**Proof:** By the definition of A, we have

\[
x \in I_\alpha \Rightarrow \alpha \leq A(x) \Rightarrow x \in A_\alpha
\]

for any \( x \in X \) and \( \alpha \in L \). Therefore \( I_\beta \subseteq A_\beta \) for all \( \beta \in L \).

Also, note that

\[
\alpha \leq \beta \Rightarrow I_\beta \subseteq I_\alpha \text{ for any } \alpha \text{ and } \beta \in L.
\]

Now, \( x \in A_\beta \Rightarrow \beta \leq A(x) = \bigvee \{ \alpha \in L : x \in I_\alpha \} \)

\[
\Rightarrow \beta = \beta \land \left( \bigvee \{ \alpha \in L : x \in I_\alpha \} \right)
\]

\[
\Rightarrow \beta = \bigvee \{ \beta \land \alpha : x \in I_\alpha \}
\]

\[
\Rightarrow I_\beta = \bigcap_{x \in I_\alpha} I_{\beta \land \alpha}
\]

\[
\Rightarrow x \in \bigcap_{x \in I_\alpha} I_{\beta \land \alpha} = I_\beta \left( \text{since } I_\alpha \subseteq I_{\beta \land \alpha} \right).
\]

Thus \( A_\beta \subseteq I_\beta \). Therefore \( A_\beta = I_\beta \) for all \( \beta \in L \). By theorem 1.3.4, A is an \( L \)– fuzzy ideal of X. The converse is clear, since, for any \( L \)– fuzzy ideal A of X, the \( \alpha \)–cuts \( A_\alpha \) are crisp ideals of X and

\[
\bigcap_{\alpha \in M} A_\alpha = A_{\bigvee_{\alpha \in M} M} \text{ for any } M \subseteq L
\]

and, for any \( x \in X \),

\[
A(x) = \bigvee \{ \alpha \in L : x \in A_\alpha \}.
\]
1.3.21 Corollary

Let \( \{0\} = I_1 \subseteq I_2 \subseteq \ldots \) be an increasing sequence of crisp ideals of a lattice \( X \) such that \( \bigcup_{n=1}^{\infty} I_n = X \) and let

\[
1 = \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots
\]

be a decreasing sequence of elements in a frame \( L \).

Define \( A : X \rightarrow L \) by

\[
A(x) = \alpha_n, \text{ where } n \text{ is the least such that } x \in I_n.
\]

Then \( A \) is an \( L \)-fuzzy ideal of \( X \) and the \( \alpha \)-cut of \( A \) is given by

\[
A_\alpha = \begin{cases} 
X & \text{if } \alpha_n \geq \alpha \text{ for all } n \\
I_n & \text{if } n \text{ is the largest such that } \alpha_n \geq \alpha
\end{cases}
\]

The following is a generalization of Theorem 1.3.18, where we have proved that, for each crisp ideal \( I \) of \( X \) and \( \alpha \in L \), there is an \( L \)-fuzzy ideal \( \alpha \).

1.3.22 Corollary

Let \( I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = X \) be a finite sequence of crisp ideals of \( X \) and \( 1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \) be a finite sequence of elements in \( L \). Define \( A : X \rightarrow L \) by

\[
A(x) = \alpha_i, \text{ where } i \text{ is the least such that } x \in I_i.
\]

Then \( A \) is an \( L \)-fuzzy ideal of \( X \).

Let us recall that any class of subsets of a set \( X \) is called a closure set system if it is closed under arbitrary intersections. It is well known that any closure set system forms a complete lattice with respect of the inclusion ordering and, conversely, any complete lattice is isomorphic to a closure set system. Further, it is known that a closure set system \( \mathcal{C} \) is an algebraic lattice if and only if \( \mathcal{C} \) is closed under unions of upward directed subclasses. The following result is a straight forward verification.
1.3.23 Theorem
Let $X = (X, \land, \lor)$ be a lattice with 0 and $I(X)$ the class of all ideal of $X$. Then $I(X)$ is a closure set system which is closed under unions of upward directed subclasses and hence $I(X)$ is an algebraic lattice.

In view of the above theorem and its preceding discussion, we introduce the following.

1.3.24 Definition
Let $C$ be a class of $L$-fuzzy subsets of a set $X$. Any subclass $\{A_i\}_{i \in \Delta}$ of $C$ is called upward directed if, for any $i$ and $j \in \Delta$ there is $k \in \Delta$ such that

$$A_i \leq A_k \text{ and } A_j \leq A_k.$$

$C$ is said to be closed under point wise supremums of upward directed subclasses if the point wise supremum of any upward directed subclass of $C$ belongs to $C$.

1.3.25 Definition
A class $C$ of $L$-fuzzy subsets of $X$ is called algebraic if it is closed under arbitrary point-wise infimums and point-wise supremums of upward directed subclasses.

1.3.26 Theorem
Let $X = (X, \land, \lor)$ be a lattice with zero and $L$ be a frame. Then the class $FI_L(X)$ of $L$-fuzzy ideals of $X$ is algebraic.

**Proof:** In theorem 1.3.6, we have proved that $FI_L(X)$ is closed under point-wise infimums. Now, let $\{A_i\}_{i \in \Delta}$ be an upward directed class of $L$-fuzzy ideals of $X$ and $A$ be the point wise supremum of $\{A_i\}_{i \in \Delta}$. That is, $A(x) = \lor_{i \in \Delta} A_i(x)$ for any $x \in X$.

We prove that $A$ is an $L$-fuzzy ideal of $X$.

Clearly $A_i \leq A$ for all $i \in \Delta$. Since, $1 = A_i(0) \leq A(0)$, we have $A(0) = 1$. 

64
Now, let \( x \) and \( y \) be arbitrarily fixed elements of \( X \).

\[ x \leq y \Rightarrow A_i(x) \geq A_i(y) \text{ for all } i \in \Delta \]

\[ \Rightarrow A(x) = \bigvee_{i \in \Delta} A_i(x) \geq \bigvee_{i \in \Delta} A_i(y) = A(y) \]

and hence \( A \) is an antitone. In particular,

\[ A(x) \geq A(x \vee y) \text{ and } A(y) \geq A(x \vee y) \]

and therefore \( A(x \vee y) \leq A(x) \wedge A(y) \).

On the other hand,

\[
A(x) \wedge A(y) = \left( \bigvee_{i \in \Delta} A_i(x) \right) \wedge \left( \bigvee_{i \in \Delta} A_i(y) \right)
\]

\[
= \bigvee_{i, j \in \Delta} \left( A_i(x) \wedge A_j(y) \right) \quad \text{(*)}
\]

by the infinite meet distributivity in \( L \).

Now, for any \( i, j \in \Delta \), there exists \( k \in \Delta \) such that

\[ A_i \leq A_k \text{ and } A_j \leq A_k \]

and hence \( A_i(x) \wedge A_j(y) \leq A_k(x) \wedge A_k(y) \)

\[ = A_k(x \vee y) \leq A(x \vee y) \]

Therefore, by (\( \ast \)), it follows that

\[ A(x) \wedge A(y) \leq A(x \vee y) \]

and hence \( A(x \vee y) = A(x) \wedge A(y) \). Thus \( A \) is an \( L \)-fuzzy ideal of \( X \) and therefore, the point-wise supremum of \( \{A_i\}_{i \in \Delta} \) is in \( \mathcal{F}_L(X) \). Thus \( \mathcal{F}_L(X) \) is algebraic.

Usually a complete lattice is said to be an algebraic lattice if it is compactly generated, in the sense that every element is the supremum of a set of compact elements. Also, a closure set system is called algebraic if it is closed under unions of upward directed subclasses. It is well known that a lattice is an algebraic lattice if and
only if it is isomorphic to an algebraic closure set system. Even though we have proved that \( F \upharpoonright L(X) \) is algebraic, note that \( F \upharpoonright L(X) \) may not be an algebraic lattice. However, if the frame \( L \) is an algebraic lattice, then we can prove that the lattice \( F \upharpoonright L(X) \) is an algebraic lattice.

Next, we prove that the distributivity (modularity) of a lattice \( X \) can be extended to that of the lattice \( F \upharpoonright L(X) \) of \( L \)-fuzzy ideals of \( X \). First of all, recall that the map

\[
x \mapsto \{x \} = \{y \in X : y \leq x\}
\]

is an embedding of \( X \) into the lattice \( \mathcal{L}(X) \) of ideals of \( X \). Also, the mapping \( I \mapsto \chi_I \) is an embedding of \( \mathcal{L}(X) \) into \( F \upharpoonright L(X) \). Thus, we have

\[
X \subseteq \mathcal{L}(X) \subseteq F \upharpoonright L(X);
\]

that is, \( X \) can be identified with a sublattice of \( \mathcal{L}(X) \) and \( \mathcal{L}(X) \) can be identified with a sublattice of \( F \upharpoonright L(X) \).

Recall that a lattice is said to be distributive if, for all elements \( a, b \) and \( c \),

\[
a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c);
\]

or equivalently \( a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \) for all elements \( a, b \) and \( c \). Now, we prove the following.

**1.3.27 Theorem**

Let \( X = (X, \wedge, \vee) \) be a lattice with zero and \( L \) a frame.

Then the following are equivalent to each other:

1. \( F \upharpoonright L(X) \) is a distributive lattice
2. \( \mathcal{L}(X) \) is a distributive lattice
3. \( X \) is a distributive lattice.
Proof: Since $X$ is isomorphic to a sublattice of $I(X)$ and $I(X)$ is isomorphic to a sublattice of $\mathcal{P}(X)$ and since any sublattice of a distributive lattice is distributive, we have $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$.

Finally, we prove $(3) \Rightarrow (1)$:

Suppose that $X$ is a distributive lattice. Note that the frame $L$ is distributive. Let $A, B$ and $C \in \mathcal{P}(X)$.

Clearly we have

$$A \land B \leq A \land (B \lor C)$$

and

$$A \land C \leq A \land (B \lor C)$$

and hence $(A \land B) \lor (A \land C) \leq A \land (B \lor C)$. To prove the other inequality, let $x \in X$ be arbitrarily fixed. Then

$$(A \land (B \lor C))(x) = A(x) \land (B \lor C)(x)$$

$$= A(x) \land \left[ \bigvee_{F \subseteq X} \left( \bigwedge_{x \leq \vee F} \left( a \in F \right) \right) \right]$$

$$= \left[ \bigvee_{F \subseteq X} \left( \bigwedge_{x \leq \vee F} \left( a \in F \right) \right) \right]$$

$$= \left[ \bigwedge_{x \leq \vee F} \left( A(x) \land B(a) \lor (A(x) \land C(a)) \right) \right]$$

where $F \subseteq X$ denotes that $F$ is a finite subset of $X$ and $\vee F$ denotes the supremum of $F$ in $X$. Now, if $F = \{a_1, a_2, \ldots, a_n\}$ and $x \leq \bigvee F = \bigvee_{i=1}^n a_i$, then

$$x = x \land \left( \bigvee_{i=1}^n a_i \right) = \bigvee_{i=1}^n \left( x \land a_i \right) \text{ (since $X$ is distributive)}$$

and

$$\bigwedge_{a \in F} \left((A(x) \land B(a)) \lor (A(x) \land C(a))\right)$$
\[
\forall i \left( (A \land B(a_i)) \lor (A \land C(a_i)) \right) \leq \bigwedge_{i=1}^n \left[ (A \land x \land a_i) \land B(x \land a_i) \lor (A \land x \land a_i) \land C(x \land a_i) \right]
\]

(since A,B,C are fuzzy ideals and hence antitone)

\[
\forall i \left( (A \land B(x \land a_i)) \lor (A \land C(x \land a_i)) \right)
\]

\[
\leq \left( (A \land B) \lor (A \land C) \right) \left( x = \bigvee_{i=1}^n (x \land a_i) \right).
\]

Therefore, by (*), we get that

\[
( A \land (B \lor C) ) \leq ( (A \land B) \lor (A \land C) ) (x).
\]

This is true for all \( x \in X \). Thus

\[
A \land (B \lor C) \leq (A \land B) \lor (A \land C)
\]

and hence \( A \land (B \lor C) = (A \land B) \lor (A \land C) \).

Thus \( \mathcal{F} \models \mathcal{L}(X) \) is a distributive lattice.

Let us recall that a lattice is called modular if, for any of its elements \( a, b \) and \( c \) with \( a \leq c \),

\[
a \lor (b \land c) = (a \lor b) \land c
\]

The following can be proved on the lines of the proof of Theorem 1.3.27.

**1.3.28 Theorem**

The following are equivalent to each other for any lattice \( X \) with zero and any frame \( \mathcal{L} \)

\[(1) \mathcal{F} \models \mathcal{L}(X) \text{ is a modular lattice}
\]

\[(2) \models \mathcal{L}(X) \text{ is a modular lattice}
\]

\[(3) X \text{ is a modular lattice.}
\]
1.4 FUZZY FILTERS

Recall that a nonempty subset $F$ of a lattice $X = (X, \wedge, \vee)$ is called a filter of $X$ if

$$a \text{ and } b \in F \Rightarrow a \wedge b \in F \text{ and } a \vee x \in F \text{ for all } x \in X.$$  

In other words, $F$ is a filter of $X$ if and only if $F$ is closed under the operation $\wedge$ and $F$ is a final segment (that is, $y \geq a \in F \Rightarrow y \in F$). Note that by interchanging the operations $\wedge$ and $\vee$ in a lattice $(X, \wedge, \vee)$, we again get a lattice $(X, \vee, \wedge)$ in which the partial order is precisely the inverse (or dual) of that in $(X, \wedge, \vee)$. For this reason, the lattice $(X, \vee, \wedge)$ is called the dual of $(X, \wedge, \vee)$. Also, note that a subset $I$ of $X$ is a filter of the lattice $(X, \wedge, \vee)$ if and only if $I$ is an ideal of the dual $(X, \vee, \wedge)$. In this section, we introduce the notion of an $L$-fuzzy filter and discuss certain properties of these, which are analogous to those of $L$-fuzzy ideals. The proofs of most of the results are simply dual to the corresponding results on $L$-fuzzy ideals. For this reason, we simply state the results and skip their proofs.

1.4.1 Theorem

The following are equivalent to each other for any lattice $X = (X, \wedge, \vee)$

1. $X$ has largest element $1$
2. $X$ has smallest filter
3. The class $\mathcal{F}(X)$ of filters of $X$ is a Moore class.

Since we intend to make fuzzy filters to form an algebraic lattice, in particular, a complete lattice, this theorem necessitates to assume that the lattice $X$ has largest element also, which is usually denoted by $1$. The lattice $X$ is said to be bounded if $X$ has both largest element $1$ and smallest element $0$. 

69
Here after wards, the lattice X is always assumed to be bounded, unless otherwise mentioned.

1.4.2 Definition
Let \( X = (X, \land, \lor, 0, 1) \) be a bounded lattice and L a frame. An L-fuzzy subset A of X is called an L-fuzzy filter of X (or simply, a fuzzy filter of X) if

\[
A(1) = 1
\]

and

\[
A(x \land y) = A(x) \land A(y)
\]

for all \( x \) and \( y \in X \).

Note that any L-fuzzy filter A of X must be necessarily an isotone, in the sense that, for any \( x \) and \( y \in X \),

\[
x \leq y \Rightarrow A(x) = A(x \land y) = A(x) \land A(y) \Rightarrow A(x) \leq A(y).
\]

In the following we prove that any crisp filter of X can be identified with an L-fuzzy filter of X.

1.4.3 Theorem
Let S be a subset of a bounded lattice X and L a frame. then S is a crisp filter of X if and only if the characteristic map \( \chi_S \) is an L-fuzzy filter of X.

1.4.4 Theorem
Let A be an L-fuzzy subset of a bounded lattice X and L a frame. Then A is an L-fuzzy filter of X if and only if the \( \alpha \)-cut \( A_\alpha \) is a crisp filter of X for each \( \alpha \in L \).

1.4.5 Definition
For any bounded lattice \( (X, \land, \lor) \) and for any frame L, let \( \mathcal{F}_L(X) \) denote the set of all L-fuzzy filters of X.

1.4.6 Theorem
\( \mathcal{F}_L(X) \) is closed under point-wise infimums.
1.4.7 Theorem

$\mathcal{F} \mathcal{F}_{L}(X)$ is a complete lattice under point-wise ordering in which, for any $\{F_i\}_{i \in \Delta}$, the glb and lub are given by

$$\wedge_{i \in \Delta} F_i(x) = \wedge_{i \in \Delta} F_i(x)$$

and

$$\bigvee_{i \in \Delta} F_i(x) = \bigwedge \{ A(x) : A \in \mathcal{F} \mathcal{F}_{L}(X) \text{ and } A_i \leq A \text{ for all } i \in \Delta \}$$

for any $x \in X$.

1.4.8 Definition

Let $X$ be a bounded lattice and $L$ a frame. For any $L$-fuzzy subset $A$ of $X$, define

$$\overline{A} : X \rightarrow L$$

by

$$\overline{A}(x) = \wedge \{ F(x) : F \in \mathcal{F} \mathcal{F}_{L}(X) \text{ and } A \leq F \},$$

for any $x \in X$.

By theorem 1.4.6, $\overline{A}$ is an $L$-fuzzy filter of $X$ and, in fact, $\overline{A}$ is the smallest $L$-fuzzy filter of $X$ containing $A$, in the sense that, for any $L$-fuzzy filter $F$ of $X$,

$$A \leq F \iff \overline{A} \leq F.$$ 

For this reason, $\overline{A}$ is called the $L$-fuzzy filter of $X$ generated by $A$. The following result gives a point-wise description of $\overline{A}$.

1.4.9 Theorem

For any $L$-fuzzy subset $A$ of a lattice $X$,

$$\overline{A}(x) = \bigvee \left\{ \bigwedge_{i=1}^{n} A(a_i) : a_1, ..., a_n \in X, \bigwedge_{i=1}^{n} a_i \leq x \right\},$$

for any $x < 1$ in $X$ and $\overline{A}(1) = 1$. 

1.4.10 Corollary

Let $\{F_i\}_{i \in \Delta}$ be a class of L-fuzzy filters of a bounded lattice $X$. Then the supremum $\vee_{i \in \Delta} F_i$ of $\{F_i\}_{i \in \Delta}$ in $\mathcal{F}_L(X)$ is given by

$$\left( \bigvee_{i \in \Delta} F_i \right)(x) = \bigvee \left\{ \prod_{j=1}^n \left( \bigvee_{i \in \Delta} F_i(a_j) \right) : a_1, \ldots, a_n \in X \text{ and } \bigwedge_{j=1}^n a_j \leq x \right\}$$

for any $x \in X$.

1.4.11 Corollary

For any L-fuzzy filters $F$ and $G$ of $X$, the glb $A \wedge B$ and the lub $A \vee B$ in $\mathcal{F}_L(X)$ are respectively given by

$$(A \wedge B)(x) = A(x) \wedge B(x)$$

and

$$(A \vee B)(x) = \bigvee \left\{ \bigwedge_{j=1}^n \left( A(a_j) \vee B(a_j) \right) : a_1, \ldots, a_n \in X \text{ and } \bigwedge_{j=1}^n a_j \leq x \right\}$$

for any $x \in X$.

1.4.12 Definition

Let $X$ be a bounded lattice, $L$ a frame, $x \in X$ and $\alpha \in L$.

Define $\alpha^x : X \to L$ by

$$\alpha^x(y) = \begin{cases} 1 & \text{if } x \leq y \\ \alpha & \text{if } x \nleq y \end{cases}$$

for any $y \in X$.

1.4.13 Theorem

For any $x \in X$ and $\alpha \in L$, $\alpha^x$ is an L-fuzzy filter of $X$ and is called the $\alpha$-level principal fuzzy filter of $X$ corresponding to $x$.

Note that $\alpha^0 = \chi_X$ and $0^1 = \chi_{\{1\}}$.

We can replace the principal crisp filter $[x)$ by a general crisp filter in $\alpha^x$ as follows.
1.4.14 Definition

Let $F$ be a crisp filter of a lattice $X$ and $L$ a frame. For any $\alpha \in L$, define $\alpha^F : X \to L$ by

$$\alpha^F(y) = \begin{cases} 1 & \text{if } y \in F \\ \alpha & \text{if } y \not\in F \end{cases}$$

Note that, for any $\alpha \in L$ the interval $[\alpha, 1] = \{ \beta \in L : \alpha \leq \beta \leq 1 \}$ is a frame and $\alpha^F$, defined above, can be considered as a characteristic map $\chi_F$ of $X$ into $[\alpha, 1]$ and hence $\alpha^F$ is an $[\alpha, 1]$-fuzzy filter of $X$ and hence an $L$-fuzzy filter of $X$. $\alpha^F$ is called the $\alpha$-level fuzzy filter of $X$ corresponding to $F$.

For any $x \in X$, consider the principal filter $[x]$ given by

$$[x] = \{ y \in X : x \leq y \}$$

Then, for any $\alpha \in L$, $\alpha^x = \alpha^{[x]}$, further, $\alpha^1 = 1 = \chi_X$.

Also, if we define $\alpha^F$ as in 1.4.14 above for any $\alpha \in L$ and for any $F \subseteq X$, then one can verify that $\alpha^F$ is an $L$-fuzzy filter of $X$ if and only if $F$ is a crisp filter of $X$.

1.4.15 Theorem

Let $X = (X, \land, \lor)$ be a bounded lattice, $L$ a frame and $1 \neq \alpha \in L$. Then $F \mapsto \alpha^F$ is an isomorphism of the lattice $\mathcal{F}(X)$ of crisp filters of $X$ onto the lattice of $\alpha$–level $L$-fuzzy filters of $X$ corresponding to crisp filters of $X$.

In the above theorem we have proved that, for any given $1 \neq \alpha \in L$, the $\alpha$–level $L$-fuzzy filters of $X$ form a sublattice of $\mathcal{F}_L(X)$, isomorphic to the lattice $\mathcal{F}(X)$ of crisp filters of $X$. In theorem 1.4.4, we have proved that the $\alpha$–cuts of any $L$-fuzzy filter of $X$ are crisp filters of $X$. In fact, these $\alpha$–cuts completely determine the $L$-fuzzy filter $A$ in the sense of the following.
1.4.16 Theorem

Let $X = (X, \land, \lor)$ be a bounded lattice and $L$ a frame.

Let $\{F_\alpha\}_{\alpha \in L}$ be a class of crisp filters of $X$ such that

$$\bigcap_{\alpha \in M} F_\alpha = F \lor_{\alpha \in M} F_\alpha$$

for any $M \subseteq L$.

For any $x \in X$, define $A(x) = \lor \{\alpha \in L : x \in F_\alpha\}$. Then $A$ is an $L$-fuzzy filter of $X$ such that the $\alpha$-cut $A_\alpha = A_{\alpha_i}$ for all $\alpha \in L$. Conversely every $L$-fuzzy filter of $X$ can be obtained as above.

1.4.17 Corollary

Let $\{1\} = F_1 \subseteq F_2 \subseteq \ldots$ be an increasing sequence of crisp filters of a lattice $X$ such that $\bigcup_{n=1}^{\infty} F_n = X$ and let

$$1 = \alpha_1 \geq \alpha_2 \geq \ldots$$

be a decreasing sequence of elements in a frame $L$.

Define $A : X \to L$ by

$$A(x) = \alpha_n,$$

where $n$ is the least such that $x \in F_n$. Then $A$ is an $L$-fuzzy filter of $X$ and the $\alpha$-cuts of $A$ are given by

$$A_{\alpha_i} = \begin{cases} X & \text{if } \alpha_n \geq \alpha_i \text{ for all } n \\ I_n & \text{if } n \text{ is the largest such that } \alpha_n \geq \alpha_i \end{cases}$$

1.4.18 Corollary

Let $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n = X$ be a finite sequence of crisp filters of $X$ and $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be a finite sequence of elements of $L$. Define $A : X \to L$ by

$$A(x) = \alpha_i,$$

where $i$ is the least such that $x \in F_i$. Then $A$ is an $L$-fuzzy filter of $X$. 
1.4.19 Theorem

The class \( F(X) \) of all crisp filters of a bounded lattice \( X \) is a closure set system which is closed under unions of upward directed subclasses and hence \( F(X) \) forms an algebraic lattice.

1.4.20 Theorem

Let \( X = (X, \wedge, \vee) \) be a bounded lattice and \( L \) a frame. Then the class \( FF_L(X) \) of \( L \)-fuzzy filters of \( X \) is an algebraic class (refer 1.3.25)

As in the case of \( L \)-fuzzy ideals, \( FF_L(X) \) may not be algebraic, as a lattice. However, if the frame \( L \) is an algebraic lattice, then we can prove that \( FF_L(X) \) is an algebraic lattice. Further, as in the case of \( L \)-fuzzy ideals, the following can be proved.

1.4.21 Theorem

The following are equivalent to each other for any bounded lattice \( (X, \wedge, \vee) \) and for any frame \( L \)

(1) \( FF_L(X) \) is a distributive lattice

(2) \( F(X) \) is a distributive lattice

(3) \( X \) is a distributive lattice.

1.4.22 Theorem

The following are equivalent to each other for any bounded lattice \( X \) and a frame \( L \)

(1) \( FF_L(X) \) is a modular lattice

(2) \( F_L(X) \) is a modular lattice

(3) \( X \) is a modular lattice.