FUZZY IDEALS AND FILTERS OF LATTICES

INTRODUCTION

Ever since Zadeh [32] introduced the notion of a fuzzy subset of a non-empty set $X$ as a function from $X$ into the unit interval $[0,1]$, several algebraists took interest in the study of fuzzy subalgebras of several algebraic structures. Rosenfeld [23] defined the concept of fuzzy subgroup of a group and since then several researchers worked on fuzzy ideals and subrings of rings (Malik and Mordeson [19] and [20]), fuzzy ideals of lattices (Attallah [2] and Lehmke [18]), Fuzzy semi prime ideals of a semiring (Dutta and Biswas [7]), Fuzzy maximal ideals of Gamma Near-rings (Jun, Kim and Ozturk [12]), Fuzzy prime ideals of lattices and hyper lattices (Koguep, Nkuimi and Lele [14] and [15]), Transfert principle in fuzzy theory (Kondo and Dudak [16]), fuzzy prime ideals of rings (Swamy, U. M. and Swamy, K. L. N. [25]), fuzzy ideals and Congruencies of lattices (Swamy, U. M. and Raju, D. V [28]), Algebraic fuzzy systems and irreducibility (Swamy, U. M. and Raju, D. V [26] and [27]), fuzzy subgroups (Anthony and Sherwood [1]), fuzzy pseudo ideals in semi groups (Dutta [8]), fuzzy vector spaces and fuzzy topological vector spaces (Katsaras and Liu [13]), fuzzy ideals and bi-ideals in semi groups (Kuroki [17]), fuzzy ideals of a ring (Murkharjee and sen [22]), fuzzy groups and level subgroups (Das [5]), fuzzy invariant subgroups and fuzzy ideals (Wang Jin-Liu [30]), Normal fuzzy subgroups (Wu wang Ming [31]), etc.

Goguen [10] realised that the unit interval $[0,1]$ is not sufficient to take the truth values of fuzzy statements. In the works mentioned above, the fuzzy
statements usually take truth values in the interval [0,1] of real numbers, while the conventional (or crisp) statements take truth values in the two-element set \{ F, T \} or \{ 0, 1 \}, where F and 0 stand for ‘false’ and T and 1 stand for ‘true’. However, [0,1] is found to be insufficient to have the truth values of certain fuzzy statements. For example, consider the statement ‘Guntur is a good city’. The truth value of this statement may not be a real number in [0, 1]. Being good city may have several components; good in environment, good in cleanliness, good in educational facilities, good in medical facilities, good in political awareness, good in civic responsibilities, good in literary among the people, good in public transport system, good in municipal administration etc. The truth value corresponding to each component may be a real number in [0, 1]. If n is the number of such components under consideration, then the truth value of the statement ‘Guntur is a good city’ is an n-tuple of real numbers in [0,1]; that is, an element of [0,1]^n, which is not a totally ordered set, when n >1, under the usual coordinate wise ordering, where [0, 1] is considered with the usual ordering of real numbers.

However, [0,1]^n satisfies certain rich lattice theoretic properties. For example [0,1]^n is a complete lattice satisfying the infinite meet distributivity; that is, for any element a and for any subset X of [0,1]^n,

\[ a \land ( \text{Sup } X ) = \text{Sup } \{ a \land x : x \in X \}. \]

For this reason, we consider an abstract complete lattice satisfying the infinite meet distributivity to have the truth values of fuzzy statements. This type of lattice is called a frame.
If $X$ denotes the set of all cities in India, then the collection $A$ of good cities in India is actually not a subset of $X$, but it is a fuzzy subset of $X$, since being good is fuzzy. That is, $A$ can be considered as a function from $X$ into a lattice $L$ of the type discussed above. Such a fuzzy subset is called an $L$-fuzzy subset of $X$. If $X$ is a lattice and $L$ is a frame, then a function $A$ of $X$ into $L$ is called an $L$-fuzzy ideal of $X$ if, for any $x$ and $y \in X$,

$$A(x \lor y) = A(x) \land A(y)$$

and $A: X \rightarrow L$ is called an $L$-fuzzy filter of $X$ if

$$A(x \land y) = A(x) \land A(y)$$

for any $x$ and $y \in X$. In this thesis we thoroughly discuss various properties of $L$-fuzzy ideals and filters of a lattice $X$ in general and, of a distributive lattice, in particular. This study replaces $[0,1]$ with an abstract frame to take the truth values of fuzzy statements in the earlier works of Koguep, N Kuimi and Lele [14] and Swamy and Raju [28].

This thesis is broadly divided into four chapters 0, 1, 2, and 3. Chapter 0 is devoted to collect all the necessary preliminaries which will be useful in our discussions in the main text of the thesis. Even though these preliminaries are well known for those working in Lattice Theory, it will be convenient for others to have all these elementary notions and results in the beginning of the thesis for the sake of ready reference. The proofs of most of the results presented in chapter 0 are either straightforward verifications or well known and hence we simply state the results and skip the proofs.
The main text of the thesis is in chapters 1, 2 and 3. Chapter 1 is on L-fuzzy ideals and filters of a general lattice, where L is a given frame. To make the thesis complete and a self contained one, we first discuss in section 1.1 about the frames, which are complete lattices satisfying the infinite distributive law. In section 1.2, we deal with L-fuzzy subsets of a given non empty set and several lattice theoretic properties of these. In section 1.3, we consider a general lattice X = (X, ∧, ∨) and a frame L and define the concept of an L-fuzzy ideal of X and prove certain important structural properties of these. Finally, in section 1.4, we define the notion of an L-fuzzy filter of a lattice X as an L-fuzzy ideal of the dual of X and extend all the results on L-fuzzy ideals to L-fuzzy filters.

Chapter 2 is on L-fuzzy prime ideals and maximal ideals of lattices. In section 2.1, we first recall the concepts of irreducible elements and prime elements in general lattices, for the simple reason that the prime (equivalently irreducible, in the case of distributive lattices) elements in the lattice of ideals of a lattice X are precisely the prime ideals of X. Section 2.2 is devoted to a brief discussion on crisp or ordinary prime ideals and maximal ideals of a lattice as a prelude to that on L-fuzzy prime ideals and L-fuzzy maximal ideals, for the reason that these play a vital role in the structure theory of lattices in general and of distributive lattices in particular. Section 2.3 is on prime L-fuzzy ideals, which are simply prime elements in the lattice of L-fuzzy ideals of a given lattice. This concept of prime fuzzy ideal of a lattice was first introduced by U. M. Swamy and D. V. Raju [28] and later studied by B. B. N Konguep, C.N Kumi and C.Lele [14], when the truth values are taken from the interval [0,1]. Here we extend these results to the case when the truth values are taken from a general frame L and obtain certain comprehensive results on these. We concentrate on prime L-fuzzy ideals of distributive lattices,
eventhough some of the results are true for general lattices. We mainly obtain a one-to-one correspondence between prime L-fuzzy ideals of a bounded distributive lattice $X$ and pairs $(I, \alpha)$, where $I$ is a prime ideal of $X$ and $\alpha$ is a prime element in the frame $L$.

In section 2.4, we introduce the notion of an L-fuzzy prime ideal, which is weaker than that of a prime L-fuzzy ideal. An L-fuzzy ideal $A$ of a lattice $X$ is called an L-fuzzy prime ideal if, for each $\alpha$ in $L$, the $\alpha$-cut $A_{\alpha}$ is either $X$ or a prime ideal of $X$, where

$$A_{\alpha} = \{ x \in X : A(x) \geq \alpha \}.$$ 

We prove here that an L-fuzzy ideal $A$ is an L-fuzzy prime ideal if and only if, for any $x$ and $y$ in $X$,

$$A(x \wedge y) = A(x) \text{ or } A(y).$$

Also, we extend the celebrated theorem of M. H. Stone [24] on prime ideals of distributive lattices to L-fuzzy prime ideals. In section 2.5, we consider maximal L-fuzzy ideals of $X$, which are, as usual, maximal members in the set of proper L-fuzzy ideals with respect to the pointwise partial ordering. We determine all the maximal L-fuzzy ideals of a bounded distributive lattice $X$ by establishing a one-to-one correspondence between maximal L-fuzzy ideals of $X$ and pairs $(M, \alpha)$, where $M$ is a maximal ideal of $X$ and $\alpha$ is a dual atom in the frame $L$. Note that, if $L = [0,1]$ or $[0,1]^n$, then $L$ has no dual atoms and hence $X$ has no maximal L-fuzzy ideals. In section 2.6, we introduce the notion of an L-fuzzy maximal ideal in such away that an ideal $I$ of $X$ is a maximal ideal if and only if its characteristic map is an L-fuzzy maximal ideal of $X$ and that a proper L-fuzzy ideal of $X$ is an L-fuzzy
maximal ideal of $X$ if and only if, for each $\alpha \in L$, the $\alpha$-cut $A_{\alpha}$ is either $X$ or a maximal ideal of $X$. Further we prove that any maximal $L$-fuzzy ideal of $X$ is always an $L$-fuzzy maximal ideal and that the converse is not true.

Chapter 3 is on primeness and maximality in $L$-fuzzy filters. It is well known that the concept of a lattice is self-dual, in the sense that, if $(X, \wedge, \vee)$ is a lattice, then the structure $(X, \vee, \wedge)$ obtained by interchanging the operations $\wedge$ and $\vee$ is again a lattice and is called the dual of $(X, \wedge, \vee)$. Also, this duality is carried to the partial orders induced by $\wedge$ and $\vee$, since we have the equality

$$a = a \wedge b \iff a \vee b = b$$

for any elements $a$ and $b$ in a lattice. The partial order $\leq$, induced by the operation $\wedge$ and defined by

$$a \leq b \text{ if and only if } a = a \wedge b,$$

is precisely the inverse (or dual) of that induced by the operation $\vee$ and defined by

$$a \geq b \text{ if and only if } a = a \vee b.$$

In fact the absorption laws are equivalent to saying that $\leq$ is equivalent to the inverse of $\geq$. Note that $\leq \leq^{-1}$ if and only if $a = a \vee (a \wedge b)$ for all $a$ and $b$ and that $\geq^{-1} \leq$ if and only if $a = a \wedge (a \vee b)$ for all $a$ and $b$. In view of all these, the concept that is dual to that of an ideal in a lattice is called a filter. That is, $I$ is called a filter of a lattice $(X, \wedge, \vee)$ if $I$ is an ideal of the dual lattice $(X, \vee, \wedge)$. In chapter 3, we discuss the notions of $L$-fuzzy filters in lattices and their primeness and maximality. Even though the proofs of
several statements are simply dual to those in fuzzy ideals, there are certain variations in the statements. Whenever proofs are simply dual to the earlier ones on ideals, they are skipped. The proofs are given only if variations are there.

In section 3.1, we recall the concepts of prime filters and maximal filters and discuss certain important fundamental properties of these, in order to facilitate a smooth discussion on primeness and maximality among L-fuzzy filters of a lattice. In section 3.2, we state all the results, without proofs, concerning prime L-fuzzy filters of a bounded distributive lattice. The proofs are skipped in view of their duality with those of prime L-fuzzy ideals. In section 3.3, we characterize L-fuzzy filters A of a lattice X for which each $\alpha$ - cut $A_{\alpha}$ is either a prime filter of X or the whole lattice X. The prime L-fuzzy filters discussed in section 3.2 satisfy the above property and, not every L-fuzzy filter satisfying the above property is a prime L-fuzzy filter. Being a prime L-fuzzy filter is stronger than saying that each $\alpha$ - cut is either a prime filter or the whole lattice; L-fuzzy filters of this type are called L-fuzzy prime filters. Finally in section 3.4, we discuss maximality among L-fuzzy filters. As in the case of ideals, we introduce the notions of maximal L-fuzzy filter and L-fuzzy maximal filter and extend the results proved for L-fuzzy ideals to those. We simply state the results on maximal L-fuzzy filters and L-fuzzy maximal filters, without proofs, since their proofs are analogous or dual to those on ideals.