The concept of prime ideals is crucial in the study of structure theory of lattices in general and, in particular, that of distributive lattices. Prime ideals play a vital role in the representation theory of distributive lattices which is a pioneering work of M.H. Stone [24]. Prime ideals in distributive lattices are precisely the meet irreducible elements in the lattice of ideals. An element \( a \) in a lattice \( L \) is called meet irreducible if \( a \) is not the largest in \( L \) and, for any \( x \) and \( y \) in \( L \), \( a = x \land y \) implies that \( a = x \) or \( a = y \). Also, \( a \) is called meet prime if, \( x \land y \leq a \) implies \( x \leq a \) or \( y \leq a \). However, in a distributive lattice an element is meet prime if and only if it is meet irreducible.

In this chapter we study the meet prime elements and dual atoms in the lattice of fuzzy ideals of a lattice. The concept of fuzzy prime ideals was introduced by U.M. Swamy and K.L.N. Swamy [25] in rings and later extended to the case of lattices by U.M. Swamy and D.V. Raju [28] and B.B.N. Koguex, C.N. Kuimi and C. Lele [14]. We present certain comprehensive results on fuzzy prime ideals and maximal ideals, where the fuzzy objects assume values in a general frame; that is, a complete lattice satisfying the infinite meet distributivity.

### 2.1 Irreducible Elements in Lattices

Before taking up discussion on fuzzy prime ideals and maximal ideals of lattices, we first recall the concepts of irreducible elements and prime elements in general lattices, for the simple reason that the prime (irreducible, in the case of
distributive lattices) elements in the lattice of ideals of a lattice X are precisely the prime (crisp) ideals of X. Let us begin with the following.

2.1.1 Definition

Let \( X = (X, \wedge, \vee) \) be a bounded lattice and \( x \in X \)

(1) \( x \) is said to be meet irreducible (or \( \wedge \)-irreducible) if \( x \neq 1 \) and, for any \( a \) and \( b \in X \),

\[
x = a \land b \implies x = a \lor x = b.
\]

(2) \( x \) is said to be meet prime (or \( \wedge \)-prime) if \( x \neq 1 \) and, for any \( a \) and \( b \in X \),

\[
a \land b \leq x \implies a \leq x \lor b \leq x.
\]

2.1.2 Theorem

Any meet prime element in any bounded lattice is meet irreducible and the converse is not true.

Proof: Let \( X \) be a bounded lattice and \( x \in X \) be a meet prime element. Then \( x \neq 1 \) and, for any \( a \) and \( b \in X \),

\[
x = a \land b \implies a \leq x \lor b \leq x
\]

\[
\implies x = a \lor x = b \quad (\text{since } x \leq a \text{ and } x \leq b).
\]

Therefore \( x \) is meet irreducible. The example given in the following establishes that a meet irreducible element need not be meet prime.

2.1.3 Example

Consider the lattice \( M_3 \) represented by the Hasse diagram given below

```
In this lattice, it can be easily checked that \( a, b \) and \( c \) are all the meet irreducible elements and none of them is meet prime, since
```
Even though a meet irreducible element need not be meet prime in general, in certain special cases every meet irreducible element is meet prime. We prove in the following that every meet irreducible element in a distributive lattice is meet prime. Note that the lattice $M_5$ given above is not distributive, since

$$a \land (b \lor c) = a \land 1 = a$$

and $(a \land b) \lor (a \land c) = 0 \lor 0 = 0 \neq a \land (b \lor c)$.

**2.1.4 Theorem**

Let $X = (X, \land, \lor)$ be a bounded distributive lattice and $x \in X$. Then $x$ is meet irreducible if and only if $x$ is meet prime.

**Proof:** Let $x$ be meet irreducible and $a$ and $b \in X$ such that $a \land b \leq x$. Then

$$x = x \lor (a \land b) = (x \lor a) \land (x \lor b)$$

and hence $x = x \lor a$ or $x = x \lor b$, so that $a \leq x$ or $b \leq x$. Thus $x$ is meet prime. The converse is proved in 2.1.2.

**2.1.5 Definition**

For any lattice $X$, let $P_\land(X)$ be the set of all meet prime elements in $X$ and, for any $x \in X$, let

$$P_\land(x) = \{ p \in P_\land(X) : x \leq p \}$$

**2.1.6 Theorem**

The following hold for any elements $x$ and $y$ of a bounded lattice $X = (X, \land, \lor)$

1. $x \leq y \Rightarrow P_\land(y) \subseteq P_\land(x)$
2. $P_\land(x) \cap P_\land(y) = P_\land(x \lor y)$
3. $P_\land(x) \cup P_\land(y) = P_\land(x \land y)$
4. $P_\land(1) = \emptyset$ and $P_\land(0) = P_\land(X)$.  

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Proof:

(1) For any $x \leq y$ and $p \in P_{\land}(y)$, we have $x \leq y \leq p \in P_{\land}(X)$.

$$\Rightarrow x \leq p \in P_{\land}(x)$$

$$\Rightarrow x \in P_{\land}(x).$$

(2) For any $p \in P_{\land}(X)$,

$$p \in P_{\land}(x) \cap P_{\land}(y) \iff x \leq p \text{ and } y \leq p$$

$$\iff x \vee y \leq p$$

$$\iff p \in P_{\land}(x \vee y)$$

and hence $P_{\land}(x) \cap P_{\land}(y) = P_{\land}(x \vee y)$.

(3) Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it follows from (1) that $P_{\land}(x) \cup P_{\land}(y) \subseteq P_{\land}(x \wedge y)$.

On the other hand,

$$p \in P_{\land}(x \wedge y) \Rightarrow x \wedge y \leq p \text{ and } p \text{ is meet prime}$$

$$\Rightarrow x \leq p \text{ or } y \leq p$$

$$\Rightarrow p \in P_{\land}(x) \cup P_{\land}(y)$$

Thus $P_{\land}(x) \cup P_{\land}(y) = P_{\land}(x \wedge y)$.

(4) For any meet prime element $p$, we have $p \neq 1$ and hence $1 \nleq p$, so that $p \notin P_{\land}(1)$.

Thus $P_{\land}(1) = \phi$. Also, since $0 \leq p$ for all $p \in P_{\land}(X)$, we get that $P_{\land}(0) = P_{\land}(X)$.

2.1.7 Theorem

The following are equivalent to each other for any bounded lattice $X = (X, \land, \lor)$.

(1) For any $x$ and $y \in X$, $P_{\land}(x) \subseteq P_{\land}(y) \Rightarrow y \leq x$

(2) For any $x$ and $y \in X$, $P_{\land}(x) = P_{\land}(y) \Rightarrow x = y$

(3) $x = \inf P_{\land}(x)$, for all $x \in X$

(4) Any element of $X$ is the infimum of a set of meet prime elements in $X$.  

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Proof:

(1) \( \Rightarrow \) (2): \( P_\wedge (x) = P_\wedge (y) \Rightarrow x \leq y \) and \( y \leq x \Rightarrow x = y \).

(2) \( \Rightarrow \) (3): Let \( x \in X \) and \( P_\wedge (x) = \{ p \in P_\wedge (x) : x \leq p \} \).

Clearly \( x \) is a lower bound of \( P_\wedge (x) \). If \( y \) is any other lower bound of \( P_\wedge (x) \) in \( X \), then

\[
y \leq p \quad \text{for all} \quad p \in P_\wedge (x)
\]

and hence \( P_\wedge (x) \subseteq P_\wedge (y) \) so that

\[
P_\wedge (x) = P_\wedge (x) \cap P_\wedge (y) = P_\wedge (x \lor y).
\]

From (2), it follows that \( x = x \lor y \) and hence \( y \leq x \).

Thus \( x \) is the greatest lower bound of \( P_\wedge (x) \).

(3) \( \Rightarrow \) (4): is clear.

(4) \( \Rightarrow \) (1): let \( x \) and \( y \in X \) and

\[
x = \text{Inf } A \quad \text{and} \quad y = \text{Inf } B
\]

for some subsets \( A \) and \( B \) of the set \( P_\wedge (X) \) of meet prime elements in \( X \). Then, clearly \( A \subseteq P_\wedge (x) \) and \( B \subseteq P_\wedge (y) \) and hence

\[
P_\wedge (x) \subseteq P_\wedge (y) \Rightarrow A \subseteq P_\wedge (y)
\]

\[
\Rightarrow y \leq p \quad \text{for all} \quad p \in A
\]

\[
\Rightarrow y \leq \text{Inf } A = x
\]

\[
\Rightarrow y \leq x.
\]

2.1.8 Corollary

Let \( (X, \wedge, \lor) \) be a bounded lattice in which every element is the infimum of a set of meet prime elements in \( X \). Then \( (X, \wedge, \lor) \) is a distributive lattice.

Proof:

For any \( x, y \) and \( z \in X \), we have

\[
P_\wedge (x \wedge (y \lor z)) = P_\wedge (x) \cup P_\wedge (y \lor z) \quad \text{(by 2.1.6(3))}
\]
\[ = P_\land(x) \cup (P_\land(y) \cap P_\land(z)) \text{ (by 2.1.6 (2))} \]
\[ = (P_\land(x) \cup P_\land(y)) \cap (P_\land(x) \cup P_\land(z)) \]
\[ = P_\land(x \land y) \cap P_\land(x \land z) \]
\[ = P_\land((x \land y) \lor (x \land z)). \]

Therefore, by theorem 2.1.7, we have
\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

Thus \( (X, \land, \lor) \) is a distributive lattice.

Let us recall that a partially ordered set \( (X, \leq) \) is called a complete lattice if every subset of \( X \) has infimum and supremum in \( X \). Note that, for any subset \( Y \) of a complete lattice \( X \) and for any element \( x \) in \( X \),
\[ \text{Sup } Y \leq x \iff y \leq x \text{ for all } y \in Y \]
and hence we have the following.

2.1.9 Theorem

Let \( X = (X, \land, \lor) \) be a complete lattice and \( Y \subseteq X \). Then
\[ \bigcap_{y \in Y} P_\land(y) = P_\land(\text{Sup } Y). \]

2.1.10 Theorem

Let \( X = (X, \land, \lor) \) be a complete lattice in which every element is the infimum of a set of meet prime elements in \( X \). Then \( X \) is a frame.

Proof: Recall that a complete lattice is called a frame if it satisfies the infinite meet distributivity. For any \( x \in X \) and \( \{y_\alpha\}_{\alpha \in \Delta} \subseteq X \), we have
\[ P_\land\left( x \land \bigvee_{\alpha \in \Delta} y_\alpha \right) = P_\land(x) \cup P_\land(\bigvee_{\alpha \in \Delta} y_\alpha) \]
\[ = P_\land(x) \cup \left( \bigcap_{\alpha \in \Delta} P_\land(y_\alpha) \right) \]
\[
\begin{align*}
\left( P_{\alpha} (x) \cup P_{\alpha} (y) \right) & = \bigcap_{\alpha \in \Delta} \left( P_{\alpha} (x) \cup P_{\alpha} (y) \right) \\
\left( x \wedge y \right) & = \bigcap_{\alpha \in \Delta} P_{\alpha} (x \wedge y) \\
\left( x \wedge y \right) & = P_{\bigvee_{\alpha \in \Delta} \left( x \wedge y \right)}.
\end{align*}
\]

Therefore, by Theorem 2.1.7, we have

\[
x \wedge \left( \bigvee_{\alpha \in \Delta} y_{\alpha} \right) = \bigvee_{\alpha \in \Delta} \left( x \wedge y_{\alpha} \right)
\]

Thus \((X, \wedge, \vee)\) is frame.

The following example establishes that the infinite join distributivity may not hold in a complete lattice in which any element is the infimum of a set meet prime elements.

**2.1.11 Example**

Let \(X\) be the set of all non-negative integers and, for any \(a\) and \(b \in X\), define

\[a \leq b\] if and only if \(b\) divides \(a\).

Then \(X\) is a complete lattice with 0 and 1 as the smallest and largest elements respectively. Also, for any \(Y \subseteq X\),

\[
\text{Sup } Y = \gcd Y
\]

and

\[
\text{Inf } Y = \text{lcm } Y.
\]

In this lattice, for any prime number \(p\) and a positive integer \(n\), \(p^n\) is meet prime in \(X\); for, if \(a\) and \(b\) are in \(X\) such that \(a \wedge b \leq p^n\), then \(p^n\) divides the LCM of \(\{a, b\}\) and hence \(p^n\) divides \(a\) or \(b\), so that \(a \leq p^n\) or \(b \leq p^n\).

Also, for any \(a \in X\),

\[a = 1 \Rightarrow a = \text{Inf } \phi,\]

\[a = 0 \Rightarrow a = \text{lcm } \{p^n : n \in \mathbb{Z}^+, p \text{ is a prime number}\} = \text{Inf } \{p^n : n \in \mathbb{Z}^+, p \text{ is a prime number}\}\]

and \(a \neq 0, 1 \Rightarrow a = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}, n_i \in \mathbb{Z}^+\) and \(p_i\)'s are distinct primes.
\[ a = \text{LCM} \{ p_1^n, p_2^n, ..., p_r^n \} \]
\[ a = \text{Inf} \{ p_1^n, p_2^n, ..., p_r^n \}. \]

Therefore, any element of \( X \) is the infimum of a set of meet prime elements in \( X \). However, the infinite join distributivity does not hold in \( X \); for consider the set \( P \) of odd prime numbers. Then we have
\[
2 \lor (\text{Inf } P) = 2 \lor (\text{LCM } P) = 2 \lor 0 = 2
\]
and hence
\[
\text{Inf} \left( \bigwedge_{p \in P} (2 \lor p) \right) \neq \bigwedge_{p \in P} (2 \lor p)
\]

The following is an important example of a frame in which every element is the infimum of a set of meet prime (equivalently irreducible) elements. This is important in view of the fact that it is a prospective candidate to take the truth values of the fuzzy objects we are investigating.

\textbf{2.1.12 Example}

Let \( n \) be any positive integer and \( X = [0,1]^n \), where \([0,1]\) is the closed unit interval in the real number system \( \mathbb{R} \). Clearly \([0,1]\) is a complete lattice under the usual ordering of the real numbers. Therefore \( X \) is a complete lattice under the point-wise ordering. In 1.1.10(3), we have observed that \([0,1]^n\) is a frame. Now, for any \( p = (p_1, p_2, ..., p_n) \in [0,1]^n \), \( p \) is meet prime if and only if there exists \( i, 1 \leq i \leq n \), such that \( p_i < 1 \) and \( p_j = 1 \) for all \( j \neq i \). For, if \( i \neq j \) and \( p_i < 1 \) and \( p_j < 1 \), then
\[
p = (p_1, p_2, ..., p_n) = (a_1, a_2, ..., a_n) \land (b_1, b_2, ..., b_n),
\]
where
\[
a_k = \begin{cases} p_k & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases} \quad \text{and} \quad b_k = \begin{cases} p_k & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}
\]
and \( p < (a_1, a_2, \ldots, a_n) \) and \( p < (b_1, b_2, \ldots, b_n) \). Also, any \((a_1, a_2, \ldots, a_n) < (1, 1, \ldots, 1)\) in \([0,1]^n\) can be written as

\[
(a_1, a_2, \ldots, a_n) = \bigwedge_{i=1}^{n} (x_{i_1}, x_{i_2}, \ldots, x_{i_n})
\]

where \( x_{ij} = \begin{cases} a_i & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases} \)

Thus any element of \([0,1]^n\) is the infimum of a set of meet prime elements.

In fact, we can extend the above discussion to a larger class of lattices. Let \( I \) be any non-empty set and \( X = [0,1]^I \), the set of all mappings of \( I \) into \([0,1]\). Then \( X \) is a complete lattice under the point-wise ordering; that is, for any \( f \) and \( g \in X \),

\[ f \leq g \text{ if and only if } f(i) \leq g(i) \text{ for all } i \in I. \]

Also, for any \( \{f_a\}_{a \in \Delta} \subseteq X \),

\[ \text{Inf} \{f_a\}_{a \in \Delta} = \bigwedge_{a \in \Delta} f_a, \text{ the point-wise infimum} \]

and \( \text{sup} \{f_a\}_{a \in \Delta} = \bigvee_{a \in \Delta} f_a, \text{ the point-wise supremum} \). Since \([0,1]\) is frame, it follows that \( X \) is also a frame. Also, for any \( f \in X = [0,1]^I \) and \( i \neq j \in I \), if we define \( g \) and \( h \in [0,1]^I \) by

\[
g(k) = \begin{cases} f(k) & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases} \quad \text{and} \quad h(k) = \begin{cases} f(k) & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}
\]

then \( f = g \wedge h \). From this it follows that \( f \) is meet prime if and only if there exists unique \( i \in I \) such that

\[ f(i) < 1 \text{ and } f(j) = 1 \text{ for all } j \neq i \text{ in } I. \]
Further, for any \( f \in [0,1]^{I} \) and \( i \in I \), if we define \( g_{i} \in [0,1]^{I} \) by

\[
g_{i}(j) = \begin{cases} 
1 & \text{if } j \neq i \\
f(i) & \text{if } j = i \text{ and } f(i) < 1 
\end{cases}
\]

then, it can be easily verified that

\[
f = \bigwedge \{ g_{i} : i \in I, f(i) < 1 \}.
\]

Thus any \( f \in [0,1]^{I} \) is the infimum of a set of meet prime elements in \([0,1]^{I}\).

## 2.2 PRIME IDEALS AND MAXIMAL IDEALS

In the structure theory of lattices, in general, and of distributive lattices in particular, the concept of a prime ideal plays a vital role, in view of the Stone’s theorem [24]. In this section, we discuss prime ideals and maximal ideals of lattices and extend these notions to the fuzzy ideals in the further sections of this chapter.

### 2.2.1 Definition

Let \( X = (X, \wedge, \vee) \) be a bounded lattice. Let \( \mathcal{I}(X) \) and \( \mathcal{P}(X) \) the sets of ideals and principal ideals, respectively of \( X \).

Recall that \( \mathcal{I}(X) \) forms a complete lattice under the inclusion ordering and that \( \mathcal{P}(X) \) is a sublattice of \( \mathcal{I}(X) \). For any \( a \) in \( X \), we have the principal ideal \( (a) \) defined by

\[
(a) = \{ x \in X : x \leq a \}.
\]

Also, for any \( a \) and \( b \) in \( X \),

\[
(a) \cap (b) = (a \wedge b) \text{ and}
\]

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\((a] \lor (b] = (a \lor b]\).

Therefore \(a \mapsto (a] \) is an isomorphism of \(X\) onto \(\mathcal{P} \setminus (X)\). Thus, we can treat \(X\) as (an isomorphic copy of) a sublattice of the lattice \(\mathcal{P} \setminus (X)\) of ideals of \(X\).

2.2.2 Definition

An ideal \(I\) of a bounded lattice \(X = (X, \land, \lor)\) is called a prime ideal of \(X\) if \(I\) is a meet prime element in \(\mathcal{I}(X)\); that is, \(I\) is not the largest element \(X\) in \(\mathcal{I}(X)\) and, for any \(J\) and \(K\) in \(\mathcal{I}(X)\),

\[ J \cap K \subseteq I \Rightarrow J \subseteq I \text{ or } K \subseteq I. \]

2.2.3 Theorem

The following are equivalent to each other for any ideal \(I\) of a bounded lattice

(1) \(I\) is a prime ideal of \(X\)

(2) \(I \neq X\) and, for any \(a\) and \(b\) in \(X\),

\[ a \land b \in I \Rightarrow a \in I \text{ or } b \in I \]

(3) \(1 \notin I\) and, for any \(a\) and \(b\) in \(X\),

\[ (a] \cap (b] \subseteq I \Rightarrow (a] \subseteq I \text{ or } (b] \subseteq I. \]

Proof: Note that the ideal \(I\) becomes the whole lattice \(X\) if and only if the largest element 1 in \(X\) is in \(I\). Also, \((a] \cap (b] = (a \land b]\). Further, \((a] \subseteq I\) if and only if \(a \in I\). Thus (2) and (3) are clearly equivalent.

(1) \(\Rightarrow\) (2): Let \(I\) be a prime ideal of \(X\). Then, by definition 2.2.2, \(I\) is a meet prime element in the lattice \(\mathcal{I}(X)\) of ideals of \(X\). Therefore \(I \neq X\), which is the largest element in \(X\) and, for any \(a\) and \(b \in I\),

\[ a \land b \in I \Rightarrow (a \land b] \subseteq I \]

\[ \Rightarrow (a] \cap (b] \subseteq I \]

\[ \Rightarrow (a] \subseteq I \text{ or } (b] \subseteq I \]
(2) \(\Rightarrow (1)\): Let \(J\) and \(K\) be ideals of \(X\) such that \(J \not\subseteq I\) and \(K \not\subseteq I\). There exist elements \(a\) and \(b\) in \(X\) such that \(a \in J\), \(a \not\in I\) and \(b \in K\), \(b \not\in I\).

Then \(a \land b \in J \land K\) and \(a \land b \not\in I\), therefore \(J \land K \not\subseteq I\).

### 2.2.4 Corollary

The following are equivalent for any proper ideal \(I\) of a bounded lattice \(X\).

(1) \(I\) is a prime ideal of \(X\).

(2) For any ideals \(I_1, I_2, \ldots, I_n\) of \(X\),

\[
I_1 \cap I_2 \cap \ldots \cap I_n \subseteq I \implies I_i \subseteq I \text{ for some } 1 \leq i \leq n.
\]

(3) For any elements \(a_1, a_2, \ldots, a_n \in X\),

\[
\bigwedge_{i=1}^n a_i \in I \implies a_i \in I \text{ for some } 1 \leq i \leq n.
\]

### 2.2.5 Theorem

Let \(X = (X, \land, \lor)\) be a bounded lattice and \(I\) an ideal of \(X\). Suppose \(P_1, P_2, \ldots, P_n\) are prime ideals of \(X\) such that

\[
I \subseteq P_1 \cup P_2 \cup \ldots \cup P_n.
\]

Then \(I \subseteq P_i\) for some \(1 \leq i \leq n\).

**Proof:** We use induction on \(n\). The theorem trivial for \(n=1\). Assume that \(n > 1\) and that the theorem is true for \(n-1\). Put

\[
Q_i = P_1 \cup \ldots \cup P_{i-1} \cup P_{i+1} \cup \ldots \cup P_n
\]

for each \(1 \leq i \leq n\). If \(I \subseteq Q_i\) for some \(i\), then by the induction hypothesis, \(I \subseteq P_j\) for some \(j \neq i\). Suppose, if possible, that \(I \not\subseteq Q_i\) for all \(1 \leq i \leq n\); then for each \(i\), \(1 \leq i \leq n\), there exists \(a_i \in I\) such that \(a_i \not\in Q_i\).

That is, \(a_1, a_2, \ldots, a_n \in I\) such that \(a_i \not\in P_j\) for all \(i \neq j\). Put, for each \(1 \leq i \leq n\),
\[ b_i = \bigwedge_{j \neq i} a_j = a_i \land \ldots \land a_{i-1} \land a_{i+1} \land \ldots \land a_n \]

and \( b = b_1 \lor b_2 \lor \ldots \lor b_n \).

Then \( b_i \in I \) for all \( 1 \leq i \leq n \) and hence \( b \in I \).

But, \( b_i \notin P_i \) (since \( a_j \notin P_i \) for \( j \neq i \) and \( P_i \) is a prime ideal) and hence \( b \notin P_i \) for all \( 1 \leq i \leq n \), which is a contradiction to the hypothesis that \( I \subseteq P_1 \cup P_2 \cup \ldots \cup P_n \).

Next, we describe prime ideals in two important examples of frames, as we are more interested in frames, which assume the truth values of the fuzzy objects we are investigating. Before going for these examples, the following is an easy verification.

**2.2.6 Theorem**

Let \( X \) and \( Y \) be two bounded lattices and \( f : X \to Y \) be a homomorphism of lattices such that \( f(1) = 1 \) and \( f(0) = 0 \). Then \( f^{-1}(P) \) is a prime ideal of \( X \) for any prime ideal \( P \) of \( Y \), where \( f^{-1}(P) = \{ x \in X : f(x) \in P \} \).

**2.2.7 Example**

Consider the interval \([0,1]\), which is a frame under the usual ordering of real numbers. Note that, for any \( a \) and \( b \) in \([0,1]\), \( a \land b = a \lor b \), depending on whether \( a \leq b \) or \( b \leq a \).

Therefore, every proper ideal of \([0,1]\) is a prime ideal. Let \( I \) be any proper ideal of \([0,1]\) and \( a = \sup I \). Then, clearly \( I \subseteq [0,a] \).

If \( a \in I \), then \( I = [0,a] \).

If \( a \notin I \), then \( I = [0,a) \) (for, if \( x < a \), then there exists \( y \in I \) such that \( x < y < a \), since \( a = \sup I \)). Thus, any prime ideal \( I \) of \([0,1]\) is of the form

\[ [0,a] \text{ for some } 0 \leq a < 1 \]

or \([0,a) \text{ for some } 0 < a \leq 1 \).

**2.2.8 Example**

Let \( n \) be a positive integer and \( X = [0,1]^n \), which is a frame under the coordinate-wise ordering (refer 1.1.10(3)). For each \( 1 \leq i \leq n \), let
$p_i: [0,1]^n \to [0,1]$ be the $i$th projection.

Then, by 2.2.6, $p_i^{-1}[0,a]$ is a prime ideal of $[0,1]^n$ for each $0 \leq a < 1$ and $p_i^{-1}[0,b]$ is also a prime ideal of $[0,1]^n$ for each $0 < b \leq 1$, since $p_i$ is a homomorphism of lattices. Also, any prime ideal of $[0,1]^n$ must be necessarily of the form

$p_i^{-1}[0,a] = \{x \in [0,1]^n : x_i \leq a\}$ for some $0 \leq a < 1$

or $p_i^{-1}[0,b] = \{x \in [0,1]^n : x_i < b\}$ for some $0 < b \leq 1$,

for some $1 \leq i \leq n$.

2.2.9 Example

Consider the example given in 1.1.12, where $\mathbb{N}$ is the set of all non-negative integers.

For any $a$ and $b$ in $\mathbb{N}$, let us define

$a \leq b$ if and only if $b$ divides $a$.

Then $(\mathbb{N}, \leq)$ is a frame in which 0 and 1 are respectively the smallest and greatest elements and for any $Y \subseteq \mathbb{N}$, LCM $Y$ and GCD $Y$ are respectively the glb $Y$ and lub $Y$. In particular, for any $a$ and $b \in \mathbb{N}$,

$a \wedge b = \text{LCM}\{a, b\}$

and $a \vee b = \text{GCD}\{a, b\}$.

For each prime number $p$ and a positive integer $n$, let us define

$P_{p,n} = \{a \in \mathbb{N} : p^n \text{ divides } a\}$.

Then $P_{p,n} = (p^n)$, the principal ideal in $\mathbb{N}$ generated by $p^n$. Since $n$ is positive, $1 \notin P_{p,n}$ and hence $P_{p,n}$ is a proper ideal of $\mathbb{N}$. For any $a$ and $b \in \mathbb{N}$,

$a \wedge b \in P_{p,n} \Rightarrow p^n \text{ divides } \text{LCM}\{a, b\}$

$\Rightarrow p^n \text{ divides } a$ or $b$

$\Rightarrow a \in P_{p,n}$ or $b \in P_{p,n}$.
Therefore \( P_{p,n} \) is a prime ideal. On the other hand we can prove that any prime ideal of \( \mathbb{N} \) is of the form \( P_{p,n} \) or \( \{0\} \). Clearly \( \{0\} \) is a prime ideal of \( \mathbb{N} \) (since \( a \neq 0 \) and \( b \neq 0 \) implies that \( a \land b (=\text{LCM} \{a, b\}) \neq 0 \)). Now, let \( P \) be any prime ideal of \( \mathbb{N} \) and \( P \neq \{0\} \). Choose \( 0 \neq a \in P \). Since \( P \) is a proper ideal, \( a \neq 1 \) and hence

\[
a = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} = \text{LCM} \{p_i^{n_i}\} = \bigwedge_{i=1}^{r} p_i^{n_i},
\]

where \( p_1, p_2, \ldots, p_r \) are distinct prime numbers and \( n_1, n_2, \ldots, n_r \) are positive integers.

Now, since \( P \) is a prime ideal and

\[
\bigwedge_{i=1}^{r} p_i^{n_i} = a \in P,
\]

it follows that \( p^n \in P \) for some prime number \( p \) and a positive integer \( n \). If \( q \neq p \) is another prime number and \( m \in \mathbb{Z}^+ \) such that \( q^m \in P \), then

\[
1 = \text{GCD} \{p^n, q^m\} = p^n \lor q^m \in P,
\]

which is a contradiction, since \( P \) is a proper ideal and 1 is the largest element in \( \mathbb{N} \).

Thus, there exists unique prime number \( p \) such that \( p^n \in P \) for some positive integer \( n \).

Let \( n \) be the least positive integer such that \( p^n \in P \).

Then \( a \in P_{p,n} \Rightarrow p^n \) divides \( a \)

\[
\Rightarrow a \leq p^n \in P
\]

\[
\Rightarrow a \in P.
\]

Therefore \( P_{p,n} \subseteq P \). Also, for any \( a \) not divisible by \( p^n \):

\[
0 \neq a \in P \Rightarrow a \neq 1, a \neq 0
\]

\[
\Rightarrow a = p^r \cdot b, \text{ where } 0 \leq r < n \text{ and } p \text{ does not divide } b
\]

\[
\Rightarrow p^r \land b = p^r b = a \in P
\]

\[
\Rightarrow b \in P \text{ (since } p^r \not\in P)
\]

\[
\Rightarrow b \in P \text{ and } p^n \in P
\]

\[
\Rightarrow 1 = p^n \lor b \in P, \text{ a contradiction.}\]
Therefore, $a \not\in P_{p,n} \Rightarrow a \not\in P$ and hence $P \subseteq P_{p,n}$. Thus $P = P_{p,n}$. Also, we have that 

$(p,n) \mapsto P_{p,n}$ is a one-to-one correspondence between $\mathbb{Q} \times \mathbb{Z}^+$ (Where $\mathbb{Q}$ is the set of prime numbers) and the set of non zero prime ideals of $\mathbb{N}$. In fact, we can prove the following

1. $P_{p,n} \neq P_{q,m}$ for all primes $p \neq q$ and $n \in \mathbb{Z}^+$

2. For any prime number $p$ and $n \in \mathbb{Z}^+$; $P_{p,n} \subseteq P_{q,m} \iff p = q$ and $m \leq n$

in the usual order.

2.2.10 Definition

An ideal I of lattice X is called irreducible if it is meet irreducible in the lattice $\mathfrak{I}(X)$ of ideals of X.

Since any meet prime element in any lattice is meet irreducible, it follows that every prime ideal of a lattice is an irreducible ideal. The converse of this is not true, as we observe in the following example.

2.2.11 Example

Consider the lattice $M_5=\{0, a, b, c, 1\}$ given in example 2.1.3. Let $I=\{0, a\}$, $J=\{0, b\}$ and $K=\{0, c\}$. Then I is an irreducible ideal, but not prime, since $b \land c = 0 \in I$, $b \not\in I$ and $c \not\in I$.

Similarly J and K are irreducible ideals which are not prime.

The following is a straightforward verification, since every ideal in a finite lattice is a principal ideal; in fact, if I is a finite ideal and $I=\{a_1, a_2, \ldots, a_n\}$, then $I=(a)$, where $a = a_1 \lor a_2 \lor \ldots \lor a_n$.

2.2.12 Theorem

Let $X = (X, \land, \lor)$ be a finite lattice and $p \in X$. Then the following hold.

1. $p$ is meet irreducible in X if and only if $(p)$ is an irreducible ideal of X.

2. $p$ is meet prime in X if and only if $(p)$ is a prime ideal of X.
Note that any prime ideal of a lattice $X$ need not be of the form $(p)$ for a meet prime element $p$ in $X$. For example, in the example given in 2.2.7, for any $0 < a \leq 1$, $[0, a)$ is a prime ideal of $[0,1]$ and is not of the form $(p)=[0,p)$. However, in any finite lattice $X$, any prime (irreducible) ideal of $X$ is of the form $(p)$ for some meet prime (irreducible) element $p$ in $X$, since every ideal of a finite lattice is principal.

Since an element in a distributive lattice is meet prime if and only if it is meet irreducible and since a lattice is distributive if and only if its ideal lattice is distributive, we have the following.

**2.2.13 Theorem**

Let $P$ be an ideal of a bounded distributive lattice. Then $P$ is a prime ideal if and only if $P$ is an irreducible ideal.

The following is a celebrated theorem of M.H. Stone and is an important tool in proving several structure theorems on distributive lattices.

**2.2.14 Theorem (M.H. Stone[24])**

Let $X = (X, \land, \lor)$ be a distributive lattice and $a$ and $b$ elements in $X$ such that $a \leq b$. Then there exists a prime ideal $P$ of $X$ such that $b \in P$ and $a \not\in P$.

The above theorem can be deduced from the following, which is a characterization of distributive lattices.

**2.2.15 Theorem**

The following are equivalent to each other for any lattice $X = (X, \land, \lor)$

1. $X$ is a distributive lattice
2. Every ideal $I$ of $X$ is the intersection of all prime ideals of $X$ containing $I$.
3. For any distinct elements $a$ and $b$ in $X$, there exists a prime ideal $P$ of $X$ containing one of $a$ and $b$ and not containing the other.
Proof: (1) $\Rightarrow$ (2):

Let $I$ be an ideal of $X$ and $I_0 = \cap \{ P : P$ is a prime ideal of $X$ and $I \subseteq P \}$.

Clearly $I \subseteq I_0$. On the other hand suppose $a$ is an element of $X$ such that $a \not\in I$. Let

$$P = \{ J : J$ is an ideal of $X$, $I \subseteq J$ and $a \not\in J \}.$$

Then $(P, \subseteq)$ is a partially ordered set (note that $I \in P$ and hence $P$ is a non-empty class of sets). We shall verify that the Zorn’s hypothesis is satisfied in $P$. Let

$$\{ J_\alpha \}_{\alpha \in \Lambda}$$

be a chain in $P$. Then $J = \bigcup_{\alpha \in \Lambda} J_\alpha$ is an ideal of $X$, $I \subseteq J$ and $a \not\in J$ and hence $J$ is an upper bound of $\{ J_\alpha \}_{\alpha \in \Lambda}$ in $P$. Therefore every chain in $P$ has an upper bound in $P$. By the Zorn’s lemma, $P$ has a maximal member, say $P$. Then $P$ is a proper ideal of $X$ (since $a \not\in P$). Also, if $b$ and $c$ are any elements in $X$ such that $b \not\in P$ and $c \not\in P$, then, by the maximality of $P$,

$$a \in P \lor (b] \text{ and } a \in P \lor (c]$$

and hence $a \in (P \lor (b]) \cap (P \lor (c]) = P \lor ((b] \cap (c])$

$$= P \lor (b \land c],$$

which implies that $b \land c \not\in P$. Thus $P$ is a prime ideal of $X$ and $I \subseteq P$ and hence $I_0 \subseteq P$.

But $a \not\in P$ and hence $a \not\in I_0$. Thus $I = I_0$

(2) $\Rightarrow$ (3) : Let $a \neq b \in X$. Then $a \not\in b$ or $b \not\in a$. We can assume, without loss of generality, that $a \not\in b$.

Then $a \not\in (b]$ and hence, by (2), there exists a prime ideal $P$ containing $(b]$ and not containing $a$.

Therefore $b \in P$ and $a \not\in P$.

(3) $\Rightarrow$ (1) : Let $a, b$ and $c \in X$. Clearly
\[(a \land b) \lor (a \land c) \leq a \land (b \lor c).\]

If \(a \land (b \lor c) \preceq (a \land b) \lor (a \land c),\) then, by (3), there exists a prime ideal \(P\) in \(X\) such that
\[(a \land b) \lor (a \land c) \in P\] and \(a \land (b \lor c) \notin P.\]

Then \(a \land b\) and \(a \land c \in P\) and \(a \notin P.\) This implies that \(b\) and \(c \in P\) (since \(P\) is a prime ideal) and hence \(b \lor c \in P.\) This leads to a contradiction, since \(a \land (b \lor c) \notin P.\) Thus, it is necessary that \(a \land (b \lor c) \leq (a \land b) \lor (a \land c)\) and hence \(a \land (b \lor c) = (a \land b) \lor (a \land c).\)

Thus \(X\) is a distributive lattice.

The above theorem 2.2.15 can be extended further. First, let us have the following.

**2.2.16 Definition**

For any ideals \(I\) and \(J\) of a lattice \(X = (X, \land, \lor),\) we define the set
\[(I : J) = \{x \in X : x \land a \in J \text{ for all } a \in I\}\]

\((I : J)\) is called the *relative annihilator of \(I\) corresponding to \(J.\)* For any elements \(a\) and \(b\) in \(X,\) let
\[(a : b) = \{x \in X : x \land a \leq b\}\]

Note that \((a : b) = (a] : (b)]\), where \((a]\) and \((b]\) are the principal ideals generated by \(a\) and \(b\) respectively.

**2.2.17 Theorem**

The following are equivalent to each other for any lattice \(X = (X, \land, \lor).\)

1. For any ideals \(I\) and \(J\) of \(X,\)
\[(I : J) = \cap \{P : P \text{ is a prime ideal of } X, J \subseteq P \text{ and } I \notin P\}\]

2. For any elements \(a\) and \(b\) of \(X,\)
\[(a : b) = \cap \{P : P \text{ is a prime ideal of } X, b \in P \text{ and } a \notin P\}\]
(3) \((a : b)\) is an ideal of \(X\) for all elements \(a\) and \(b\) in \(X\).

(4) \(X\) is a distributive lattice.

**Proof:** (1) \(\Rightarrow\) (2): is clear, if we take \(I = (a)\) and \(J = (b)\).

(2) \(\Rightarrow\) (3): is also clear, since the intersection of any class of ideals is an ideal, provided it is non empty. Note that \(b \in (a : b)\) for any \(a\) and \(b\).

(3) \(\Rightarrow\) (4): Let \(x, y\) and \(z \in X\). Put \(b = (x \wedge y) \vee (x \wedge z)\).

Then \((x \wedge y) \leq b\) and \((x \wedge z) \leq b\) and hence
\[ y \text{ and } z \in (x : b). \]

Since \((x : b)\) is an ideal of \(X\) (by (3)), we get that
\[ y \vee z \in (x : b) \]

and hence \(x \wedge (y \vee z) \leq b = (x \wedge y) \vee (x \wedge z)\). It is always true that
\( (x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)\). Therefore
\[ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \]

Thus \(X\) is a distributive lattice.

(4) \(\Rightarrow\) (1): Recall that the lattice \(I(X)\) of ideal of \(X\) is distributive if and only if \(X\) is distributive. Let \(I\) and \(J\) be any ideals of \(X\) and
\[ \bigcap \{ P : P \text{ is a prime ideal of } X, J \subseteq P, I \not\subseteq P \} = K. \]

Then, for any prime ideal \(P\) of \(X\) such that \(J \subseteq P\) and \(I \not\subseteq P\), there exists \(a \in I\) such that \(a \notin P\) and hence
\[ x \in (I : J) \Rightarrow x \wedge a \in P \Rightarrow x \in P. \]

Therefore \((I : J) \subseteq K\). On the other hand, suppose that \(x \notin (I : J)\). Then there exists \(a \in I\) such that \(x \wedge a \notin J\). Consider
\[ P = \{ R : R \text{ is an ideal of } X, J \subseteq R \text{ and } x \wedge a \notin R \}. \]
Since $J \in \mathcal{P}$, $\mathcal{P}$ is a non-empty class of sets, which is closed under unions of chains. Therefore, by the Zorn’s lemma, $\mathcal{P}$ has a maximal member, say $M$. Then $M$ is a proper (since $x \land a \not\in M$) ideal of $X$. Also, for any $b$ and $c \in X$,
\[
b \not\in M \text{ and } c \not\in M \implies x \land a \in M \lor (b) \text{ and } x \land a \in M \lor (c)\]
\[
\implies x \land a \in (M \lor (b)) \cap (M \lor (c))\]
\[
\implies x \land a \in M \lor (b \land c)\]
\[
\implies b \land c \not\in M.
\]
Therefore $M$ is a prime ideal of $X$. Also $J \subseteq M$ and $I \not\subseteq M$ (since $a \in I$ and $a \not\in M$).

Further $x \not\in M$ (since $x \land a \not\in M$). All these imply that $x \not\in K$. Thus
\[
(I : J) = K.
\]

2.2.18 Definition

For any ideal $I$ of a bounded lattice $X$, let us define the annihilator of $I$ as the set
\[
I^* = \{ I : \{0\} \} = \{ x \in X : x \land a = 0 \text{ for all } a \in I \}
\]

2.2.19 Corollary

Let $X = (X, \land, \lor)$ be a bounded distributive lattice and $I$ an ideal of $X$. Then
\[
I^* = \bigcap \{ P : P \text{ is a prime ideal of } X \text{ and } I \not\subseteq P \}
\]

Proof: We get this by taking $J = \{0\}$ in 2.2.17 (1).

2.2.20 Corollary

In any distributive lattice $X$, every ideal $J$ of $X$ is the intersection of all prime ideals of $X$ containing $J$.

Proof: We can take $I = X$ in 2.2.17 (1). Note that any prime ideal is a proper ideal.

Next we have a brief discussion on maximal ideals of lattices. First, we recall the following.
2.2.21 Definition

Let \( X = (X, \wedge, \vee) \) be a lattice. A proper ideal which is maximal among proper ideals of \( X \) is called a \textit{maximal ideal} of \( X \); that is, a proper ideal \( M \) of \( X \) is called a maximal ideal of \( X \) if, for any ideal \( I \) of \( X \), \( M \subseteq I \) implies that either \( M = I \) or \( I = X \).

Note that the existence of maximal ideals in a lattice \( X \) (even when \( X \) is distributive) is not guaranteed. Consider the following example.

2.2.22 Example

Let \( X \) be a totally ordered set without largest element; for example \( \mathbb{R} \) or an open interval with respect to the usual ordering of real numbers. If \( M \) is a proper ideal of \( X \), then we can choose \( a \in X - M \) and another element \( b \in X \) such that \( a < b \). Then, it can be easily checked that \( \{a\} \) is a proper ideal of \( X \) containing \( M \) properly; that is,

\[
M \subset \{a\} \subset X
\]

and hence \( M \) is not a maximal ideal of \( X \). Thus \( X \) has no maximal ideals.

However, if the lattice \( X \) has the largest element, then, using Zorn’s lemma, we can prove the existence of maximal ideals in \( X \). If \( X \) is non-trival, then the class of proper ideals of \( X \) satisfies the hypothesis of the Zorn’s lemma, since an ideal \( I \) of \( X \) is proper if and only if the largest element 1 is not in \( I \).

A maximal ideal of a lattice need not be a prime ideal, for, consider the following.

2.2.23 Example

Consider the lattice \( M_5 = \{0, a, b, c, 1\} \) discussed in example 2.1.3. Then \( M = \{0, a\} \) is a maximal ideal, since \( a \lor x = 1 \) for any \( x \notin M \). But \( M \) is not a prime ideal, since

\[
b \land c = 0 \in M \quad \text{and} \quad b \notin M \quad \text{and} \quad c \notin M.
\]
However, if the lattice $X$ is distributive, then every maximal ideal of $X$ is a prime ideal.

2.2.24 Theorem

Let $X = (X, \wedge, \vee)$ be a distributive lattice and $M$ a maximal ideal of $X$. Then $M$ is a prime ideal.

**Proof:** Since $M$ is maximal, it is a proper ideal. Let $a$ and $b \in X$ such that $a \wedge b \in M$ and $a \notin M$. Then, by the maximality of $M$, $M \vee (a) = X$ and hence $b \in M \vee (a)$ which implies that $b \leq m \vee a$ for some $m \in M$. Now,

$$b = b \wedge (m \vee a) = (b \wedge m) \vee (b \wedge a) \in M,$$

since $b \wedge m \in M$ and $b \wedge a \in M$. Thus $M$ is a prime ideal of $X$.

The converse of the above theorem is not true. For, consider the real number system $\mathbb{R}$ together with the usual ordering. Then any proper ideal of $\mathbb{R}$ is a prime ideal. However, $\mathbb{R}$ has no maximal ideals at all. Even when $X$ is a bounded distributive lattice, a prime ideal of $X$ need not be maximal. In $[0, 1]$ every proper ideal is prime and $[0, 1)$ is the only maximal ideal in $[0, 1]$.

It is known that a bounded distributive lattice is complemented (and hence a Boolean algebra) if and only if every prime ideal of it is maximal. The proof of this is included in the next result. Before this, let us recall that a bounded lattice $X$ is called complemented if, for each $a \in X$, there exists $b \in X$ such that $a \wedge b = 0$ and $a \vee b = 1$.

A complemented distributive lattice is called a Boolean algebra.

2.2.25 Theorem

Let $X = (X, \wedge, \vee)$ be a nontrivial bounded distributive lattice. Then the following are equivalent to each other.

(1) $X$ is a Boolean algebra
(2) Every prime ideal of $X$ is maximal

(3) Every prime ideal of $X$ is a minimal prime ideal

(4) For any prime ideal $P$ of $X$ and $a \in P$, there exists $b \in X - P$ such that $a \land b = 0$.

**Proof:** (1) $\Rightarrow$ (2): Suppose that $X$ is a Boolean algebra. Let $P$ be a prime ideal of $X$ and $I$ be any ideal of $X$ properly containing $P$. Choose $a \in I$ such that $a \notin P$. Let $b \in X$ be such that $a \land b = 0$ and $a \lor b = 1$. Since $a \land b = 0$ and $a \notin P$, it follows from the primeness of $P$ that $b \in P \subseteq I$ and hence

$$1 = a \lor b \in I.$$ 

Therefore $I = X$. Thus $P$ is a maximal ideal.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4): Let $P$ be a prime ideal of $X$ and $a \in P$. Let

$$A = \{x \in X : y \land a \leq x \text{ for some } y \in X - P\}.$$ 

We prove that $0 \in A$ and hence $y \land a = 0$ for some $y \in X - P$. Suppose, if possible that $0 \notin A$. Consider the class

$$\mathcal{P} = \{I : I \text{ is an ideal of } X \text{ and } I \cap A = \emptyset\}.$$ 

Since $0 \notin A, \{0\} \in \mathcal{P}$ and hence $\mathcal{P}$ is a nonempty class of ideals and is closed under unions of chains. By the Zorn’s lemma, $\mathcal{P}$ has a maximal member, say $M$. We prove that $M$ is a prime ideal. Since $M \cap A = \emptyset$ and $a \in A$, we get that $a \notin M$ and hence $M$ is a proper ideal of $X$. Let $b$ and $c \in X$ such that $b \notin M$ and $c \notin M$. Then, by the maximality of $M$, $M \lor (b)$ and $M \lor (c)$ are not members in $\mathcal{P}$ and hence

$$(M \lor (b)) \cap A \neq \emptyset \text{ and } (M \lor (c)) \cap A \neq \emptyset.$$ 

Choose $x \in (M \lor (b)) \cap A$ and $x' \in (M \lor (c)) \cap A$. Then

$$y \land a \leq x \leq m \lor b \text{ for some } y \in X - P \text{ and } m \in M$$ 

and $y' \land a \leq x' \leq m' \lor c$ for some $y' \in X - P$ and $m' \in M$. 

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Now, \( y \wedge y' \wedge a \leq x \wedge x' \leq (m \vee b) \wedge (m' \vee c) = m'' \wedge (b \wedge c) \) for some \( m'' \in M \)
(by using the distributivity and the fact that \( M \) is an ideal). Also, note that
\( y \wedge y' \in X - P \), since \( P \) is a prime ideal. If \( b \wedge c \in M \), then \( x \wedge x' \in M \cap A \), which is a contradiction, since \( M \cap A = \phi \). Therefore \( b \wedge c \not\in M \). Thus \( M \) is a prime ideal. Also, clearly \( X - P \subseteq A \) and \( a \in A \) and hence, from the disjointness of \( M \) and \( A \),
\[ M \cap (X - P) = \phi = M \cap \{a\}. \]
This implies that \( M \) is a prime ideal properly contained in \( P \) (since \( a \in P, a \not\in M \) and \( M \subseteq P \) ), which is a contradiction to (3). Thus \( 0 \in A \) and hence \( y \wedge a = 0 \) for some \( y \in X - P \).

(4) \( \Rightarrow \) (1): Let \( a \in X \). We have to prove that there exists \( b \in X \) such that \( a \wedge b = 0 \) and \( a \vee b = 1 \). We can assume that \( a \neq 0 \) and \( a \neq 1 \). Let
\[ I = \{x \in X : x \leq a \vee b \text{ for some } b \in X \text{ with } a \wedge b = 0\} \]
Then \( I \) is an ideal of \( X \). We prove that \( I = X \).

Suppose, if possible, that \( I \) is a proper ideal of \( X \). By Zorn’s lemma, we can get a maximal ideal (and hence a prime ideal) \( P \) such that \( I \subseteq P \). Now, \( a \in I \) and hence \( a \in P \). By (4), there exists \( b \in X - P \) such that \( a \wedge b = 0 \) which is a contradiction since \( b \in I \subseteq P \). Therefore, \( I = X \) and, in particular, \( 1 \in I \) so that \( a \vee b = 1 \) for some \( b \in X \) with \( a \wedge b = 0 \). Thus \( X \) is a complemented lattice and hence a Boolean algebra.

### 2.3 PRIME L – FUZZY IDEALS

The concept of prime fuzzy ideal of a lattice was first introduced by U.M. Swamy and D V. Raju [28] and later B.B.N Koguep, C.N.Kuimi and C.Lele [14] discussed certain properties of prime fuzzy ideals of lattices, when the truth values are
taken from the interval [0,1]. In this section, we extend these results to the case when
the truth values are taken from a general frame L and obtain certain comprehensive
results on these. We concentrate on prime fuzzy ideals of distributive lattices, even
though some of the results are true for general lattices. As such, throughout the
remaining part of this thesis, X always denotes a nontrivial bounded distributive lattice
\((X, \land, \lor)\), unless otherwise stated. As usual \(L = (L, \land, \lor)\) denotes a frame; that is, L is a
complete lattice satisfying the infinite meet distributivity.

Let us recall from 1.3.2 that an \(L\)-fuzzy subset \(A\) of a lattice \(X\) is called an
\(L\)-fuzzy ideal of \(X\) if \(A(0) = 1\) and
\[ A(x \lor y) = A(x) \land A(y) \quad \text{for all } x, y \in X \]
and that an \(L\)-fuzzy ideal is always an antitone.

2.3.1 Definition
An \(L\)-fuzzy ideal \(A\) of a lattice \(X\) is called proper if \(A\) is not the constant map \(1\), that
is, \(A(x) \neq 1\) for some \(x \in X\).

2.3.2 Definition
Let \(X = (X, \land, \lor)\) be a bounded distributive lattice and \(L\) a frame. An \(L\)-fuzzy ideal \(P\)
of \(X\) is called a prime \(L\)-fuzzy ideal if \(P\) is proper and, for any \(L\)-fuzzy ideals \(A\) and
\(B\) of \(X\),
\[ A \land B \leq P \Rightarrow A \leq P \text{ or } B \leq P. \]

Note that the set \(\mathcal{F}_L(X)\) of all \(L\)-fuzzy ideals of \(X\) forms a complete
distributive lattice under the point-wise ordering (refer 1.3.7 and 1.3.27) and that the
prime \(L\)-fuzzy ideals of \(X\) are precisely the meet prime elements of \(\mathcal{F}_L(X)\). Also,
since \(\mathcal{F}_L(X)\) is a distributive lattice, an element of \(\mathcal{F}_L(X)\) is meet prime if and only
if it is meet irreducible (refer 2.1.4) and hence we have the following.
2.3.3 Theorem
Let \( X \) be a bounded distributive lattice and \( L \) a frame. Then an \( L \)-fuzzy ideal \( P \) of \( X \) is a prime \( L \)-fuzzy ideal if and only if, for any \( L \)-fuzzy ideals \( A \) and \( B \) of \( X \),

\[ P = A \land B \Rightarrow P = A \text{ or } P = B. \]

In the following three results we characterise prime \( L \)-fuzzy ideals by establishing a one-to-one correspondence between them and the pairs \((I, \alpha)\) where \( I \) is prime ideal of \( X \) and \( \alpha \) is an irreducible (prime) element of the frame \( L \).

First, recall that for any (crisp) ideal \( I \) of \( X \) and \( \alpha \in L \), we have defined an \( L \)-fuzzy ideal \( I_\alpha \) of \( X \) by

\[ I_\alpha(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases} \]

and that \( I_\alpha \) is called the \( \alpha \)-level fuzzy ideal corresponding to \( I \) (refer 1.3.17 and 1.3.18).

2.3.4 Theorem
Let \( X = (X, \land, \lor) \) be a bounded distributive lattice and \( L \) a frame. Suppose that \( I \) is an ideal of \( X \) and \( \alpha \in L \). Then the \( \alpha \)-level fuzzy ideal \( I_\alpha \) is a prime \( L \)-fuzzy ideal of \( X \) if and only if \( I \) is a prime ideal of \( X \) and \( \alpha \) is a prime element in \( L \).

**Proof:** Recall that \( \alpha_i : X \to L \) is defined by

\[ \alpha_i(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases} \]

First note that \( \alpha_i \) is a proper \( L \)-fuzzy ideal if and only if \( I \) is a proper ideal of \( L \) and \( \alpha \neq 1 \) in \( L \). Therefore we can assume that \( I \) is a proper ideal of \( X \) and \( \alpha \neq 1 \) in \( L \). Then \( \alpha_i \) is proper. Now, suppose that \( \alpha_i \) is a prime \( L \)-fuzzy ideal of \( X \). If \( J \) and \( K \) are ideals of \( X \) such that \( J \cap K \subseteq I \), then by theorem 1.3.19,
\[ \alpha_j \wedge \alpha_K = \alpha_{j \wedge K} \leq \alpha_1 \]

and hence \( \alpha_j \leq \alpha_1 \) or \( \alpha_K \leq \alpha_1 \), so that \( J \subseteq I \) or \( K \subseteq I \). Therefore I is a prime ideal of \( X \). Also, for any \( \beta \) and \( \gamma \in L \),

\[ \beta \wedge \gamma \leq \alpha \Rightarrow (\beta \wedge \gamma)_I \leq \alpha_1 \]

\[ \Rightarrow \beta_1 \wedge \gamma_1 \leq \alpha_1 \]

\[ \Rightarrow \beta_1 \leq \gamma_1 \text{ or } \gamma_1 \leq \alpha_1 \]

\[ \Rightarrow \beta \leq \alpha \text{ or } \gamma \leq \alpha . \]

Therefore \( \alpha \) is a prime element in \( L \).

Conversely, suppose that I is a prime ideal of \( X \) and \( \alpha \) is a prime element in \( L \). Then clearly \( \alpha_1 \) is a proper \( L \) – fuzzy ideal of \( X \). Let A and B be \( L \) – fuzzy ideals of \( X \) such that \( A \nleq \alpha_1 \) and \( B \nleq \alpha_1 \). Then, there exist \( x \) and \( y \in X \) such that

\[ A(x) \nleq \alpha_1 \text{ (} x \text{) and } B(y) \nleq \alpha_1 \text{ (} y \text{).} \]

Now \( x \nleq I \) (otherwise, \( \alpha_j(x) = 1 \geq A(x) \)) and similarly \( y \nleq I \) and since I is a prime ideal \( x \wedge y \nleq I \). Also, since \( \alpha \) is a prime element in \( L \) and \( A(x) \nleq \alpha_1(x) = \alpha \) and \( B(y) \nleq \alpha_1(y) = \alpha \), it follows that

\[ A(x) \wedge B(y) \nleq \alpha \]

Now, since A and B are antitones, we have

\[ (A \wedge B)(x \wedge y) = A(x \wedge y) \wedge B(x \wedge y) \]

\[ \geq A(x) \wedge B(y) \]

and therefore \( \alpha \nleq (A \wedge B)(x \wedge y) \). Hence

\[ (A \wedge B)(x \wedge y) \nleq \alpha_1(x \wedge y) (= \alpha, \text{ since } x \wedge y \nleq I) . \]

This implies that \( A \wedge B \nleq \alpha_1 \). Thus \( \alpha_1 \) is a prime \( L \) – fuzzy ideal of \( X \).
2.3.5 Theorem

Let \(X = (X, \land, \lor)\) be a bounded distributive lattice and \(L\) a frame. A proper \(L\)-fuzzy ideal \(P\) of \(X\) is prime if and only if the following are satisfied.

1. \(P\) assumes exactly two values
2. \(P(1)\) is a prime element in \(L\)
3. \(\{ x \in X : P(x) = 1 \}\) is a prime ideal of \(X\)

Proof: Suppose that \(P\) is a prime \(L\)-fuzzy ideal of \(X\)

(1): First we observe that \(P\) assumes at least two values; for otherwise \(P\) is not a proper \(L\)-fuzzy ideal; note that \(P(0) = 1\). Now, suppose that \(P\) assumes more than two values. Then there exist \(x\) and \(y \in X\) such that \(P(x), P(y)\) and \(P(0) = 1\) are all distinct.

Put \(P(x) = \alpha, P(y) = \beta\).

Define \(L\)-fuzzy subsets \(A\) and \(B\) of \(X\) as follows.

\[
A(z) = \begin{cases} 
1 & \text{if } z \leq x \\
0 & \text{if } z \not\leq x 
\end{cases}
\]

and \(B(z) = \begin{cases} 
1 & \text{if } z = 0 \\
\alpha & \text{if } z \neq 0 
\end{cases}\)

Then \(A = 0_{\{x\}}\) and \(B = \alpha_{\{0\}}\) and hence, but 1.3.18, \(A\) and \(B\) are \(L\)-fuzzy ideals of \(X\).

Also,

\((A \land B)(z) \leq P(z)\) for all \(z \in X\);

for, \(z = 0 \Rightarrow (A \land B)(z) = 1 = P(z)\)

\(0 \not\leq z \leq x \Rightarrow A(z) \land B(z) = 1 \land \alpha = P(x) \leq P(z)\)

and \(z \not\leq x \Rightarrow A(z) \land B(z) = 0 \land \alpha = 0 \leq P(z)\).

Therefore \(A \land B \leq P\). Since \(P\) is prime, \(A \leq P\) or \(B \leq P\).
But $A \not\leq P$, since $A(x) = 1, P(x) = \alpha$ and $1 \not\leq \alpha$. Therefore $B \leq P$; in particular,

$B(y) \leq P(y) = \beta$. Since $P(y) \neq P(0)$, we get that $y \neq 0$ and hence $B(y) = \alpha$.

Therefore $\alpha \leq \beta$. Similarly, we can prove that $\beta \leq \alpha$. Thus $\alpha = \beta$ and hence $P(x) = P(y)$, a contradiction to our assumption. Thus $P$ assumes exactly two values.

(2) Since $P$ is proper, $P(1) \neq P(0) = 1$. Let $\alpha$ and $\beta \in L$ such that $\alpha \wedge \beta \leq P(1)$.

Define $A$ and $B$ as follows.

$$A(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } x \neq 0 \end{cases}$$

and

$$B(x) = \begin{cases} 1 & \text{if } x = 0 \\ \beta & \text{if } x \neq 0 \end{cases}$$

Then, it can be easily checked that $A$ and $B$ are $L$–fuzzy ideals of $X$ and $A \wedge B \leq P$.

Since $P$ is prime, we get that either $A \leq P$ or $B \leq P$ and hence $A(1) \leq P(1)$ or $B(1) \leq P(1)$.

Therefore $\alpha \leq P(1)$ or $\beta \leq P(1)$. Thus $P(1)$ is a prime element in $L$.

(3) Let $I = \{x \in X : P(x) = 1\}$. Since $P$ is proper, we get that $I$ is a proper ideal of $X$.

Let $\alpha$ be the other value of $P$.

Then

$$P(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases}$$

and hence $P = \alpha_1$. By theorem 2.3.4, we get that $I$ is a prime ideal of $X$. Conversely suppose that $P$ is an $L$–fuzzy ideal of $X$ satisfying the properties (1), (2) and (3).

Since $P(0) = 1$ and $P$ assumes exactly two values, there exists $\alpha \neq 1$, such that, for each $x \in X, P(x) = 1$ or $\alpha$. Then $P(1) = \alpha$. By (2), $\alpha$ is a prime element in $L$. Let
\[ I = \{ x \in X : P(x) = 1 \}. \]

By (3), I is a prime ideal of X. Now, observe that

\[ P = \alpha_i \]

and hence, again by theorem 2.3.4, P is a prime \( L \)–fuzzy ideal of X.

**2.3.6 Theorem**

An \( L \)–fuzzy ideal \( P \) of a bounded distributive lattice \( X \) is a prime \( L \)–fuzzy ideal if and only if \( P = \alpha_i \) for some prime ideal \( I \) of \( X \) and a prime element \( \alpha \) in \( L \).

**Proof:** Let \( P \) be an \( L \)–fuzzy ideal of \( X \). Suppose that \( P \) is a prime \( L \)–fuzzy ideal of \( X \). Then, by theorem 2.3.5, \( P \) assumes exactly two values, Since \( P(0) = 1 \), let \( \alpha \) be the only value of \( P \) other than 1. Then \( P(1) \neq 1 \) (otherwise \( P(1) = 1 \) implies that \( P(x) \geq P(1) = 1 \) and hence \( P(x) = 1 \) for all \( x \in X \), so that \( P \) is not proper) and therefore \( P(1) = \alpha \). Now, by (2) and (3) of theorem 2.3.5, \( \alpha \) is a prime element in \( L \) and \( I = \{ x \in X : P(x) = 1 \} \) is a prime ideal of \( X \). Therefore

\[ P(x) \neq 1 \iff P(x) = \alpha \text{ for any } x \in X \]

and hence \( P = \alpha_i \). Conserve follows from theorem 2.3.4.

**2.3.7 Theorem**

Let \( I \) and \( J \) be proper ideals of \( X \) and \( \alpha \) and \( \beta \in L \) such that \( \alpha \neq 1 \) and \( \beta \neq 1 \). Then

\[ \alpha_i \leq \beta_j \iff I \subset J \text{ and } \alpha \leq \beta \]

**Proof:** suppose that \( \alpha_i \leq \beta_j \). Then

\[ x \in I \implies 1 = \alpha_i(x) \leq \beta_j(x) \]

\[ \implies \beta_j(x) = 1 \]

\[ \implies x \in J \]

and hence \( I \subset J \). If \( x \) is any element of \( X \) not in \( J \) (since \( J \) is proper, there is one such),
then \( x \not\in I \) and
\[
\alpha = \alpha_I(x) \leq \beta_J(x) = \beta
\]
Thus \( I \subseteq J \) and \( \alpha \leq \beta \). Conversely suppose that \( I \subseteq J \) and \( \alpha \leq \beta \). For any \( x \in X \),
\[
x \not\in J \Rightarrow x \not\in I \quad \text{and} \quad x \not\in J
\]
\[
\Rightarrow \alpha_I(x) = \alpha \leq \beta = \beta_J(x)
\]
and \( x \in J \Rightarrow \alpha_I(x) \leq 1 = \beta_J(x) \)
Thus \( \alpha_I \leq \beta_J \).

2.3.8 Definition
For any bounded distributive lattice \( X \) and a frame \( L \), let
\[
P \upharpoonright (X) = \text{The set of all prime ideals of } X,
\]
\[
P E(L) = \text{The set of all prime elements of } L
\]
and \( F P \upharpoonright \mathcal{L}(X) = \text{The set of all } L \text{-fuzzy prime ideals of } X. \)

2.3.9 Corollary
Let \( X = (X, \wedge, \vee) \) be a bounded distributive lattice and \( L \) a frame. Then \( (I, \alpha) \mapsto \alpha_I \) is a one-to-one correspondence between \( P \upharpoonright (X) \times P E(L) \) onto \( F P \upharpoonright \mathcal{L}(X) \).

2.3.10 Example
If \( L = [0, 1] \), the closed unit interval in \( \mathbb{R} \), then any \( \alpha < 1 \) in \( L \) is a prime element and hence \( P \) is a prime \( L \)-fuzzy ideal of a lattice \( X \) if and only if there exists unique prime ideal \( I \) of \( X \) and an element \( \alpha < 1 \) in \( L \) such that \( P = \alpha_I \).
2.4 L-FUZZY PRIME IDEALS

We have proved in Theorem 1.3.4 that an L-fuzzy subset A of a lattice X is an L-fuzzy ideal of X if and only if the \( \alpha \)– cut

\[
A_{\alpha} = \{ x \in X : A(x) \geq \alpha \}
\]

is a crisp ideal of X for all \( \alpha \in L \). In the following we characterize L-fuzzy ideal A for which each \( \alpha \)– cut \( A_{\alpha} \) is either a prime ideal of X or the whole lattice X.

2.4.1 Theorem

Let \( X = (X, \wedge, \vee) \) be a bounded distributive lattice and \( L \) a frame. Then the following are equivalent to each other for any L-fuzzy ideal A of X.

1. For each \( \alpha \in L \), \( A_{\alpha} = X \) or \( A_{\alpha} \) is a prime ideal of X.
2. For any \( x \) and \( y \in X \), \( A(x \wedge y) = A(x) \) or \( A(y) \)
3. For any \( x \) and \( y \in X \), \( A(x \wedge y) = A(x) \vee A(y) \) and either \( A(x) \leq A(y) \) or \( A(y) \leq A(x) \).

Proof: First note that (1) is equivalent to the following.

For any \( x \) and \( y \in X \) and \( \alpha \in L \),

\[
x \wedge y \in A_{\alpha} \Rightarrow x \in A_{\alpha} \text{ or } y \in A_{\alpha}
\]

(1) \( \Rightarrow \) (2): Let \( x \) and \( y \in X \) and \( \alpha = A(x \wedge y) \). Then

\[
x \wedge y \in A_{\alpha} \text{ and hence } x \in A_{\alpha} \text{ or } y \in A_{\alpha}
\]

If \( x \in A_{\alpha} \), then \( A(x \wedge y) = \alpha \leq A(x) \leq A(x \wedge y) \)

(Since A is an antitone ) and hence \( A(x \wedge y) = A(x) \).

If \( y \in A_{\alpha} \), then \( A(x \wedge y) = \alpha \leq A(y) \leq A(x \wedge y) \) and hence \( A(x \wedge y) = A(y) \).

(2) \( \Rightarrow \) (3): Let \( x \) and \( y \in X \). Then, by (2), \( A(x \wedge y) = A(x) \) or \( A(y) \)

\[
A(x \wedge y) = A(x) \Rightarrow A(y) \leq A(x \wedge y) = A(x)
\]

\[
A(x \wedge y) = A(y) \Rightarrow A(x) \leq A(x \wedge y) = A(y).
\]

Therefore \( A(y) \leq A(x) \) or \( A(x) \leq A(y) \). Also, by (2),
\[ A(x \land y) = A(x) \text{ or } A(y) \]

and, since \( A(x) \) and \( A(y) \leq A(x \land y) \), if follows that

\[
A(x \land y) = \max \{ A(x), A(y) \} = A(x) \lor A(y).
\]

(3) \( \Rightarrow \) (1): Since \( A \) is an L-fuzzy ideal of \( X \), \( A_\alpha \) is an ideal of \( X \) for any \( \alpha \in L \).

Let \( \alpha \in L \) be fixed. Suppose that \( A_\alpha \neq X \); that is, \( A_\alpha \) is a proper ideal of \( X \). Also,

\[
x \land y \in A_\alpha \Rightarrow \alpha \leq A(x \land y) = A(x) \lor A(y) = A(x) \text{ or } A(y)
\]

\[
\Rightarrow \alpha \leq A(x) \text{ or } \alpha \leq A(y)
\]

\[
\Rightarrow x \in A_\alpha \text{ or } y \in A_\alpha
\]

Thus \( A_\alpha \) is a prime ideal of \( X \).

**2.4.2 Definition**

A proper L-fuzzy ideal \( A \) of \( X \) is called an *L-fuzzy prime ideal* of \( X \) if \( A \) satisfies one (and hence all) of the three properties (1), (2) and (3) given in the above theorem 2.4.1

That is, a proper L-fuzzy ideal \( A \) of \( X \) is an L-fuzzy prime ideal of \( X \) if, for any \( x \) and \( y \in X \),

\[
A(x \land y) = A(x) \text{ or } A(y),
\]

or equivalently, \( A(x \land y) = A(x) \lor A(y) \) and either \( A(x) \leq A(y) \) or \( A(y) \leq A(x) \). Consider the examples given below.

**2.4.3 Examples**

Let \( X \) be the 4-element Boolean algebra \{ 0, a, b, 1 \} represented by the Hasse diagram given below.

![Hasse diagram](image)
Here \( a \) and \( b \) are complements to each other and 0 and 1 are respectively the smallest and greatest elements in \( X \). Let \( L \) be the closed unit interval \([0, 1]\) in the real number system. \( L \) is a frame in which, for any \( \alpha \) and \( \beta \),

\[
\alpha \land \beta = \text{minimum} \{ \alpha, \beta \}
\]

and

\[
\alpha \lor \beta = \text{maximum} \{ \alpha, \beta \}.
\]

Let \( A \) and \( B \) be the \( L \)-fuzzy subsets of \( X \) defined by

\[
A(0) = 1; \quad A(a) = 0.5; \quad A(b) = 1 \quad \text{and} \quad A(1) = 0.5
\]

and

\[
B(0) = 1; \quad B(a) = 0; \quad B(b) = 0.2 \quad \text{and} \quad B(1) = 0.
\]

Then \( A \) and \( B \) are proper \( L \)-fuzzy ideals of \( X \), \( A \) is an \( L \)-fuzzy prime ideal of \( X \), while \( B \) is not an \( L \)-fuzzy prime ideal of \( X \) (Since \( B(a \land b) \) is not equal to \( \max \{ B(a), B(b) \} \)).

Let us recall from 1.3.17 that we have defined an \( L \)-fuzzy ideal \( \alpha_I \) of a lattice \( X \) by

\[
\alpha_I(x) = \begin{cases} 
1 & \text{if } x \in I \\
\alpha & \text{if } x \notin I
\end{cases}
\]

for any ideal \( I \) of \( X \) and an element \( \alpha \) in the frame \( L \). We slightly generalize this in the following.

**2.4.4 Definition**

Let \( X \) be a bounded lattice, \( I \) an ideal of \( X \) and \( \alpha \) and \( \beta \) elements of a frame \( L \). Then define an \( L \)-fuzzy subset \( (\alpha, \beta)_I \) of \( X \) by

\[
(\alpha, \beta)_I(x) = \begin{cases} 
1 & \text{if } x = 0 \\
\alpha & \text{if } 0 \neq x \in I \\
\beta & \text{if } x \notin I
\end{cases}
\]

Note that \( (1, \beta)_I = \beta_I \) (refer 1.3.17) and that \( (1, 0)_I = \chi_I \), the characteristic map corresponding to \( I \).

The following can be easily proved.
2.4.5 Theorem
For any $\alpha$ and $\beta \in L$ and for any proper ideal $I$ of $X$, $(\alpha, \beta)_I$ is an $L$-fuzzy ideal of $X$ if and only if $\beta \leq \alpha$ and, in this case, $(\alpha, \beta)_I$ is proper if and only if $\beta < 1$.

In the following we prove that the prime ideals of $X$ can be identified with fuzzy prime ideals of $X$.

2.4.6 Theorem
Let $X$ be a bounded distributive lattice and $L$ a frame. Let $I$ be a proper ideal of $X$.

Then the following are equivalent to each other.

1. $I$ is a prime ideal of $X$
2. $(1, \beta)_I$ is an $L$-fuzzy prime ideal of $X$ for each $\beta < 1$ in $L$.
3. The characteristic map $\chi_I$ is an $L$-fuzzy prime ideal of $X$.

Proof: (1) $\Rightarrow$ (2): Suppose that $I$ is a prime ideal of $X$ and let $\beta \in L$ be such that $\beta < 1$. Put $A = (1, \beta)_I$ ($= \beta_I$).

Let $x$ and $y$ be any elements of $X$, Then

$x \wedge y \in I \Rightarrow A(x \wedge y) = 1$ and $x \in I$ or $y \in I$

$\Rightarrow A(x \wedge y) = 1 = A(x)$ or $A(x \wedge y) = 1 = A(y)$

and $x \wedge y \notin I \Rightarrow x \notin I$ and $y \notin I$ and $A(x \wedge y) = \beta = A(x)$.

Therefore, by Theorem 2.4.1 and Definition 2.4.2, $A$ is an $L$-fuzzy prime ideal of $X$.

Note that, since $I$ is a proper ideal of $X$ and $\beta < 1$, $A$ is a proper $L$-fuzzy ideal of $X$.

(2) $\Rightarrow$ (3) is clear, Since $\chi_I = (1,0)_I$

(3) $\Rightarrow$ (1): Suppose that $\chi_I$ is an $L$-fuzzy prime ideal of $X$, we are already given that $I$ is a proper ideal of $X$. Let $x$ and $y$ be elements in $X$ such that $x \wedge y \in I$.

Then
2.4.7 Theorem

Suppose that I is a proper ideal of the lattice X and 0 be a prime element in X. Then I is a prime ideal of X if and only if \((\alpha, \beta)_I\) is an L-fuzzy prime ideal of X for all \(1 \neq \beta \leq \alpha\) in L.

**Proof**: Suppose that I is a prime ideal of X and \(1 \neq \beta \leq \alpha\) in L.

Let \(x\) and \(y\) \(\in\) X. Then

\[
x \land y = 0 \Rightarrow x = 0 \text{ or } y = 0
\]

\[
\Rightarrow (\alpha, \beta)_I (x \land y) = 1 = (\alpha, \beta)_I (x) \text{ or } (\alpha, \beta)_I (y)
\]

\[
0 \neq x \land y \in I \Rightarrow 0 \neq x \in I \text{ or } 0 \neq y \in I
\]

\[
\Rightarrow (\alpha, \beta)_I (x \land y) = \alpha = (\alpha, \beta)_I (x) \text{ or } (\alpha, \beta)_I (y)
\]

and \(x \land y \notin I \Rightarrow x \notin I \text{ and } y \notin I\)

\[
\Rightarrow (\alpha, \beta)_I (x \land y) = \beta = (\alpha, \beta)_I (x) = (\alpha, \beta)_I (y).
\]

Therefore, by 2.4.1, \((\alpha, \beta)_I\) is an L-fuzzy prime ideal of X.

Converse follows from Theorem 2.4.6.

The following explains an inter-relationship between the prime L-fuzzy ideals and L-fuzzy prime ideals of any lattice.

2.4.8 Theorem

Let X be a bounded distributive lattice and L a frame. Then every prime L-fuzzy ideal of X is an L-fuzzy prime ideal of X. The converse of this is not true.

**Proof**: Let A be a prime L-fuzzy ideal of X. Then by Theorem 2.3.6, there exists a prime ideal I of X and a prime element \(\alpha\) in L such that \(A = \alpha_I\); That is,
\[ A(x) = \begin{cases} 
1 & \text{if } x \in I \\
\alpha & \text{if } x \notin I 
\end{cases} \]

Since \( \alpha < 1 \), \( A \) is a proper \( L \)-fuzzy ideal of \( X \). Let \( x \) and \( y \in X \).

Then

\[ x \land y \in I \Rightarrow A(x \land y) = 1 \text{ and } x \in I \text{ or } y \in I \]

\[ \Rightarrow A(x \land y) = 1 = A(x) \text{ or } A(x \land y) = 1 = A(y) \]

and \( x \land y \notin I \Rightarrow x \notin I \text{ and } y \notin I \)

\[ \Rightarrow A(x \land y) = \alpha = A(x) = A(y). \]

Therefore, by Theorem 2.4.1 and 2.4.2, \( A \) is an \( L \)-fuzzy prime ideal of \( X \). In the following, we give an example of an \( L \)-fuzzy prime ideal which is not a prime fuzzy ideal.

**2.4.9 Example**

Let \( X \) be the 5-element distributive lattice \( \{ 0, 1, a, b, c \} \) represented by the Hasse diagram given below and let \( L \) be the three element chain \( \{ 0, s, 1 \} \) with \( 0 < s < 1 \).

![Hasse diagram]

Define an \( L \)-fuzzy subset \( A \) of \( X \) by

\[ A(0) = 1, A(a) = A(c) = s \text{ and } A(1) = A(b) = 0. \]

For any \( x \) and \( y \in X \), we have

\[ x \leq y \Rightarrow A(y) \leq A(x) \]

and hence \( A \) is an antitone. Also, observe that

\[ A(a \lor b) = A(1) = 0 = s \land 0 = A(a) \land A(b) \]
Therefore A is a proper L-fuzzy ideal of X. Also, for any \( x \) and \( y \in X \),
\[
A(x \land y) = A(x) \text{ or } A(y).
\]
Therefore A is an L-fuzzy prime ideal of X. However A is not a prime L-fuzzy ideal, since A is not two-valued (refer 2.3.5).

2.4.10 Theorem

If A is an L-fuzzy prime ideal of a lattice X, then the 1-cut \( A_1 \) is a prime ideal of X.

Proof: Let A be an L-fuzzy prime ideal of X. Recall that the 1-cut of A is given by
\[
A_1 = \{ x \in X : A(x) = 1 \}.
\]
Since A is a proper L-fuzzy ideal, there exists \( x \in X \) such that \( A(x) \neq 1 \) and hence \( x \notin A_1 \). Therefore \( A_1 \) is a proper ideal of X. Also, for any \( x \) and \( y \in X \),
\[
x \land y \in A_1 \implies A(x \land y) = 1
\]
\[
\implies 1 = A(x \land y) = A(x) \text{ or } A(y) \quad \text{(by 2.4.1)}
\]
\[
\implies x \in A_1 \text{ or } y \in A_1
\]
Thus \( A_1 \) is a prime ideal of X.

The converse of the above theorem is not true; for, consider the example given below.

2.4.11 Example

Let X be the lattice \{ 0,1,a,b,c,d \} represented by the Hasse diagram given below.

![Hasse diagram](image-url)
Then X is a bounded distributive lattice. Let L be the lattice X itself. Then L (and X) is a frame; note that any finite distributive lattice is a frame. Define the L – fuzzy subset A of X as follows.

\[ A(0) = 1, A(c) = d, A(a) = b \]

\[ A(b) = a, A(d) = c \quad \text{and} \quad A(1) = 0. \]

Then \( A_1 = \{ x \in X : A(x) = 1 \} = \{ 0 \} \) which is a prime ideal of X. Since

\[ A(a \land b) = A(c) = d \neq A(a) \quad \text{and} \quad \neq A(b), \]

it follows from 2.4.1 that A is not an L – fuzzy prime ideal of X. This proves that the converse of 2.4.10 is not true.

**2.4.12 Definition**

Let A be an L – fuzzy subset of a lattice X and \( \alpha \in L \). Then we define the L – fuzzy subsets \( A \lor \alpha \) and \( A \land \alpha \) by

\[ (A \lor \alpha)(x) = A(x) \lor \alpha \]

and \( (A \land \alpha)(x) = A(x) \land \alpha \)

for all \( x \in X \).

**2.4.13 Theorem**

For any L - fuzzy ideal A of a lattice X and \( \alpha \in L \), \( A \lor \alpha \) is an L- fuzzy ideal.

**Proof**: First, observe that

\[ (A \lor \alpha)(0) = A(0) \lor \alpha = 1 \lor \alpha = 1. \]

Now, for any \( x \) and \( y \) \( \in X \),

\[ (A \lor \alpha)(x \lor y) = A(x \lor y) \lor \alpha \]

\[ = (A(x) \land A(y)) \lor \alpha \]

\[ = (A(x) \lor \alpha) \land (A(y) \lor \alpha) \]
\[ = (A \lor \alpha)(x) \land (A \lor \alpha)(y). \]

Thus \( A \lor \alpha \) is an \( L \)-fuzzy ideal of \( X \).

Unlike \( A \lor \alpha \), \( A \land \alpha \) is not an \( L \)-fuzzy ideal, unless \( \alpha = 1 \). For \( A \land \alpha \) to be an \( L \)-fuzzy ideal, it is necessary that \( (A \land \alpha)(0) = 1 \) and hence

\[ \alpha = 1 \land \alpha = A(0) \land \alpha = (A \land \alpha)(0) = 1. \]

Note that, if \( \alpha = 1 \), then \( A \lor \alpha \) is the constant map \( \overline{1} \). Therefore, for \( A \lor \alpha \) to be a proper \( L \)-fuzzy ideal, it is necessary that \( \alpha < 1 \).

2.4.14 Theorem

Let \( A \) be an \( L \)-fuzzy prime ideal of a lattice \( X \) and \( \alpha \) be an element in a frame \( L \) such that \( A(1) \leq \alpha < 1 \). Then \( A \lor \alpha \) is an \( L \)-fuzzy prime ideal of \( X \).

**Proof:** In theorem 2.4.13, we have proved that \( A \lor \alpha \) is an \( L \)-fuzzy ideal of \( X \). Also,

\[ (A \lor \alpha)(1) = A(1) \lor \alpha = \alpha < 1 \]

and hence \( A \lor \alpha \) is a proper \( L \)-fuzzy ideal of \( X \).

Further, for any \( x \) and \( y \) in \( X \), we have

\[ A(x \land y) = A(x) \lor A(y) \]

and hence \( (A \lor \alpha)(x \land y) = A(x \land y) \lor \alpha \)

\[ = A(x) \lor \alpha \lor A(y) \lor \alpha \]

\[ = (A \lor \alpha)(x) \lor (A \lor \alpha)(y) \]

Thus \( A \lor \alpha \) is an \( L \)-fuzzy prime ideal of \( X \).

The above theorem is used to extend an important theorem on prime ideals to the \( L \)-fuzzy prime ideals of a distributive lattice. First, we recall the following theorem, in whose proof the Zorn’s lemma is used.

2.4.15 Theorem

Let \( X \) be a bounded distributive lattice and \( I \) a proper ideal of \( X \). Then \( I \) is contained in a prime ideal of \( X \).
Proof: Consider the class

\[ P = \{ J : J \text{ is a proper ideal of } X \text{ and } I \subseteq J \} \]

Note that an ideal \( J \) belongs to \( P \) if and only if \( I \subseteq J \) and \( 1 \notin J \). Clearly \( P \) satisfies the Zorn’s hypothesis; that is \( P \) is closed under unions of chains and \( P \) is non-empty. Therefore, by Zorn’s lemma, \( P \) has a maximal member, say \( M \). It is enough if we prove that \( M \) is a prime ideal of \( X \). Since \( M \in P \), \( M \) is a proper ideal of \( X \). Also, let \( x \) and \( y \in X \) such that \( x \land y \in M \). Then

\[ x \notin M \implies 1 \in M \lor (x) \quad (\text{by the maximality of } M) \]

\[ \implies 1 = m \lor x, \quad m \in M \]

\[ \implies y = y \land (m \lor x) = (y \land m) \lor (y \land x) \in M \]

Thus \( M \) is a prime ideal of \( X \) and \( I \subseteq M \).

2.4.16 Theorem

Let \( X \) be a bounded distributive lattice and \( L \) frame. Let \( A \) be a proper \( L \)-fuzzy ideal of \( X \) such that \( \text{sup} \{ A(x) : x \in X \text{ and } A(x) < 1 \} < 1 \). Then there exists an \( L \)-fuzzy prime ideal \( B \) of \( X \) such that \( A \leq B \).

Proof: Put \( \alpha = \text{Sup} \{ A(x) : x \in X \text{ and } A(x) < 1 \} \). Then, by hypothesis, \( \alpha < 1 \). Now, put

\[ I = \{ x \in X : A(x) = 1 \}. \]

Then \( I \) is a proper ideal of \( X \), since \( A \) is a proper \( L \)-fuzzy ideal. By the above theorem, there exists a prime ideal \( P \) of \( X \) such that \( I \subseteq P \). Then, consider \( \chi_P \lor \alpha \), where \( \chi_P \) is the characteristic function corresponding to \( P \). Note that \( \chi_P \lor \alpha \) is precisely equal to \( \alpha_p \) which is defined by

\[ \alpha_p(x) = \begin{cases} 1 & \text{if } x \in P \\ \alpha & \text{if } x \notin P \end{cases} \]
By Theorem 2.4.14 (or, by theorem 2.4.6), $\alpha$ is an L-fuzzy prime ideal of $X$. Now,

$$x \in P \Rightarrow A(x) \leq 1 = \alpha(x)$$

and

$$x \notin P \Rightarrow x \notin I \Rightarrow A(x) < 1 \Rightarrow A(x) \leq \alpha = \alpha_p(x)$$

Thus $A \leq \alpha_p$.

Theorem 2.4.1 and definition 2.4.2 suggest a method for constructing L-fuzzy prime ideals of a given lattice. This method is described in the following.

**2.4.17 Theorem**

Let $X = (X, \wedge, \lor)$ be a bounded distributive lattice and $L$ a frame. Let $C$ be a chain in $L$ such that $1 \in C$ and $C$ is closed under arbitrary supremums. Let $\{I_\alpha\}_{\alpha \in C}$ be a class of ideals of $X$ such that, for each $\alpha \in C$, either $I_\alpha = X$ or $I_\alpha$ is a prime ideal of $X$ and, for any $D \subseteq C$, $\bigcap_{\alpha \in D} I_\alpha = I_{\sup D}$. Define an L-fuzzy subset $P$ of $X$ by

$$P(x) = \text{Sup} \{ \alpha \in C : x \in I_\alpha \}$$

for any $x \in X$. Then $P$ is an L-fuzzy prime ideal of $X$ if $P$ is proper.

**Proof**: Since $C$ is closed under arbitrary supremums and the supremum of the empty set is the least element $0$ in $L$, it follows that $0 \in C$. For any $\alpha \in C$ and $x$ and $y$ in $X$, we have

$$x \wedge y \in I_\alpha \iff x \in I_\alpha \text{ and } y \in I_\alpha$$

(Since $I_\alpha$ is an ideal of $X$) and therefore, we have

$$P(x) \wedge P(y) = \sup\{ \alpha \in C : x \in I_\alpha \} \wedge \sup\{ \beta \in C : y \in I_\beta \}$$

$$= \sup\{ \alpha \wedge \beta : \alpha, \beta \in C, x \in I_\alpha \text{ and } y \in I_\beta \}$$

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\[= \sup \left\{ \gamma \in C : x \vee y \in I_\alpha \right\}\]

\[= P(x \vee y).\]

In the above argument, notice that we have used the infinite meet distributivity in L and that \(I_\alpha \subseteq I_{\alpha \wedge \beta}\) and \(I_\beta \subseteq I_{\alpha \wedge \beta}\). Thus P is an L-fuzzy ideal of X. Observe that, for any \(x \in X\) and \(\alpha \in C\),

\[x \in I_\alpha \Rightarrow \alpha \leq P(x) \Rightarrow x \in P_\alpha\]

Also, on the other hand,

\[x \in P_\alpha \Rightarrow \alpha \leq P(x) = \sup \left\{ \beta \in C : x \in I_\beta \right\}\]

\[\Rightarrow \alpha = \alpha \wedge P(x) = \sup \left\{ \alpha \wedge \beta : x \in I_\beta \right\}\]

\[\Rightarrow I_\alpha = \bigcap \left\{ I_{\alpha \wedge \beta} : x \in I_\beta \right\}\]

\[\Rightarrow x \in I_\alpha \left( \text{since } I_\beta \subseteq I_{\alpha \wedge \beta} \right)\]

Therefore \(P_\alpha = I_\alpha\) for any \(\alpha \in C\). Note that, for any \(\alpha \in C\) and \(x, y \in X\),

\[x \wedge y \in I_\alpha \iff x \in I_\alpha \text{ or } y \in I_\alpha.\]

Therefore, \(P(x) \vee P(y) = P(x \wedge y)\) and, since \(P(x)\) and \(P(y)\) \(\in C\) and \(C\) is a chain, we have \(P(x) \leq P(y)\) or \(P(y) \leq P(x)\). Thus, if \(P\) is a proper L-fuzzy ideal of \(X\), then \(P\) is an L-fuzzy prime ideal of \(X\).

2.4.18 Remark

In the above, notice that \(P\) is proper if and only if \(I_\alpha \neq X\) for at least one \(\alpha \in C\);

That is, \(P\) is the constant map \(\overline{1}\) if and only if \(I_\alpha = X\) for all \(\alpha \in C\).

Next we extend an important theorem of M.H.Stone [24] on prime ideals of distributive lattices to L-fuzzy prime ideals. First let us recall the following.
2.4.19 Theorem

Let $X$ be a bounded distributive lattice, $I$ an ideal of $X$ and $J$ a filter of $X$ such that $I \cap J$ is empty. Then there exists a prime ideal $P$ of $X$ such that $I \subseteq P$ and $P \cap J$ is empty.

**Proof:** Consider the class

$$\mathcal{P} = \{ K : K \text{ is an ideal of } X, I \subseteq K \text{ and } K \cap J = \emptyset \}$$

Since $I \in \mathcal{P}$, $\mathcal{P}$ is a non-empty class of sets which is closed under unions of chains. By Zorn’s lemma, $\mathcal{P}$ has a maximal member. Let $P$ be a maximal member in $\mathcal{P}$. It is enough if we prove that $P$ is a prime ideal of $X$, since $P \in \mathcal{P}$ and hence $I \subseteq P$ and $P \cap J = \emptyset$.

Let $x$ and $y \in X$ such that $x \notin P$ and $y \notin P$. By the maximality of $P$, it follows that

$$(P \cup \{x\}) \cap J \neq \emptyset$$

and hence there exists $a$ and $b \in J$ such that

$$a \in P \cup \{x\} \text{ and } b \in P \cup \{y\}.$$ 

This implies that $a \leq p \lor x$ and $b \leq q \lor y$ for some $p$ and $q \in P$ and hence

$$a \land b \leq (p \lor x) \land (q \lor y) = (p \land q) \lor (p \land y) \lor (x \land y) \lor (x \land y).$$

If $x \land y \in P$, then $a \land b \in P \cap J$ (since $a$ and $b \in J$ and $J$ is a filter), which is a contradiction, since $P \cap J = \emptyset$.

Thus, $x \land y \notin P$. Therefore $P$ is a prime ideal of $X$, $I \subseteq P$ and $P \cap J = \emptyset$.

2.4.20 Theorem

Let $X$ be a bounded distributive lattice and $L$ a frame. Let $A$ be an $L$-fuzzy ideal of $X$ and $F$ an $L$-fuzzy subset of $X$ such that $F(1) = 1$ and $F(x \land y) = F(x) \land F(y)$ for all $x$ and
\(y \in X\). Let \(\alpha\) be a meet prime element in \(L\) such that \(A \land F \leq \alpha\). Then there exists an \(L\)-fuzzy prime ideal \(P\) of \(X\) such that \(A \leq P\) and \(P \land F \leq \alpha\).

**Proof:** we are given that \(A(x) \land F(x) \leq \alpha\) for all \(x \in X\).

Put \(I = \{x \in X : A(x) \not\leq \alpha\}\)

and \(J = \{x \in X : F(x) \not\leq \alpha\}\)

Since \(A(0) = I \not\leq \alpha\) (because \(\alpha\) is meet prime), \(0 \in I\) and hence \(I\) is not empty.

Also, since \(F(1) = I \not\leq \alpha\), \(1 \in J\) and hence \(J\) is not empty.

\[x, y \in I \Rightarrow A(x) \not\leq \alpha \text{ and } A(y) \not\leq \alpha\]

\[\Rightarrow A(x \lor y) = A(x) \land A(y) \not\leq \alpha\]

\[\Rightarrow x \lor y \in I\].

Also, \(y \leq x \in I \Rightarrow A(y) \geq A(x) \text{ and } A(x) \not\leq \alpha \Rightarrow A(y) \not\leq \alpha\)

\[\Rightarrow y \in I\]. Therefore \(I\) is an ideal of \(X\). Further,

\[x, y \in J \Rightarrow F(x) \not\leq \alpha \text{ and } F(y) \not\leq \alpha\]

\[\Rightarrow F(x \land y) = F(x) \land F(y) \not\leq \alpha\]

\[\Rightarrow x \land y \in J\]

Also, \(y \geq x \in J \Rightarrow A(y) \geq A(x) \text{ and } A(x) \not\leq \alpha\)

\[\Rightarrow A(y) \not\leq \alpha \Rightarrow y \in J\]

Therefore \(J\) is a filter of \(X\). Since \(\alpha\) is meet prime and \(A(x) \land F(x) \leq \alpha\), it follows that \(A(x) \leq \alpha\) or \(F(x) \leq \alpha\) and hence \(x \not\in I\) or \(x \not\in J\). Therefore \(I \cap J = \emptyset\). By theorem 2.4.19, there exists a prime ideal \(K\) of \(X\) such that \(I \subseteq K\) and \(K \cap J = \emptyset\). Now \(a_k\) is an \(L\)-fuzzy prime ideal of \(X\) and \(A \leq a_k\) (since \(x \not\in K \Rightarrow x \not\in I \Rightarrow A(x) \leq a_k(x)\))

and \(a_k \land F \leq \alpha\). Since, for any \(x \in K\), \(x \not\in J\) and hence \(F(x) \leq \alpha\).