CHAPTER 1

Introduction and Preliminaries

1.1 GENERAL INTRODUCTION

One can acknowledge that the word “PARTITION” has numerous meanings in Mathematics. The first discoveries of any depth were made in the eighteenth century A.D. when L.Euler proved many beautiful and significant partition theorems. Euler indeed laid the foundations for the theory of partitions. Many other great mathematicians including Cayley, Gauss, Hardy, Lagrange, Littlewood, Rademacher, Schur, Sylvester, in particular Srinivasa Ramanujan have contributed to the development of partition theory.

The great Indian mathematician, Srinivasa Ramanujan has left the sign of his brilliance in Number theory throughout his entire life. He has also made some extraordinary contributions to the fields of Hypergeometric series, Elliptic functions, Prime numbers, Bernoulli`s numbers, Divergent series, Continued fractions, Elliptic Modular equations, Highly Composite numbers, Riemann Zeta functions, Partition of numbers, Mock-Theta functions etc. Actually, apart from a few elementary ones, most of the contributions of Srinivasa Ramanujan belong to a higher realm of mathematics that is often referred to as "Higher Mathematics". In fact, one can find it quite difficult to understand Srinivasa Ramanujan`s mathematics if one does not have the basic foundation in various mathematical subjects. In order to calculate the value of $\pi$ up to 17 million
places using a computer, the present day mathematicians actually use Srinivasa. Ramanujan’s fastest step-by-step method. The mathematical contributions of Srinivasa Ramanujan have also been widely used in solving various problems in higher scientific fields of specialization. Ramanujan was also considered as the master of numbers. His most outstanding contribution was his formula for \( p(n) \), the number of \( \text{`partitions'} \) of \( `n` \). Just an year before Srinivasa Ramnujan passed away, he prepared a list of partitions for each of the first 200 integers in 1920. For all these reasons, Ramanujan is hailed as an all time great mathematician like Euler, Gauss or Jacobi for his natural genius.

Partition theory has many unique and novel features and challenges both in terms of combinatorial proofs and generating functions of \( \text{`partitions'} \). Many of the mathematical sciences have seen applications of partitions recently. Various permutation problems in probability and statistics are intimately linked with Simon Newcomb problem. Nonparametric statistics requires \( \text{`restricted partitions'} \). Particle physics uses partition asymptotics and partition identities. \( \text{`Partition'} \) of a convex polygon yields solution to traffic control problems.

Still research is on, developing the role of various types of \( \text{`partitions'} \) like \( \text{`restricted partitions', `overpartitions' and `rooted partitions'} \) etc., in \( q - \text{series and combinatorics} \). Adopting concepts and theory from the famous book on \( \text{`partitions'} \) by George Andrews [3] and recent publications related to \( \text{`partitions'} \), we have made an attempt to develop combinatorial proofs for several theorems. This work takes a further step in introducing new terminology for establishment of new results and getting connection between different types of \( \text{`partitions'} \). We notice that much attention was not paid by the Number Theorists in contributing the \( \text{`partition'} \) identities of any type of \( \text{`partition'} \) of \( n \) in terms of \( \text{`r - partitions'} \) of \( n \). Hence we have chosen to work in this direction as well.

This dissertation revolves around the elementary theory on \( \text{`r - partitions'} \) of \( n \) and presents some new methods to enumerate \( \text{`partitions'} \) of \( n \) with the help of it’s \( \text{`r - partitions'} \). In addition, certain results of various types of \( \text{`partitions'} \) are presented to enrich the knowledge of enumeration of \( \text{`partitions'} \).
This chapter is divided into 4 sections. Section (1.2) deals with presenting survey of available literature, while section (1.3) describes problems of investigation undertaken in this dissertation. Finally some elementary results along with definitions and notations in concern with partition theory required and used are presented for the purpose of record in the last section(1.4) under the heading “PRELIMINARIES”.

1.2 SURVEY OF LITERATURE

Some of the important works are surveyed and presented in this section as per chronological order.

will appear [25]. Andrews’ investigations on the number of the smallest parts in the partitions of $n$ were considered which will appear in J. Reine Angew. Math [4].

1.3 PROBLEMS OF INVESTIGATION

Glancing the survey of literature, it appears that scant attention was paid to develop the elementary theory on $r$ – partitions of a positive integer in various types of partitions. Based on [3] we standardize the notations that are necessary in developing the results. The present thesis consists of the following six chapters

1. Introduction and Preliminaries
2. $r$ – Partitions of $n$
3. Restricted $r$ – Partitions of $n$
4. Relation between the Smallest and the Greatest parts of Overpartitions of $n$.
5. Relation between the Smallest and the Greatest parts of Rooted Partitions of $n$.

In addition to general introduction and survey of literature, all the required Mathematical concepts to develop the remaining chapters are included in Chapter 1. Chapter 2 is devoted to establish a recurrence relation for $r$ – partitions of $n$ with a specified least part. We consider a set $S$ with elements that are in A.P. and discuss certain aspects when parts are in $S$ with a gain of many results. We make a mention about generating functions of partitions of $n$ with parts in $S$. Some results on partitions of $n$ with specified least part are also established. Chapter 3 deals with some more results by imposing conditions on parts and also propose new notations. We extend these results for enumeration of partitions of $n$. We discuss various methods to express partitions of $n$ in terms of $r$ – partitions of $n$. We study partitions of $n$ with least part $k$ and the greatest part $s$ by furnishing results for $r$-partitions of $n$ and then for partitions of $n$. In Chapter 4 we prove results on the greatest parts by employing the conjugacy of the partitions and also establish a relation between the smallest and the greatest parts of the partitions. We extend the results of the previous section to the overpartitions which are generalized partitions. In Chapter 5 we introduce the necessary terminology conveniently
for proving the results on the *rooted partitions*. We devote to develop the relation between the smallest parts and the greatest parts of the *rooted partitions* of \(n\) as the extension of the results proved in the previous Chapter. We also present some results on *rooted partitions* by applying the tool, \(r – \text{partitions of } n\). In Chapter 6 we present results on *partitions* of Convex Polygons by imposing conditions on vertices. Finally we focus on the problems for the scope of further work concentrating on the applications of *partitions*.

### 1.4 PRELIMINARIES

#### 1.4.1 Notation and Definitions

We adopt mostly the common notation on partitions used by Andrews [3] and some of them are given below.

- \(N, \theta, \varepsilon\) stand respectively for the sets of natural, odd natural and even natural numbers. Unless otherwise specified \(n\) stands for a natural number.

- A *partition* of a positive integer \(n\) is a finite nonincreasing sequence of positive integers \(\lambda_1, \lambda_2, \ldots, \lambda_r\) such that \(\sum_{i=1}^{r} \lambda_i = n\) and is denoted by \(n = (\lambda_1, \lambda_2, \ldots, \lambda_r)\), \(n = \lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_r\) or \(\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \lambda_3^{f_3}, \ldots)\) when \(\lambda_1\) repeats \(f_1\) times, \(\lambda_2\) repeats \(f_2\) times and so on. The \(\lambda_i\) are called the parts of the *partition*. In what follows \(\lambda\) stands for a *partition* of \(n\). \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r\).

We notice that for every *partition* \((\lambda_1, \lambda_2, \ldots, \lambda_r)\) of \(n\):

- a) \(1 \leq r \leq n\)
- b) \(1 \leq \lambda_i \leq n\) \(\forall i \in \{1, 2, \ldots n\}\)

- set of all partitions is represented by \(\xi(n)\)

- \(p(n) = \begin{cases} 1 & \text{if } n = 0 \\ \text{number of partitions of } n & \text{if } n \in N \\ 0 & \text{if } n \text{ is negative} \end{cases}\)
if $1 \leq r \leq n$ then $\xi_r(n)$ is set of partitions of $n$ with $r$ parts and its cardinality is denoted by $p_r(n)$. A partition of $n$ with exactly $r$ parts is called as $r$-partition of $n$. We define

$$p_r(n) = \begin{cases} 
0 & \text{if } r = 0 \\
\text{number of } r \text{-partitions of } n & \text{if } 0 < r \leq n \\
0 & \text{if } r > n 
\end{cases}$$

Let $H$ be a set of positive integers. Let "$H$" denote the set of all partitions whose parts lie in $H$. Consequently, $p("H",n)$ is the number of partitions of $n$ that have all their parts in $H$. For our convenience $p("H",n)$ is denoted by $p(H;n)$.

number of partitions of $n$ with least part greater than or equal to $k$ is represented by $p(k,n)$.

number of parts in the partition $\lambda$ is represented by $l(\lambda)$ or $#(\lambda)$ and is called the length of $\lambda$.

rank of partition $\lambda$ is denoted by $r(\lambda) = \lambda_1 - l(\lambda)$.

1.4.2 Ferrers Diagram: Ferrer introduced representation of a partition by a diagram made of dots ‘.’ or squares ‘□’ as follows. Each $\lambda_i$ is represented as a row of $\lambda_i$ dots (or squares) and these are arranged in parallel rows in the decreasing order of $\lambda_i$’s. For example the diagrammatic representation for $(5,3,2,1)$ is given by Figs 1,2 below.

These diagrams help us in formulating the definition of the conjugate $\lambda^*$ of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_r)$. The conjugate $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*, ..., \lambda_r^*)$ is a
$s$–partition of $n$, where $\lambda^*_i$ is the number of dots (squares) in the $i^{th}$ column in the Ferrer diagram for $\lambda$, equivalently the number of $j$ such that $\lambda_j \geq i$. Thus if $\lambda = (5,3,2,1)$, $\lambda^* = (4,3,2,1,1)$

**Fig 1**

**Fig 2**

1.4.3 The values of the function $p_r(n)$ when $1 \leq n \leq 15$ and $1 \leq r \leq n$

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1.4.4 Intermediate Function: One way of getting a hand on the partition function involves an intermediate function \( p(k,n) \) which represents the number of partitions of \( n \) using only natural numbers at least as large as \( k \).

For any given value of \( k \), partitions counted by \( p(k,n) \) fit into exactly one of the following categories:

1. smallest part is \( k \)
2. smallest part is strictly greater than \( k \)

The number of partitions satisfying the first condition is \( p(k,n-k) \). The partition function in terms of the intermediate function, is

\[
1 + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} p(k,n-k) = p(n) \quad \text{where } [x] \text{ is the step function}
\]

\[
[x] = n \text{ where } n \leq x < n+1 \text{ and } n \text{ is an integer.}
\]

The number of partitions satisfying the second condition is \( p(k+1,n) \).

Since the two conditions are mutually exclusive, the number of partitions satisfying any one of the conditions is \( p(k+1,n) + p(k,n-k) \). □

The base cases of this recursively defined function are as follows:
1.4.5 The values of this function $p(k,n)$:

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**Theorem 1.4.6** [3]: The number of partitions of $n$ with at most $m$ parts equals the number of partitions of $n$ in which no part exceeds $m$.

**Proof**: We may set up a one-to-one correspondence between the two classes of partitions under consideration by merely mapping each partition onto its conjugate. The mapping is certainly one-to-one, and by considering the graphical representation we see that under conjugation the condition “at most $m$ parts” is transformed into “no part exceeds $m$” and vice versa. ■

As an example, let us consider the partitions of 6, first into at most three parts and then into parts none of which exceeds 3. We shall list conjugates opposite each other.

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Remark 1.4.7: We calculate the number of all partitions of \( n \) by the formula
\[
p(n) = p_n(n) + p_{n-1}(n) + \ldots + p_1(n)
\]

1.4.8 Generating function: The generating function \( f(q) \) for the sequence \( a_0, a_1, a_2, \ldots \) is the power series \( f(q) = \sum a_n q^n \).

Theorem 1.4.9: Let \( H \) be a set of positive integers, and let
\[
f(q) = \sum_{n \geq 0} p(H^n, n) q^n.
\]
\[
f_d(q) = \sum_{n \geq 0} p(H^n(\leq d), n) q^n.
\]
Then for \( |q| < 1 \)
\[
f(q) = \prod_{n \in H} \left(1-q^n\right)^{-1} \quad \text{............................................(1)}
\]
\[
f_d(q) = \prod_{n \in H} \left(1+q^n+\ldots+q^{dn}\right) \ns = \prod_{n \in H} \left(1-q^{(d+1)n}\right)(1-q^n)^{-1} \quad \text{.....................(2)}
\]

Proof: We shall proceed in a formula manner to prove (1) and (2);
At the conclusion of our proof we shall sketch how to justify our steps analytically. Let
\( H = \{h_1, h_2, h_3, \ldots\} \). Then
\[
\prod_{n \in H} \left(1-q^n\right)^{-1} = \prod_{n \in H} \left(1+q^n+q^{2n}+\ldots\right) \\
= \left(1+q^{h_1}+q^{2h_1}+\ldots\right) \\
\times \left(1+q^{h_2}+q^{2h_2}+\ldots\right) \\
\times \left(1+q^{h_3}+q^{2h_3}+\ldots\right) \ldots \ldots \\
= \sum_{a_1 \geq 0} \sum_{a_2 \geq 0} \sum_{a_3 \geq 0} \ldots q^{a_1h_1+a_2h_2+a_3h_3+\ldots}
\]
and we observe that the exponent of \( q \) is just the partition \( (h_1^{a_1}, h_2^{a_2}, h_3^{a_3}, \ldots, h_n^{a_n}) \) is just a partition of the exponent of \( q \).
Hence \( q^n \) will occur in the foregoing summation once for each partition of \( n \) into parts taken from \( H \). There fore

\[
\prod_{n \in H} (1 - q^n)^{-1} = \sum_{n \geq 0} p("H",n)q^n.
\]

The proof of (2) is similar to that of (1) except that the infinite geometric series is replaced by the finite geometric series:

\[
\prod_{n \in H} \left(1 + q^n + \ldots + q^{dn}\right) = \sum_{d \geq 0} \sum_{d \geq a_2} \sum_{d \geq a_1} \ldots q^{a_1h_1 + a_2h_2 + a_3h_3 + \ldots}
\]

If we are to view the foregoing procedure as operations with convergent infinite products, then the multiplication of infinitely many series together requires some justification. The simplest procedure is to \( \prod_{i=1}^n (1 - q^{h_i})^{-1} \). This truncated product will generate those partitions whose parts are among \( h_1, h_2, \ldots, h_n \). The multiplication is now perfectly valid since only a number of absolutely convergent series are involved. Now assume \( q \) is real and \( 0 < q < 1 \), then if \( M = h_n \),

\[
\sum_{j=0}^{M} p("H",j)q^j \leq \prod_{i=1}^{n} (1 - q^{h_i})^{-1} \leq \prod_{i=1}^{\infty} (1 - q^{h_i})^{-1} < \infty
\]

Thus the sequence of partial sums \( \sum_{j=0}^{M} p("H",j)q^j \) is a bounded increasing sequence and must there fore converge. On the other hand

\[
\sum_{j=0}^{\infty} p("H",j)q^j \geq \prod_{i=1}^{n} (1 - q^{h_i})^{-1} \rightarrow \prod_{i=1}^{\infty} (1 - q^{h_i})^{-1} \quad \text{as} \quad n \rightarrow \infty
\]

There fore

\[
\sum_{j=0}^{\infty} p("H",j)q^j = \prod_{i=1}^{n} (1 - q^{h_i})^{-1} = \prod_{n \in H} \left(1 - q^n\right)^{-1}.
\]

Similar justification can be given for the proof of (2). ☐

**Corollary 1.4.10:** (Glaisher). Let \( N_d \) denote the set of those positive integers not divisible by \( d \). Then

\[
p("N_{d+1}\), n) = p("N"(\leq d), n) \quad \text{for all} \quad n.
\]
Proof: \[ p("N\)(\leq d), n) = \prod_{n=1}^{\infty} \left( \frac{1 - q^{(d+1)n}}{1 - q^n} \right) \]

\[ = \prod_{n=1}^{\infty} \left( \frac{1}{1 - q^n} \right) \]

\[ = p("N_{d+1}\", n) \]

Hence \[ p("N_{d+1}\", n) = p("N\)(\leq d), n) \]

1.4.11 Two variable generating function:

Let \( r, n \in \mathbb{N} \) and \( S = \{am + b \mid a \in \mathbb{N}, b \in \mathbb{Z} \text{ and } m = 1, 2, \ldots, n\} \) be a set of positive integers, then the two variable generating function is

\[ f(z; q) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} p_r(S, n) z^r q^n \]

\[ = \sum_{\lambda \in \Phi} z^{\#(\lambda)} q^{\sigma(\lambda)} \]

Where if \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), \( \#(\lambda) = r \), \( \sigma(\lambda) = \lambda_1 + \lambda_2 + \ldots + \lambda_r \), and \( \Phi \) is set of all partitions of \( n \) whose parts are elements of \( S \).