CHAPTER III
SOLUTION OF FMADM BY TOPSIS AND FUZZY TOPSIS METHOD

3.1. Introduction

In Multiple Attribute Decision Making (MADM) problem, decision makers often face the problem of selecting among alternatives that have complex criteria. A theoretical approach, giving fundamental conditions for the construction of order preserving additive representation of multiple aspects alternatives is described by Fishburn [30]. Many real world problems are fuzzy in nature and not the random; the probability applications have not been very satisfactory in a lot of cases.

The proposed study focused on the selection decision which is usually based on both quantitative and qualitative performance rating pertaining to desired closeness among the available facilities. The aim of MADM is to obtain the optimum alternative that has the highest degree of satisfaction for all the relevant attributes.

In real life, decision-making is the process of finding best option from all the feasible alternatives. The selection of alternative and the weight of each criterion are described. A Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and fuzzy TOPSIS are proposed and tried to obtain ideal solution. Finally, an example is illustrated for proposed method.
3.2. Proposed methodology

3.2.1 Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) Method

In TOPSIS alternatives are ranked based on their distance from an ideal solution and a negative-ideal solution, Yoon and Hwang [83].

Algorithm

Step 1- For decision matrix \( D = [x_{ij}]_{m \times n} \) matrix of attainments for ‘m’ alternative with respect to ‘n’ attributes) calculate the normalized (unit free) decision matrix \( D_N = [r_{ij}]_{m \times n} \). The normalized value \( r_{ij} \) is defined as:

\[
r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad i=1,2,...,m, \; j=1,2,...,n
\]  

(3.1)

Where \( x_{ij} \) is the attainment of attribute ‘j’ by alternative ‘i’

Step 2- Compute weighted normalized decision matrix \( V_{ij} \). The weighted normalized value \( V_{ij} \) is defined as:

\[
V_{ij} = w_j \cdot r_{ij}
\]  

(3.2)

Where \( w_j \) is the importance weight of attribute ‘j’.

Step 3- Determine the ideal \( A^+ \) and negative ideal \( A^- \) solutions:

\[
A^+ = (V_{1}^+, V_{2}^+, \ldots, V_{n}^+) = [V_j^+]_{1 \times n}
\]  

(3.3)

\[
A^- = (V_{1}^-, V_{2}^-, \ldots, V_{n}^-) = [V_j^-]_{1 \times n}
\]  

(3.4)

Where \( V_{j}^+ \) and \( V_{j}^- \) are defined as the followings:
\[ V_{ij}^+ = \text{Max}(V_{ij}) \quad \text{for } i = 1,2,\ldots,m \text{ for benefit attribute.} \] (3.5)

\[ V_{ij}^- = \text{Min}(V_{ij}) \quad \text{for } i = 1,2,\ldots,m \text{ for cost attribute.} \] (3.6)

Step 4- Compute separation measures (distance to ideal and negative-ideal solutions):

\[
S_{i}^+ = \sqrt{\sum_{j=1}^{n} (V_{ij}^+ - V_{ij}^+)^2}
\] (3.7)

\[
S_{i}^- = \sqrt{\sum_{j=1}^{n} (V_{ij}^- - V_{ij}^-)^2}
\] (3.8)

Step 5- The relative closeness to the ideal solution is defined as:

\[
RC_{i}^+ = \frac{S_{i}^-}{S_{i}^- + S_{i}^+}
\] (3.9)

Step 6- Rank the alternatives in descending order of their relative closeness.

3.2.2 Fuzzy Technique for Order Preference by Similarity to Ideal Solution (FTOPSIS) Method

It is often difficult for a decision maker to assign a precise performance rating to an alternative for the consideration. The merit of using a fuzzy approach is to assign the relative importance of attributes using fuzzy numbers instead of precise numbers. This section extends the TOPSIS to the fuzzy environment. The development of fuzzy TOPSIS as follows:

Definition 3.1. A fuzzy set \( \tilde{a} \) in a universe of discourse \( X \) is characterized by a membership function \( \mu_{\tilde{a}}(x) \) which associates with each element \( x \) in \( X \), a real number in the interval \([0, 1]\). The function value is termed the grade of membership of \( x \) in \( \tilde{a} \) [83].
The present study uses triangular fuzzy numbers. A triangular fuzzy number \( \tilde{a} \) can be defined by a triplet \((a_1, a_2, a_3)\). Its conceptual schema and mathematical form are shown by Eq. (3.10) [97];

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
0, & x \leq a_1, \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\
\frac{a_3-x}{a_3-a_1}, & a_2 \leq x \leq a_3, \\
0, & x \geq a_3,
\end{cases}
\]  
\text{(3.10)}

**Definition 3.2.** Let \( \tilde{a} = (a_1, a_2, a_3) \) and \( \tilde{b} = (b_1, b_2, b_3) \) be two triangular fuzzy numbers, then the vertex method is defined to calculate the distance between them,

\[
d(\tilde{a}, \tilde{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}
\]  
\text{(3.11)}

**Property 3.1.** Assuming that both \( \tilde{a} = (a_1, a_2, a_3) \) and \( \tilde{b} = (b_1, b_2, b_3) \) are real numbers, then the distance measurement \( d(\tilde{a}, \tilde{b}) \) is identical to the Euclidean distance [12].

**Property 3.2.** Let \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) be three triangular fuzzy numbers. The fuzzy number \( \tilde{b} \) is closer to fuzzy number \( \tilde{a} \) than the other fuzzy number \( \tilde{c} \) if, and only if, \( d(\tilde{a}, \tilde{b}) < d(\tilde{a}, \tilde{c}) \) [12].

The basic operations on fuzzy triangular numbers are as follows:

\[
\tilde{a} \times \tilde{b} = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3) \text{ for multiplication} \quad \text{(3.12)}
\]

\[
\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \text{ for addition} \quad \text{(3.13)}
\]

The fuzzy MADM can be concisely expressed in matrix format as Eqs. (3.13) and (3.14).
\[
\tilde{D} = \begin{bmatrix}
C_1 & C_2 & \ldots & C_n \\
A_1 & \tilde{x}_{11} & \tilde{x}_{12} & \ldots & \tilde{x}_{1n} \\
A_2 & \ldots & \ldots & \ldots & \ldots \\
A_m & \tilde{x}_{m1} & \tilde{x}_{m2} & \ldots & \tilde{x}_{mn}
\end{bmatrix}
\]

(3.14)

\[
\tilde{W} = [\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n],
\]

(3.15)

where, \(\tilde{x}_{ij}\), \(i = 1, 2, \ldots, m\), \(j = 1, 2, \ldots, n\) and \(\tilde{w}_j\), \(j = 1, 2, \ldots, n\) are linguistic triangular fuzzy numbers, \(\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})\) and \(\tilde{w}_j = (\tilde{w}_{j1}, \tilde{w}_{j2}, \tilde{w}_{j3})\). Note that \(\tilde{x}_{ij}\) is the performance rating of the \(i\)th alternative, \(A_i\), with respect to the \(i\)th attribute, \(C_i\) and \(\tilde{w}_j\) represents the weight of the \(j\)th attribute, \(C_j\).

The normalized fuzzy decision matrix is shown as below

\[
\tilde{R} = [\tilde{r}_{ij}]_{m \times n}
\]

(3.16)

The weighted fuzzy normalized decision matrix is shown as Eq. (3.17)

\[
\tilde{V} = \begin{bmatrix}
\tilde{v}_{11} & \tilde{v}_{12} & \ldots & \tilde{v}_{1n} \\
\tilde{v}_{21} & \tilde{v}_{22} & \ldots & \tilde{v}_{2n} \\
\ldots & \ldots & \ldots & \ldots \\
\tilde{v}_{i1} & \tilde{v}_{i2} & \ldots & \tilde{v}_{in} \\
\ldots & \ldots & \ldots & \ldots \\
\tilde{v}_{m1} & \tilde{v}_{m2} & \ldots & \tilde{v}_{mn}
\end{bmatrix} = \begin{bmatrix}
\tilde{w}_1\tilde{r}_{11} & \tilde{w}_1\tilde{r}_{12} & \ldots & \tilde{w}_1\tilde{r}_{1n} \\
\tilde{w}_2\tilde{r}_{21} & \tilde{w}_2\tilde{r}_{22} & \ldots & \tilde{w}_2\tilde{r}_{2n} \\
\ldots & \ldots & \ldots & \ldots \\
\tilde{w}_i\tilde{r}_{i1} & \tilde{w}_i\tilde{r}_{i2} & \ldots & \tilde{w}_i\tilde{r}_{in} \\
\ldots & \ldots & \ldots & \ldots \\
\tilde{w}_n\tilde{r}_{n1} & \tilde{w}_n\tilde{r}_{n2} & \ldots & \tilde{w}_n\tilde{r}_{nn}
\end{bmatrix}
\]

(3.17)

Given the above fuzzy theory, the proposed fuzzy TOPSIS procedure is then defined as follows:

Step 1: Choose the linguistic rating \((\tilde{x}_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)\) for alternatives with respect to criteria and the appropriate linguistic variables \((\tilde{w}_j, j = 1, 2, \ldots, n)\) for the weight of the criteria.

The fuzzy linguistic rating \((\tilde{x}_{ij})\) preserves the property that the ranges of normalized triangular fuzzy numbers belong to \([0, 1]\); thus, there is no
need for a normalization procedure. For this instance, the $D$ defined by Eq. (3.14) is equivalent to the $R$ defined by Eq. (3.16).

Step 2: Construct the weighted normalized fuzzy decision matrix. The weighted normalized value $\tilde{V}$ is calculated by Eq. (3.17).

Step 3: identify positive ideal ($A^+$) and negative ideal ($A^-$) solutions. The fuzzy positive ideal solution (FPIS, $A^+$) and the fuzzy negative ideal solution (FNIS, $A^-$) are shown as Eqs. (3.18) and (3.19):

$$A^+ = (v_1^+, v_2^+, ..., v_n^+) = \{\max_{j} v_{ij} \mid i = 1, 2, ..., m, j = 1, 2, ..., n\}$$  \hspace{1cm} (3.18)

$$A^- = (v_1^-, v_2^-, ..., v_n^-) = \{\max_{j} v_{ij} \mid i = 1, 2, ..., m, j = 1, 2, ..., n\}$$  \hspace{1cm} (3.19)

Step 4: Calculate separation measures. The distance of each alternative from $A^+$ and $A^-$ can be currently calculated using Eqs. (3.20) and (3.21):

$$d^+ = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_{ij}^+), i = 1, 2, ..., m$$  \hspace{1cm} (3.20)

$$d^- = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_{ij}^-), i = 1, 2, ..., m$$  \hspace{1cm} (3.21)

Step 5: Calculate similarities to ideal solution. This step solves the similarities to an ideal solution by Eq. (3.22):

$$CC_{ij} = \frac{d^-}{d^+ + d^-}$$  \hspace{1cm} (3.22)

Step 6: Rank preference order. Choose an alternative with maximum $CC_{ij}^+$ or rank alternatives according to $CC_{ij}^+$ in descending order.

The decision makers use the linguistic variables to evaluate the importance of attributes and the rating of alternatives with respect to various attributes. The present study has only precise values for the performance rating and for the attribute weights. In order to illustrate the
idea of fuzzy MADM, we deliberately transform the existing precise values to five-levels, fuzzy linguistic variables- very low (VL), low(L), medium(M), high(H) and very high(VH). The purpose of the transformation process has two fold as (i) fuzzy TOPSIS method (ii) to benchmark the empirical results with other precise value methods in later analysis.

Among the commonly used fuzzy numbers, triangular and trapezoidal fuzzy numbers are likely to be the most adoptive ones due to their simplicity in modeling and easy of interpretation. Both triangular and trapezoidal fuzzy numbers are applicable to the present study. We feel a triangular fuzzy number can be adequately representing the five level fuzzy linguistic variables and thus, is used for the analysis hereafter.

As a rule of thumb, each rank is assigned an evenly spread membership function that has an interval of 0.30 or 0.25. Based on these assumptions, a transformation table can be found as shown in table 3.4. For example, the fuzzy variable- low has its minimum of 0.00, mode of 0.10 and maximum of 0.30. The same definition is then applied to the other fuzzy variables.

3.3. Numerical example

Case I- For TOPSIS Method

A decision maker wants to choose a shop for that he decides four alternatives (S₁, S₂, S₃ and S₄). When making a decision, the attributes considered are:

R₁: Monthly rent cost (Rs.’000)
R₂: Size of shop (Sq. ft.)
R₃: distance from house to shop (Km.)
R₄: Locality of area. (Fuzzy assessment value)
Among the four attributes $R_1$ and $R_3$ are cost type and $R_2$ and $R_4$ are of benefit type. For linguistic attribute $R_4$ the decision maker appropriately makes use of the scales suggested by Chen and Hwang [16]. The decision matrix with four alternatives and four attributes are presented as follows:

Table 3.1

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>9</td>
<td>300</td>
<td>10</td>
<td>Rich (0.865)</td>
</tr>
<tr>
<td>$S_2$</td>
<td>7</td>
<td>250</td>
<td>8</td>
<td>Upper Middle (0.590)</td>
</tr>
<tr>
<td>$S_3$</td>
<td>3</td>
<td>150</td>
<td>6</td>
<td>Lower Middle (0.410)</td>
</tr>
<tr>
<td>$S_4$</td>
<td>6</td>
<td>200</td>
<td>4</td>
<td>Middle (0.500)</td>
</tr>
</tbody>
</table>

The decision matrix from table 3.1 is used for the TOPSIS method. Based on first step of the procedure, each element is normalized by Eq. (3.1). The resulting normalized matrix is shown in table 3.2.

Table 3.2: Normalized decision matrix

\[
R_{ij} = \begin{bmatrix}
0.6803 & 0.6469 & 0.6681 & 0.7029 \\
0.5292 & 0.5392 & 0.5344 & 0.7232 \\
0.2267 & 0.3235 & 0.4008 & 0.3332 \\
0.4535 & 0.4313 & 0.2672 & 0.4063
\end{bmatrix}
\]

With the set of weights $W = (0.30, 0.25, 0.25, 0.20)$, the weighted normalized matrix $V_{ij}$ is given by:

Table 3.3: Weighted normalized matrix

\[
V_{ij} = \begin{bmatrix}
0.2041 & 0.1617 & 0.1670 & 0.1406 \\
0.1587 & 0.1348 & 0.1336 & 0.1446 \\
0.0680 & 0.0808 & 0.1002 & 0.0666 \\
0.1361 & 0.1078 & 0.0668 & 0.0813
\end{bmatrix}
\]
The separation measures are
\[ S_1^+ = 0.1691 \quad S_2^+ = 0.1158 \quad S_3^+ = 0.1172 \quad S_4^+ = 0.1075 \]
\[ S_1^- = 0.1096 \quad S_2^- = 0.1103 \quad S_3^- = 0.1516 \quad S_4^- = 0.1249 \]

The relative closeness to the ideal solution is
\[ RC_1 = 0.3933, \quad RC_2 = 0.4878, \quad RC_3 = 0.5639, \quad RC_4 = 0.5374. \]

Therefore ranking the alternatives in descending order of their relative closeness; we get
\[ S_3 > S_4 > S_2 > S_1. \]

**Case II- For Fuzzy TOPSIS Method**

Table 3.1 numeric performance ratings are adopted again for the fuzzy TOPSIS analysis. In order to transform the performance rating in table 3.1 are normalized into the range of \([0, 1]\) by Eqs. (3.23) and (3.24) [19]:

(i) The larger the better type
\[ r_{ij} = \frac{x_i - \min \{x_j\}}{\max \{x_i\} - \min \{x_j\}}, \quad (3.23) \]

(ii) The smaller- the- better type
\[ r_{ij} = \frac{\max \{x_i\} - x_j}{\max \{x_i\} - \min \{x_j\}}, \quad (3.24) \]

For the present study, \( R_1 \) and \( R_3 \) are the smaller the better type, the others \( R_2 \) and \( R_4 \) belongs to the larger the better type. Then, table 3.4 can be transformed into table 3.5.
Table 3.4: Transformation for fuzzy membership function

<table>
<thead>
<tr>
<th>Membership Function</th>
<th>Membership Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Low (VL)</td>
<td>(0, 0, 0.1)</td>
</tr>
<tr>
<td>Low (L)</td>
<td>(0, 0.1, 0.3)</td>
</tr>
<tr>
<td>Medium low (ML)</td>
<td>(0.1, 0.3, 0.5)</td>
</tr>
<tr>
<td>Medium (M)</td>
<td>(0.3, 0.5, 0.7)</td>
</tr>
<tr>
<td>Medium High (MH)</td>
<td>(0.5, 0.7, 0.9)</td>
</tr>
<tr>
<td>High (H)</td>
<td>(0.7, 0.9, 1.0)</td>
</tr>
<tr>
<td>Very high (VH)</td>
<td>(0.9, 1.0, 1.0)</td>
</tr>
</tbody>
</table>

The next step uses the fuzzy membership function discussed in section 3.3.2 to transform table 3.5 into table 3.6 as explained by following example. If the numeric rating is 0.64, then its fuzzy linguistic variable is M. This transformation is also applied the weight \( W = (0.30, 0.25, 0.25, 0.20) \), for \( R_1, R_2, R_3, R_4 \) respectively.

Table 3.5: Decision matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0.3333 & 0.6666 & 0.3333 & 0.3956 \\
1 & 0 & 0.6666 & 0 \\
0.5 & 0.3333 & 1 & 0.1978 \\
\end{bmatrix}
\]

Then, the resulting fuzzy linguistic variables are show as table 3.6. The fuzzy attribute weight is also collected in table 3.6.

Table 3.6: Fuzzy Linguistic decision matrix with attribute weights

\[
\begin{bmatrix}
\text{VL} & \text{VH} & \text{VL} & \text{VH} \\
\text{ML} & \text{MH} & \text{ML} & \text{ML} \\
\text{VH} & \text{VL} & \text{MH} & \text{VL} \\
\text{M} & \text{ML} & \text{VH} & \text{L} \\
\end{bmatrix}
= w_j = [\text{MH} \quad \text{M} \quad \text{M} \quad \text{ML}]\]
The linguistic variable is then transformed into a fuzzy triangular membership function as shown in Table 3.7. This is the first step of the fuzzy TOPSIS analysis.

<table>
<thead>
<tr>
<th>Table 3.7: Fuzzy decision matrix and fuzzy attribute weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0, 0.0, 0.3) (0.9, 1.0, 0.1) (0.0, 0.0) (0.9, 1.0, 0.3)</td>
</tr>
<tr>
<td>(0.1, 0.3, 0.5) (0.5, 0.7, 0.9) (0.1, 0.3, 0.5) (0.1, 0.3, 0.5)</td>
</tr>
<tr>
<td>(0.9, 1.0) (0.0, 0.1) (0.5, 0.7, 0.9) (0.0, 0.1)</td>
</tr>
<tr>
<td>(0.3, 0.5, 0.7) (0.1, 0.3, 0.5) (0.9, 1.0) (0.0, 1.0, 0.3)</td>
</tr>
<tr>
<td>$w_j = [(0.5, 0.7, 0.9) (0.3, 0.5, 0.7) (0.3, 0.5, 0.7) (0.1, 0.3, 0.5)]$</td>
</tr>
</tbody>
</table>

The second step in the analysis is to find the weighted fuzzy decision matrix. Using Eq. (3.12), the fuzzy multiplication equation, the resulting fuzzy weighted decision matrix is shown as Table 3.8.

<table>
<thead>
<tr>
<th>Table 3.8: Fuzzy normalized weighted decision matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 0, 0.09) (0.27, 0.5, 0.7) (0, 0, 0.7) (0.09, 0.3, 0.5)</td>
</tr>
<tr>
<td>(0.05, 0.21, 0.45) (0.15, 0.35, 0.63) (0.03, 0.15, 0.35) (0.01, 0.09, 0.25)</td>
</tr>
<tr>
<td>(0.45, 0.7, 0.9) (0, 0, 0.7) (0.15, 0.35, 0.63) (0, 0, 0.05)</td>
</tr>
<tr>
<td>(0.15, 0.35, 0.63) (0.03, 0.15, 0.35) (0.27, 0.5, 0.7) (0, 0.03, 0.15)</td>
</tr>
</tbody>
</table>

According to table 3.8, we know that the elements $\tilde{v}_i$, $\forall i, j$ are normalized positive triangular fuzzy numbers and their range belongs to the closed interval [0, 1]. Thus, we can define the fuzzy positive ideal solution (FPIS, $A^+$) and the fuzzy negative solution (FNIS, $A^-$) as: $\tilde{v}_i^+ = (1, 1, 1)$ and $\tilde{v}_i^- = (0, 0, 0)$. This is third step of fuzzy TOPSIS analysis.

For the fourth step, the distance of each alternative from $A^+$ and $A^-$ can be currently calculated using Eqs. (3.20) and (3.21).

$$d_1^+ = 2.7701 \quad d_2^+ = 3.1571 \quad d_3^+ = 2.8381 \quad d_4^+ = 2.969$$
$$d_1^- = 0.9523 \quad d_2^- = 1.0871 \quad d_3^- = 1.2018 \quad d_4^- = 1.2543$$

The fifth step solves the similarities to an ideal solution by Eq. (3.22).

$CC_1 = 0.2560 \quad CC_2 = 0.2561 \quad CC_3 = 0.2975 \quad CC_4 = 0.2969$

The resulting fuzzy TOPSIS analysis found the preference for the four alternatives are $3 > 4 > 2 > 1$. 

78
3.4. Conclusions

From the TOPSIS method the best ideal solution from four alternatives is $S_3$. $S_4$ is second choice for selection of shop on rent. The decision maker will be choosing the third alternative according to minimum cost point of view. As we compared with second approach i.e. fuzzy TOPSIS method we get same ranking or preferences. Then the decision maker chooses the best alternative i.e. $S_3$.

If decision maker gives the different weights to the attribute the result may differ to change the priorities. By using proposed method decision maker may choose best optimal alternative selection everywhere. It has large applicability in agriculture too. It can be used by farmers in different situations such as for selection of water motor pump, variety of seeds, fertilizers, pesticides and so on. The proposed method gives good results for best selection.

All the methods lead to the choice of $S_3$ as the final selection. The fuzzy TOPSIS method concludes with the same top three alternatives as those of TOPSIS method. TOPSIS is a viable method for the proposed problem as it is suitable for the use of precise and numerical data, but when ratings are vague and inaccurate, then the fuzzy TOPSIS is the preferred technique.