Chapter 2

Characterization of Fuzzy Regular Languages

2.1 Introduction

In this chapter, we introduce the concept of fuzzy regular language and show that if $L$ is a fuzzy regular language, then every $\alpha$-cut $L_\alpha (\alpha \in [0, 1])$ is a regular language. We also give a characterization of fuzzy regular languages. The results of this chapter have been published in the International Journal of Computer Science and Network Security, VOL 8, No. 12, December 2008, page number 306-308.

2.2 Definitions and Theorems

Consider a finite nonempty set $A$. As mentioned in chapter 1, a fuzzy automaton over $A$ is a 4-tuple $M = (Q, \mathcal{F}, I, F)$ where $Q$ is a finite nonempty set, $\mathcal{F}$ is a fuzzy subset of $Q \times A \times Q$, $I$ and $F$ are fuzzy subsets of $Q$. Let $S$ be a free monoid with identity element $e$ generated by $A$. If $s \in S$, then $s$ can be written as $a_1a_2...a_n$ where $a_i \in A$. Here $n$ is called the length of $s$ and we write $|s| = n$. In the case of max-min composition, $\mathcal{F}$ can be extended to a function $\mathcal{F}^*: Q \times S \times Q \rightarrow [0, 1]$ as follows.

$$\mathcal{F}^*(q, e, p) = 1 \text{ if } q = p$$
$$= 0 \text{ otherwise.}$$

$$\mathcal{F}^*(q, sa, p) = \bigvee_{r \in Q} \left[ \mathcal{F}^*(q, s, r) \land \mathcal{F}(r, a, p) \right] \text{ (s } \in S, a \in A)$$

**Theorem 2.2.1:**

For $p, q \in Q, a \in A$, $\mathcal{F}^*(q, a, p) = \mathcal{F}(q, a, p)$

**Proof:**

We have,

$$\mathcal{F}^*(q, sa, p) = \bigvee_{r \in Q} \left[ \mathcal{F}^*(q, s, r) \land \mathcal{F}(r, a, p) \right] \text{ (s } \in S, a \in A)$$

If $s = e$, then (1) becomes
\[ f'(q, a, p) = f'(q, ea, p) = \bigvee \left[ f'(q, e, r) \land f(r, a, p) \right] (a \in A) \]

\[ r \in Q \]

We know \( f'(q, e, r) = 1 \) when \( r = q \)

= 0 otherwise.

Hence, \( f'(q, a, p) = f'(q, ea, p) = [ f'(q, e, q) \land f(q, a, p) ] \]

= \( f(q, a, p) \)

**Theorem 2.2.2:**

For any two elements \( s, t \in S \) and for all \( p, q \in Q \),

\[ f'(p, st, q) = \bigvee \left[ f'(p, s, r) \land f'(r, t, q) \right]. \]

\[ r \in Q \]

**Proof:**

First we prove the result when \(|t| = 1\).

If \(|t| = 1\) then \( t = a \), we have

\[ f'(p, st, q) = f'(p, sa, q) \]

= \( \bigvee \left[ f'(p, s, r) \land f(r, a, q) \right] \)

\[ r \in Q \]

= \( \bigvee \left[ f'(p, s, r) \land f'(r, a, q) \right] \)

\[ r \in Q \]

= \( \bigvee \left[ f'(p, s, r) \land f'(r, t, q) \right] \)

\[ r \in Q \]

Now assuming the result is true for \(|t| = k\), we will prove the result for \(|t| = k + 1\).

Let \( t = a_1a_2a_3....a_k a_{k+1} \)

= \( u a_{k+1} \) where \( u = a_1a_2a_3....a_k \).

We have \(|u| = k\)

\[ f'(p, st, q) = f'(p, sa_1a_2a_3....a_{k+1}, q) \]

= \( \bigvee \left[ f'(p, sa_1a_2a_3....a_k, r) \land f(r, a_{k+1}, q) \right] \) (by definition of \( f' \))

\[ r \in Q \]

= \( \bigvee \left[ f'(p, su, r) \land f(r, a_{k+1}, q) \right] \)

\[ r \in Q \]
\( f'(p, su, r) = \bigvee \left[ f'(p, s, v) \land f'(v, u, r) \right] \) (by inductive hypothesis)

\[ \forall v \in Q \]

\( f'(p, st, q) = \bigvee \left[ f'(p, s, v) \land f'(v, u, r) \land f(r, ak+1, q) \right] \)

\[ \forall r \in Q, v \in Q \]

\[ = \bigvee f'(p, s, r) \land f'(r, ua_{k+1}, q) \]

\[ \forall r \in Q \]

\[ = \bigvee [f'(p, s, r) \land f'(r, t, q)] \]

\[ r \in Q \]

In the case of max-product composition, \( f \) can be extended to a function \( f: Q \times S \times Q \rightarrow [0,1] \) as shown below.

\[ f'(q, e, p) = 1 \text{ if } q = p \]

\[ = 0 \text{ otherwise.} \]

\[ f'(q, sa, p) = \bigvee \left[ f'(q, s, r) \times f(r, a, p) \right] (s \in S, a \in A) \]

\[ r \in Q \]

**Theorem 2.2.3:**

For \( p, q \in Q, a \in A \), \( f'(q, a, p) = f(q, a, p) \)

**Proof:**

We have,

\[ f'(q, sa, p) = \bigvee \left[ f'(q, s, r) \times f(r, a, p) \right] (s \in S, a \in A) \]

\[ r \in Q \]

If \( s = e \), then (2) becomes

\[ f'(q, a, p) = f'(q, ea, p) = \bigvee \left[ f'(q, e, r) \times f(r, a, p) \right] (a \in A) \]

\[ r \in Q \]

We know \( f'(q, e, r) = 1 \) when \( r = q \)

\[ = 0 \text{ otherwise.} \]

Hence, \( f'(q, a, p) = f'(q, ea, p) = \left[ f'(q, e, q) \times f(q, a, p) \right] \)

\[ = [1 \times f(q, a, p)] \]

\[ = f(q, a, p) \]
Theorem 2.2.4:

For any two elements \( s, t \in S \) and for all \( p, q \in Q \),
\[
\hat{f}^* (p, st, q) = \bigvee \{ f^* (p, s, r) \times f^* (r, t, q) \}.
\]
\( r \in Q \)

Proof:

First we prove the result when \( |t| = 1 \).

If \( |t| = 1 \) then \( t = a \), we have
\[
\hat{f}^* (p, St, q) = \hat{f}^* (p, sa, q)
\]
\[
= \bigvee \{ f^* (p, sa, r) \times f (r, a, q) \}.
\]
\( reQ \)
\[
= \bigvee \{ f^* (p, s, v) \times f (v, a, q) \}.
\]
\( reQ \)

Now assuming the result is true for \( |t| = k \), we will prove the result for \( |t| = k + 1 \).

Let \( t = a_1 a_2 a_3 \ldots a_k a_{k+1} \)
\[
= u a_{k+1} \text{ where } u = a_1 a_2 a_3 \ldots a_k.
\]

We have \( |u| = k \)
\[
\hat{f}^* (p, St, q) = \hat{f}^* (p, sa_1 a_2 a_3 \ldots a_k a_{k+1}, q)
\]
\[
= \bigvee \{ f^* (p, sa_1 a_2 a_3 \ldots a_k, r) \times f (r, a_{k+1}, q) \} \text{ (by definition of } \hat{f}^*)
\]
\( reQ \)
\[
= \bigvee \{ f^* (p, su, r) \times f (r, a_{k+1}, q) \}
\]
\( reQ \)
\[
\hat{f}^* (p, su, r) = \bigvee \{ f^* (p, s, v) \times f^* (v, u, r) \} \text{ (by inductive hypothesis)}
\]
\( veQ \)
\[
\hat{f}^* (p, st, q) = \bigvee \bigvee \{ f^* (p, s, v) \times f^* (v, u, r) \} \wedge f (r, a_{k+1}, q) \}
\]
\( reQ \ veQ \)
\[
= \bigvee \bigvee \{ f^* (p, s, r) \times f^* (r, a_{k+1}, q) \}
\]
\( reQ \)
\[
= \bigvee \{ f^* (p, s, r) \times f^* (r, t, q) \}.
\]
\( reQ \)
In the case of max-average composition, \( f \) can be extended to a function \( \tilde{f}: Q \times S \times Q \rightarrow [0,1] \) as shown below.

\[
\tilde{f}(q, e, p) = 1 \text{ if } q = p \\
= 0 \text{ otherwise.}
\]

\[
f'(s, a, p) = \frac{1}{2} \vee \left[ f^*(s, a, q) + f(q, a, p) \right] \quad (s \in S, a \in A)
\]

In the case of max-average composition, it does not follow that \( \tilde{f} (q, a, p) = f (q, a, p) \).

Hence for max-average composition, we define a fuzzy automaton over \( S \) to be a 4-tuple \( M = (Q, \tilde{f}, I, F) \) where \( Q \), \( I \) and \( F \) are the same as in other compositions and \( f^* \) is a fuzzy subset of \( Q \times S \times Q \). We define the language accepted by \( M \) denoted by \( L(M) \) to be a fuzzy subset of \( S \) defined as,

\[
L(M)(s) = I \circ f^* \circ F \quad \text{for all } s \in S.
\]

Here \( \circ \) denotes either max-min composition or max-product composition or max-average composition as the case may be.

**Definition 2.2.1:**

A fuzzy subset \( L \) of \( S \) is said to be a fuzzy regular language if \( L = L(M) \) where \( M \) is a fuzzy automaton over \( S \). In what follows, we will assume that \( L \) is a fuzzy regular language. Since \( L \) is a fuzzy regular language, we have \( L = L(M) \) where \( M = (Q, \tilde{f}, I, F) \) is a fuzzy automaton over \( S \). \( L \) is a fuzzy subset of \( S \) defined as \( L(s) = I \circ f^* \circ F \) for all \( s \in S \). Here \( \circ \) denotes max-min composition and \( f^*: Q \times Q \rightarrow [0,1] \) is defined as \( f^*(p, q) = f^*(p, s, q) \) for all \( p, q \in Q \). Here \( \circ \) denotes either max-min composition or max-product composition or max-average composition as the case may be.

Define \( D_{\alpha} (M) = (Q, d_{\alpha}, I_{\alpha}, F_{\alpha}) \) where \( d_{\alpha}: Q \times S \rightarrow 2^Q \) is defined as

\[
d_{\alpha}(q, s) = \{ p \in Q \mid f'(q, s, p) \geq \alpha \}, \quad I_{\alpha} = \{ p \in Q \mid I(p) \geq \alpha \} \quad \text{and}
\]

\[
F_{\alpha} = \{ p \in Q \mid F(p) \geq \alpha \}.
\]

Define a relation \( R_{\alpha} \) as follows.

For all \( s, t \in S \), \( s R_{\alpha} t \) if and only if \( f'(p, s, q) \geq \alpha \) only when \( f^*(p, t, q) \geq \alpha \) for all \( p, q \in Q \).
Lemma 2.2.1: \( R_\alpha \) is a congruence relation.

Proof: \( R_\alpha \) is reflexive because \( f^*(p, s, q) \geq \alpha \) only when \( f^*(p, s, q) \geq \alpha \) obviously holds for all \( p, q \in Q \). If \( s R_\alpha t \), then \( f^*(p, s, q) \geq \alpha \) if and only if \( f^*(p, t, q) \geq \alpha \) for all \( p, q \in Q \). This means for all \( p, q \in Q \), \( f^*(p, t, q) \geq \alpha \) if and only if \( f^*(p, s, q) \geq \alpha \) proving that \( t R_\alpha s \). Hence \( R_\alpha \) is also symmetric.

Suppose \( s R_\alpha t \) and \( t R_\alpha u \). Since \( s R_\alpha t \), for all \( p, q \in Q \), \( f^*(p, s, q) \geq \alpha \) only when \( f^*(p, t, q) \geq \alpha \). Hence \( f^*(p, s, q) \geq \alpha \) only when \( f^*(p, u, q) \geq \alpha \) so that \( s R_\alpha u \). Hence \( R_\alpha \) is transitive. Since \( R_\alpha \) is reflexive, symmetric and transitive, \( R_\alpha \) is an equivalence relation.

To prove \( R_\alpha \) is a congruence relation, assume that \( s R_\alpha t \) and \( w \in S \). We will prove that \( sw R_\alpha tw \) and \( ws R_\alpha wt \).

To prove \( sw R_\alpha tw \), we have to prove that for all \( p, q \in Q \), \( f^*(p, sw, q) \geq \alpha \) only when \( f^*(p, tw, q) \geq \alpha \). Suppose \( f^*(p, sw, q) \geq \alpha \). We will prove that \( f^*(p, tw, q) \geq \alpha \). We have

\[
\alpha \leq f^*(p, sw, q) = \bigvee_{r \in Q} [ f^*(p, s, r) \land f^*(r, w, q)].
\]

Hence \( f^*(p, s, r) \land f^*(r, w, q) \geq \alpha \) for some \( r \in Q \). This means \( f^*(p, s, r) \geq \alpha \) from which it follows that \( f^*(p, t, r) \geq \alpha \) since \( s R_\alpha t \). Also \( f^*(r, w, q) \geq \alpha \). Now

\[
f^*(p, tw, q) = \bigvee_{z \in Q} [ f^*(p, t, z) \land f^*(z, w, q)] \geq \alpha.
\]

Similarly, we can prove that if \( f^*(p, tw, q) \geq \alpha \) then \( f^*(p, sw, q) \geq \alpha \). This proves that \( sw R_\alpha tw \). Using exactly a similar argument, we can prove that \( ws R_\alpha wt \) proving further that \( R_\alpha \) is a congruence relation.

Let \( E_\alpha = \{[s]_\alpha \mid s \in S \} \) where \([s]_\alpha \) denotes the equivalence class of \( s \). Define a binary operation \( *_\alpha \) on \( E_\alpha \) as \( [s]_\alpha *_\alpha [t]_\alpha = [st]_\alpha \).

Lemma 2.2.2: \((E_\alpha, *_\alpha)\) is a monoid.

Proof: We first have to prove that \( *_\alpha \) is well defined. Suppose \( u \in [s]_\alpha \) and \( v \in [t]_\alpha \).

We have to prove that \([st]_\alpha = [uv]_\alpha \). i.e., \( st R_\alpha uv \). We have \( s R_\alpha u \) and \( t R_\alpha v \). Suppose \( f'(p, st, q) \geq \alpha \) where \( p, q \in Q \). We will prove that \( f'(p, uv, q) \geq \alpha \). We have
\[ \alpha \preceq f^*(p, st, q) = \bigvee_{r \in Q} f^*(p, s, r) \land f^*(r, t, q). \]

Hence \[ f^*(p, s, r) \land f^*(r, t, q) \geq \alpha \] for some \( r \in Q \) which means \[ f^*(p, s, r) \geq \alpha \] and \[ f^*(r, t, q) \geq \alpha \]. Since \( s R_{\alpha} u \) and \( t R_{\alpha} v \), we obtain \[ f^*(p, u, r) \geq \alpha \] and \[ f^*(r, v, q) \geq \alpha \] so that \[ f^*(p, uv, q) = \bigvee_{z \in Q} (f^*(p, u, z) \land f^*(z, v, q)) \geq \alpha. \]

Similarly, we can prove that if \[ f^*(p, uv, q) \geq \alpha \], then \[ f^*(p, st, q) \geq \alpha \]. Hence \( st R_{\alpha} uv \) and \([st]_{\alpha} = [uv]_{\alpha}\). It is easy to see that \(*_{\alpha}\) is associative and \([e]_{\alpha}\) is the identity element. Hence \((E_{\alpha}, *_{\alpha})\) is a monoid.

**Theorem 2.2.5:** If \( L \) is a fuzzy regular language, then for every \( \alpha \in [0, 1] \), the \( \alpha \) - cut \( L_{\alpha} \) is a regular language.

**Proof:** Since \( L \) is a fuzzy regular language, \( L = L(M) \) for some fuzzy automaton \( M=(Q, f^*, I, F) \). We will prove that \( L_{\alpha} = L(D_{\alpha}(M)) \). This will mean that \( L_{\alpha} \) is the language accepted by a non deterministic automaton and hence a regular language.

Let \( s \in L_{\alpha} \). Then \( L(s) = L(M)(s) \geq \alpha \) i.e. \( I \circ f_{\alpha} \circ F = \bigvee_{p \in Q} \left( (f_{\alpha} \circ F)(p) \land I(p) \right) \geq \alpha \) which means \( (f_{\alpha} \circ F)(p) \land I(p) \geq \alpha \) for some \( p \in Q \). Hence \( (f_{\alpha} \circ F)(p) \geq \alpha \) and \( I(p) \geq \alpha \) so that \( p \in I_{\alpha} \). Again

\[ \alpha \preceq (f_{\alpha} \circ F)(p) = \bigvee_{r \in Q} (f_{\alpha}(p, r) \land F(r)) \] and hence \( f_{\alpha}(p, r) \land F(r) \geq \alpha \) so that \( F(r) \geq \alpha \) implying that \( r \in F_{\alpha} \) and \( f_{\alpha}(p, r) \geq \alpha \) implying that \( f^*(p, s, r) \geq \alpha \).

Thus \( r \in d_{\alpha}(p, s) \). We have thus proved that \( \exists p \in I_{\alpha} \) such that \( d_{\alpha}(p, s) \cap F_{\alpha} \neq \emptyset \)

which means \( s \in L(D_{\alpha}(M)) \). Thus \( L_{\alpha} \subseteq L(D_{\alpha}(M)) \).

Conversely, let \( s \in L(D_{\alpha}(M)) \). Then there exists \( p \in I_{\alpha} \) such that \( d_{\alpha}(p, s) \cap F_{\alpha} \neq \emptyset \). Let \( q \in d_{\alpha}(p, s) \cap F_{\alpha} \). Now \( p \in I_{\alpha} \) means \( I(p) \geq \alpha \), \( q \in d_{\alpha}(p, s) \) means \( f_{\alpha}(p, s, q) \geq \alpha \) i.e. \( f_{\alpha}(p, q) \geq \alpha \). Now \( q \in F_{\alpha} \) means \( F(q) \geq \alpha \).

Hence \( f_{\alpha}(p, q) \land F(q) \geq \alpha \) so that

\[ (f_{\alpha} \circ F)(p) = \bigvee_{r \in Q} (f_{\alpha}(p, r) \land F(r)) \geq \alpha. \]

Again \( I(p) \land (f_{\alpha} \circ F)(p) \geq \alpha \) means

\[ L(s) = L(M)(s) = I \circ f_{\alpha} \circ F = \bigvee_{t \in Q} \left( I(t) \land (f_{\alpha} \circ F)(t) \right) \geq \alpha. \]

This proves that \( s \in L_{\alpha} \). Thus \( L(D_{\alpha}(M)) \subseteq L_{\alpha} \). This together with \( L_{\alpha} \subseteq L(D_{\alpha}(M)) \) proves that \( L_{\alpha} = L(D_{\alpha}(M)) \).
Theorem 2.2.6: \( L_\alpha = \bigcup [s]_\alpha \) where the union is taken over all equivalence classes of \( s \) for which there exists \( p \in I_\alpha \) such that \( d_\alpha (p, s) \cap F_\alpha \neq \emptyset \).

Proof: Suppose \( t \in L_\alpha \). Since \( L_\alpha = L(D_\alpha (M)) \), there exists \( p \in I_\alpha \) such that \( d_\alpha (p, t) \cap F_\alpha \neq \emptyset \). Clearly \( t \in [t]_\alpha \). Conversely, assume that \( t \in \bigcup [s]_\alpha \) where the union is taken over all equivalence classes of \( s \) for which there exists \( p \in I_\alpha \) such that \( d_\alpha (p, s) \cap F_\alpha \neq \emptyset \). Then \( t \in [s]_\alpha \) for some \( s \). We know \( d_\alpha (p, s) \cap F_\alpha \neq \emptyset \) for some \( p \in I_\alpha \). Hence \( s \in L(D_\alpha (M)) = L_\alpha \) which means \( L(s) \geq \alpha \).

Let \( q \in d_\alpha (p, s) \cap F_\alpha \). Then \( f^*(p, s, q) \geq \alpha \) so that \( f^*(p, t, q) \geq \alpha \) since \( s \Rightarrow t \).

Hence \( q \in d_\alpha (p, t) \). Also \( q \in F_\alpha \) and hence \( d_\alpha (p, t) \cap F_\alpha \neq \emptyset \) where \( p \in I_\alpha \). This means \( t \in L(D_\alpha (M)) = L_\alpha \).

With every fuzzy regular language \( L \) and \( \alpha \in (0, 1] \), we associate a fuzzy subset \( \alpha_L \) (also denoted as \( \alpha.L_\alpha \)) of \( S \) as shown below.

If \( x \in S \), define \( \alpha_L (x) = \alpha \) if \( L(x) \geq \alpha \)

\[ 0 \] otherwise.

Result 2.2.1: \( L = \bigcup \alpha_L \) where \( \bigcup \) denotes fuzzy union.

Proof: Let \( s \in S \) and assume that \( L(s) = \beta \in (0, 1] \). Then \( \beta_L (s) = \beta \) so that

\[ \beta \leq \max \alpha_L (s) \] where the maximum is taken over all \( \alpha \in (0,1] \). This proves that

\[ L(s) = \beta \leq \bigcup \alpha_L (s) \quad \alpha \in (0, 1] \]

Now take any \( \gamma \in (0, 1] \). If \( \gamma \leq \beta = L(s) \), then \( \gamma_L (s) = \gamma \leq \beta \).

If \( \gamma > \beta = L(s) \), then \( \gamma_L (s) = 0 \leq \beta \).

Thus \( \gamma_L (s) \leq \beta \) for any \( \gamma \in (0, 1] \) so that \( \max \alpha_L (s) \leq \beta \).

\[ \alpha \in (0, 1] \]

In other words, \( \bigcup \alpha_L (s) \leq \beta \leq \bigcup \alpha_L (s) \) proving that \( \bigcup \alpha_L (s) = \beta = L(s) \)

\[ \alpha \in (0, 1] \quad \alpha \in (0, 1] \quad \alpha \in (0, 1] \]
2.3 Example

We will illustrate the above theorems with an example. Take $\Sigma = \{0, 1\}$ and let $S = \Sigma^*$, the set of all strings over the alphabet $\Sigma$. Consider the fuzzy automaton $M = (Q, f, I, F)$ where $Q = \{q_0, q_1, q_2\}$, $f$ is the fuzzy subset of $Q \times \Sigma \times Q$ defined as $f(q_0, 0, q_1) = 0.2$, $f(q_0, 0, q_2) = 0.6$, $f(q_0, 1, q_1) = 0.7$, $f(q_0, 1, q_2) = 0.8$, $f(q_1, 0, q_2) = 0.7$ and $f(q_1, 1, q_2) = 0.6$. $I = \{q_0\}$ and $F$ is the fuzzy subset of $Q$ defined as $F(q_1) = 0.5$ and $F(q_2) = 0.8$. We have

$$f_{00}^* (q_0, q_2) = f^* (q_0, 00, q_2)$$
$$= f(q_0, 0, q_1) \land f(q_1, 0, q_2)$$
$$= 0.2 \land 0.7$$
$$= 0.2$$

$$f_{01}^* (q_0, q_2) = f^* (q_0, 01, q_2)$$
$$= f(q_0, 0, q_1) \land f(q_1, 1, q_2)$$
$$= 0.2 \land 0.6$$
$$= 0.2$$

$$f_{10}^* (q_0, q_2) = f^* (q_0, 10, q_2)$$
$$= f(q_0, 1, q_1) \land f(q_1, 0, q_2)$$
$$= 0.7 \land 0.7$$
$$= 0.7$$

$$f_{11}^* (q_0, q_2) = f^* (q_0, 11, q_2)$$
$$= f(q_0, 1, q_1) \land f(q_1, 1, q_2)$$
$$= 0.7 \land 0.6$$
$$= 0.6$$

Also for any string $s$ of length two or more, $f^* (q_0, s, q_1) = 0$ and for any string $s$ of length three or more, $f^* (q_0, s, q_2) = 0$.

$L(0) = L(M)(0) = I \circ f_0^* \circ F$

$$= \lor [(f_0^* \circ F) (p) \land I(p)]$$
$$= (f_0^* \circ F) (q_0) \land I(q_0)$$
$$= (f_0^* \circ F) (q_0)$$
$$= [f_0^*(q_0, q_1) \land F(q_1)] \lor [f_0^*(q_0, q_2) \land F(q_2)]$$
$$= (0.2 \land 0.5) \lor (0.6 \land 0.8)$$
$$= 0.6.$
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$L(1) = L(M)(1) = I \circ f_1^* \circ F$

$= \bigvee [(f_1^* \circ F)(p) \land I(p)]$

$= (f_1^* \circ F)(q_0) \land I(q_0)$

$= [f_1^*(q_0, q_1) \land F(q_1)] \lor [f_1^*(q_0, q_2) \land F(q_2)]$

$= (0.7 \land 0.5) \lor (0.8 \land 0.8)$

$= 0.8$

$L(00) = L(M)(00) = I \circ f_{00}^* \circ F$

$= \bigvee [(f_{00}^* \circ F)(p) \land I(p)]$

$= (f_{00}^* \circ F)(q_0) \land I(q_0)$

$= [f_{00}^*(q_0, q_1) \land F(q_1)] \lor [f_{00}^*(q_0, q_2) \land F(q_2)]$

$= (0 \land 0.5) \lor (0.2 \land 0.8)$

$= 0.2$

$L(01) = L(M)(01) = I \circ f_{01}^* \circ F$

$= \bigvee [(f_{01}^* \circ F)(p) \land I(p)]$

$= (f_{01}^* \circ F)(q_0) \land I(q_0)$

$= [f_{01}^*(q_0, q_1) \land F(q_1)] \lor [f_{01}^*(q_0, q_2) \land F(q_2)]$

$= (0 \land 0.5) \lor (0.2 \land 0.8)$

$= 0.2$

$L(10) = L(M)(10) = I \circ f_{10}^* \circ F$

$= \bigvee [(f_{10}^* \circ F)(p) \land I(p)]$

$= (f_{10}^* \circ F)(q_0) \land I(q_0)$

$= [f_{10}^*(q_0, q_1) \land F(q_1)] \lor [f_{10}^*(q_0, q_2) \land F(q_2)]$

$= (0 \land 0.5) \lor (0.7 \land 0.8)$

$= 0.7$

$L(11) = L(M)(11) = I \circ f_{11}^* \circ F$

$= \bigvee [(f_{11}^* \circ F)(p) \land I(p)]$

$= (f_{11}^* \circ F)(q_0) \land I(q_0)$

$= (f_{11}^* \circ F)(q_0)$
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\[ = [f_{11}(q_0, q_1) \land F(q_1)] \lor [f_{11}(q_0, q_2) \land F(q_2)] \]

\[ = (0 \land 0.5) \lor (0.6 \land 0.8) \]

\[ = 0.6 \]

For any string \( s \) of length three or more, \( L(s) = 0. \)

Values of \( L(s) \) after arranging them in non-decreasing order are \( 0.2, 0.6, 0.7 \) and \( 0.8 \).

Suppose \( 0 < \alpha \leq 0.2 \).

Let \( D_\alpha(M) = M_\alpha \) denote the non deterministic automaton corresponding to \( \alpha \). Then it can be easily seen that \( I_\alpha = \{q_0\}, F_\alpha = \{q_1, q_2\}, d_\alpha(q_0, 0) = d_\alpha(q_0, 1) = \{q_1, q_2\}, \)

\( d_\alpha(q_0, 00) = d_\alpha(q_0, 01) = d_\alpha(q_0, 10) = d_\alpha(q_0, 11) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\} \).

Also \( L(D_\alpha(M)) = \alpha L(0) = \alpha L(1) = \alpha L(00) = \alpha L(01) = \alpha L(10) = \alpha L(11) = \alpha. \)

\( L(D_\alpha(M)) = L_\alpha \)

Furthermore, \( [0]_\alpha = \{0, 1\}, [00]_\alpha = \{00, 01, 10, 11\} \)

\( L_\alpha = U \{s\}_\alpha \)

\[ = [0]_\alpha U [00]_\alpha \]

\[ = \{0, 1, 00, 01, 10, 11\} \]

\( 0.2 < \alpha \leq 0.6. \)

Let \( D_\alpha(M) = M_\alpha \) denote the non deterministic automaton corresponding to \( \alpha \). Then it can be easily seen that \( I_\alpha = \{q_0\}, F_\alpha = \{q_2\}, d_\alpha(q_0, 0) = d_\alpha(q_0, 1) = \{q_2\}, d_\alpha(q_0, 10) = d_\alpha(q_0, 11) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\}. \)

Also \( L(D_\alpha(M)) = L_\alpha = \{0, 1, 10, 11\}, \alpha L(0) = \alpha L(1) = \alpha L(00) = \alpha L(01) = \alpha L(10) = \alpha L(11) = \alpha. \)

\( L(D_\alpha(M)) = L_\alpha \)

Furthermore, \( [0]_\alpha = \{0, 1\}, [10]_\alpha = \{10, 11\} \)

\( L_\alpha = U [s]_\alpha \)

\[ = [0]_\alpha U [10]_\alpha \]

\[ = \{0, 1, 10, 11\} \]

\( 0.6 < \alpha \leq 0.7. \)

Let \( D_\alpha(M) = M_\alpha \) denote the non deterministic automaton corresponding to \( \alpha \). Then it can be easily seen that \( I_\alpha = \{q_0\}, F_\alpha = \{q_2\}, d_\alpha(q_0, 1) = \{q_2\}, d_\alpha(q_0, 10) = \{q_2\}. \)
Also \( L(D_{\alpha}(M)) = L_{\alpha} = \{1, 10\} \), \( \alpha_L(1) = \alpha_L(10) = \alpha \), \( \alpha_L(0) = \alpha_L(00) = \alpha_L(01) = 0 \).

\( L(D_{\alpha}(M)) = L_{\alpha} \)

Furthermore, \([1]_{\alpha} = \{1\}, \ [10]_{\alpha} = \{10\}\)

\( L_{\alpha} = U_{[s]_{\alpha}} \)

\( = [1]_{\alpha} U [10]_{\alpha} \)

\( = \{1, 10\} \)

\( 0.7 < \alpha \leq 0.8 \)

Let \( D_{\alpha}(M) = M_{\alpha} \) denote the non deterministic automaton corresponding to \( \alpha \). Then it can be easily seen that \( I_{\alpha} = \{q_0\}, F_{\alpha} = \{q_2\}, d_{\alpha}(q_0, 1) = \{q_2\}, \)

Also \( L(D_{\alpha}(M)) = L_{\alpha} = \{1\}, \alpha_L(1) = \alpha, \)

\( \alpha_L(0) = \alpha_L(11) = \alpha_L(00) = \alpha_L(01) = \alpha_L(10) = 0 \)

\( L(D_{\alpha}(M)) = L_{\alpha} \)

Furthermore, \([1]_{\alpha} = \{1\}, \)

\( L_{\alpha} = U_{[s]_{\alpha}} \)

\( = [1]_{\alpha} \)

\( = \{1\} \)

If \( \alpha > 0.8 \), then there is no corresponding non deterministic automaton and

\( L(D_{\alpha}(M)) = L_{\alpha} = \emptyset \). Also,

\( L = \cup \alpha_L \) where \( \cup \) denotes fuzzy union.

\( \alpha \in (0, 1] \)

\( (\cup \alpha_L)(0) = \vee \alpha_L(0) \)

\( = 0.2 \)

\( = L(0) \)

\( (\cup \alpha_L)(1) = \vee \alpha_L(1) \)

\( = 0.2 \vee 0.6 \vee 0.7 \vee 0.8 \)

\( = 0.8 \)

\( = L(1) \)
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\((\cup \alpha_L)(00) = \vee \alpha_L(00)\)
\[= 0.2\]
\[= L(00)\]

\((\cup \alpha_L)(01) = \vee \alpha_L(01)\)
\[= 0.2\]
\[= L(01)\]

\((\cup \alpha_L)(10) = \vee \alpha_L(10)\)
\[= 0.2 \vee 0.6 \vee 0.7\]
\[= 0.7\]
\[= L(10)\]

\((\cup \alpha_L)(11) = \vee \alpha_L(11)\)
\[= 0.2 \vee 0.6\]
\[= 0.6\]
\[= L(11)\]

This verifies that \(L = \cup \alpha_L\).