Chapter 5

Myhill Nerode Theorem for Fuzzy automata
(max-average composition)

5.1 Introduction

In this chapter, Myhill Nerode theorem of finite automaton has been extended to fuzzy automaton where the composition considered is max-average composition. In max-product composition, we have observed that if $L$ is a fuzzy regular language, then for any $\alpha \in [0, 1]$, $L_\alpha \subseteq L(D_\alpha (M))$ though the two are not equal as in max-min composition. As can be seen in the following example, in the case of max-average composition, $L_\alpha$ need not even be contained in $L(D_\alpha (M))$.

While writing this thesis, we observed a small problem. Suppose we define a fuzzy automaton as a 4-tuple $M = (Q, f, I, F)$ as in max-min and max-product compositions. We then define $f^*: Q \times S \times Q \rightarrow [0, 1]$ as follows.

$$f^*(q, e, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{otherwise} \end{cases}$$

$$f^*(q, sa, p) = \frac{1}{2} \vee \left[f^*(q, s, r) + f(r, a, p)\right] \quad (s \in S, a \in A)$$

It so happens that $f^*(q, a, p)$ is not equal to $f(q, a, p)$. Hence we define a fuzzy automaton to be a 4-tuple $M = (Q, f^*, I, F)$ where $Q, I$ and $F$ are same as in max-min and max-product compositions but $f^*$ is a fuzzy subset of $Q \times S \times Q$. Now we are not sure whether $L(M)(s) (s \in S)$ assumes only finitely many values, a fact which has been used in the proof of Myhill-Nerode theorem. This needs further investigation which we hope to complete by the time of viva.

5.2 Example

Let $\Sigma = \{0, 1\}$ and $S = \Sigma^*$, the set of all strings over the alphabet $\Sigma$. Consider the fuzzy automaton $M = (Q, f, I, F)$ where $Q = \{q_0, q_1, q_2\}$, $f$ is the fuzzy subset of $Q \times \Sigma \times Q$ defined as $f(q_0, 0, q_1) = 0.5$, $f(q_0, 0, q_2) = 0.6$, $f(q_0, 1, q_1) = 0.3$, $f(q_0, 1, q_2) = 0.4$, $f(q_1, 0, q_2) = 0.7$ and $f(q_1, 1, q_2) = 1$. $I = \{q_0\}$ and $F$ is the fuzzy subset of $Q$ defined as $F(q_1) = 0.3$ and $F(q_2) = 0.9$. 

64
We have
\[ f_{00}^* (q_0, q_2) = f^* (q_0, 00, q_2) = \frac{1}{2} [f(q_0, 0, q_1) + f(q_1, 0, q_2)] = \frac{1}{2} [0.5 + 0.7] = 0.6 \]

Similarly,
\[ f_{01}^* (q_0, q_2) = f^* (q_0, 01, q_2) = \frac{1}{2} [f(q_0, 0, q_1) + f(q_1, 1, q_2)] = \frac{1}{2} [0.5 + 1.0] = 0.75 \]
\[ f_{10}^* (q_0, q_2) = f^* (q_0, 10, q_2) = \frac{1}{2} [f(q_0, 1, q_1) + f(q_1, 0, q_2)] = \frac{1}{2} [0.3 + 0.7] = 0.5 \]
\[ f_{11}^* (q_0, q_2) = f^* (q_0, 11, q_2) = \frac{1}{2} [f(q_0, 1, q_1) + f(q_1, 1, q_2)] = \frac{1}{2} [0.3 + 1.0] = 0.65 \]

Also for any string \( s \) of length two or more, \( f^* (q_0, s, q_1) = 0 \) and for any string \( s \) of length three or more, \( f^* (q_0, s, q_2) = 0 \).

\[ L (0) = L (M)(0) = I o f_0^* o F \]
\[ = \frac{1}{2} [(f_0^* o F) (p) + l(p)] = \frac{1}{2} [(f_0^* o F) (q_0) + l(q_0)] = \frac{1}{2} [(f_0^* o F) (q_0)] = \frac{1}{2} [f_0^* (q_0, q_1) + f(q_1)] \lor \frac{1}{2} [f_0^* (q_0, q_2) + f(q_2)] = \frac{1}{2} [f(q_0, 0, q_1) + f(q_1)] \lor \frac{1}{2} [f(q_0, 0, q_2) + f(q_2)] = \frac{1}{2} (0.5 + 0.3) \lor \frac{1}{2} (0.6 + 0.9) = 0.4 \lor 0.75 = 0.75 \]
Studies in Fuzzy Automata

\[ L(1) = L(M)(1) = 1 \circ f^*_1 \circ F \]
\[ = \frac{1}{2} [f^*_1(q_0, q_1) + F(q_1)] \vee \frac{1}{2} [f^*_1(q_0, q_2) + F(q_2)] \]
\[ = \frac{1}{2} [f(q_0, 1, q_1) + F(q_1)] \vee \frac{1}{2} [f(q_0, 1, q_2) + F(q_2)] \]
\[ = \frac{1}{2} (0.3 + 0.3) \vee \frac{1}{2} (0.4 + 0.9) \]
\[ = 0.3 \vee 0.65 \]
\[ = 0.65 \]

\[ L(00) = L(M)(00) = 1 \circ f^*_{00} \circ F \]
\[ = \frac{1}{2} [f^*_{00}(q_0, q_1) + F(q_1)] \vee \frac{1}{2} [f^*_{00}(q_0, q_2) + F(q_2)] \]
\[ = \frac{1}{2} [f(q_0, 00, q_1) + F(q_1)] \vee \frac{1}{2} [f(q_0, 00, q_2) + F(q_2)] \]
\[ = \frac{1}{2} (0 + 0.3) \vee \frac{1}{2} (0.6 + 0.9) \]
\[ = 0.15 \vee 0.75 \]
\[ = 0.75 \]

\[ L(01) = L(M)(01) = 1 \circ f^*_{01} \circ F \]
\[ = \frac{1}{2} [f^*_{01}(q_0, q_1) + F(q_1)] \vee \frac{1}{2} [f^*_{01}(q_0, q_2) + F(q_2)] \]
\[ = \frac{1}{2} [f(q_0, 01, q_1) + F(q_1)] \vee \frac{1}{2} [f(q_0, 01, q_2) + F(q_2)] \]
\[ = \frac{1}{2} (0 + 0.3) \vee \frac{1}{2} (0.75 + 0.9) \]
\[ = 0.15 \vee 0.825 \]
\[ = 0.825 \]

\[ L(10) = L(M)(10) = 1 \circ f^*_{10} \circ F \]
\[ = \frac{1}{2} [f^*_{10}(q_0, q_1) + F(q_1)] \vee \frac{1}{2} [f^*_{10}(q_0, q_2) + F(q_2)] \]
\[ = \frac{1}{2} [f(q_0, 10, q_1) + F(q_1)] \vee \frac{1}{2} [f(q_0, 10, q_2) + F(q_2)] \]
\[ = \frac{1}{2} (0 + 0.3) \vee \frac{1}{2} (0.75 + 0.9) \]
\[ = 0.15 \vee 0.825 \]
\[ = 0.825 \]

\[ L(11) = L(M)(11) = 1 \circ f^*_{11} \circ F \]
\[ = \frac{1}{2} [f^*_{11}(q_0, q_1) + F(q_1)] \vee \frac{1}{2} [f^*_{11}(q_0, q_2) + F(q_2)] \]
\[ = \frac{1}{2} [f(q_0, 11, q_1) + F(q_1)] \vee \frac{1}{2} [f(q_0, 11, q_2) + F(q_2)] \]
\[ = \frac{1}{2} (0 + 0.3) \vee \frac{1}{2} (0.65 + 0.9) \]

66
For any string \( s \) of length three or more, \( L(s) = 0 \).
The possible values of \( \delta_i \) (after arranging them in nondecreasing order) are 0.65, 0.75, 0.775 and 0.825.

When \( \delta_1 = 0.65 \), \( L_{0.65} = \{0, 1, 00, 01, 10, 11\} \), \( L(D_{0.65}(M)) = \{01, 11\} \) proving that \( L_{\delta_1} \subset L(D_{\delta_1}(M)) \).

When \( \delta_2 = 0.75 \), \( L_{0.75} = \{0, 00, 01, 10, 11\} \), \( L(D_{0.75}(M)) = \{01\} \) proving that \( L_{\delta_2} \subset L(D_{\delta_2}(M)) \).

When \( \delta_3 = 0.775 \), \( L_{0.775} = \{01, 10, 11\} \), \( L(D_{0.75}(M)) = \emptyset \) proving that \( L_{\delta_3} \subset L(D_{\delta_3}(M)) \).

When \( \delta_4 = 0.825 \), \( L_{0.825} = \{01, 10\} \), \( L(D_{0.825}(M)) = \emptyset \) proving that \( L_{\delta_4} \subset L(D_{\delta_4}(M)) \).

Nevertheless, it is true (as in the case of max-min and max-product compositions) that \( L = (\delta_1)_L \cup (\delta_2)_L \cup (\delta_3)_L \cup (\delta_4)_L \) which is verified below.

\[
L(0) = 0.75 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L](0) \\
= \max \{(0.65)_L(0), (0.75)_L(0), (0.775)_L(0), (0.825)_L(0)\} \\
= \max \{0.65, 0.75, 0, 0\} \\
= 0.75
\]

\[
L(1) = 0.65 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L](1) \\
= \max \{(0.65)_L(1), (0.75)_L(1), (0.775)_L(1), (0.825)_L(1)\} \\
= \max \{0.65, 0, 0, 0\} \\
= 0.65
\]

\[
L(00) = 0.75 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L](00)
\]
Studies in Fuzzy Automata

\[ L(01) = 0.825 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L] (01) \]
\[ = \max \{ (0.65)_L (01), (0.75)_L (01), (0.775)_L (01), (0.825)_L (01) \} \]
\[ = \max \{ 0.65, 0.75, 0.775, 0.825 \} \]
\[ = 0.825 \]

\[ L(10) = 0.825 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L] (10) \]
\[ = \max \{ (0.65)_L (10), (0.75)_L (10), (0.775)_L (10), (0.825)_L (10) \} \]
\[ = \max \{ 0.65, 0.75, 0.775, 0.825 \} \]
\[ = 0.825 \]

\[ L(11) = 0.775 \cdot [(0.65)_L \cup (0.75)_L \cup (0.775)_L \cup (0.825)_L] (11) \]
\[ = \max \{ (0.65)_L (11), (0.75)_L (11), (0.775)_L (11), (0.825)_L (11) \} \]
\[ = \max \{ 0.65, 0.75, 0.775, 0 \} \]
\[ = 0.775 \]

This verifies that \( L = \bigcup \alpha_L \)
\[ \alpha \in [0, 1] \]

5.3 Myhill Nerode theorem for fuzzy automata

Let \( S \) be a monoid with identity element \( e \) and \( L \) be a fuzzy subset of \( S \). Then the following statements are equivalent.

(i) \( L \) is a fuzzy regular language.

(ii) \( L \) can be expressed as a fuzzy union

\[ L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L \]

where \( \delta_1, \delta_2, \ldots, \delta_t \in [0, 1] \). For each \( i = 1, \ldots, t \), \( (\delta_i)_L = \delta_i \in L_{\delta_i} \) where \( L_{\delta_i} = \bigcup [s]_{\delta_i} \).

This union is a set theoretic union and \( [s]_{\delta_i} \) denotes the equivalence class of \( s \) of a right invariant equivalence relation of finite index in \( L_{\delta_i} \).

(iii) Define a relation \( R_L \) as follows.
If s, t ∈ S, then s RL t if and only if for all u ∈ S and for all α ∈ [0,1], L(su) ≥ α only when L(tu) ≥ α. Then RL is a right invariant equivalence relation of finite index.

Proof of (i) → (ii)
Since L is a fuzzy regular language, we have L = L(M) where M = (Q, f*, I, F) is a fuzzy automaton. Consider any α ∈ [0,1]. With M and α, we associate a non deterministic automaton Dα(M) = (Q, dα, Iα, Fα) where dα: Q × S → 2Q is defined as
dα(q, s) = {p ∈ Q | f*(q, s, p) ≥ α}, Iα = {p ∈ Q | I(p) ≥ α} and Fα = {p ∈ Q | F(p) ≥ α}. For the sake of simplicity, we will denote L(Dα(M)) by Lα(M).
Let s ∈ Lα. Then L(s) = L(M)(s) ≥ α. i.e (L ⊕ f* ⊕ F) > α which means
\[ \forall \text{ } p \in Q \left[ (f_\ast p + F)(p) + I(p) \right] \geq \alpha \]
Hence (f_\ast p + F)(p) + I(p) ≥ 2α for some p ∈ Q. Following three cases now occur.

**Case A:** (f_\ast p + F)(p) ≥ α and I(p) ≥ α
**Case B:** (f_\ast p + F)(p) ≥ α and I(p) < α
**Case C:** I(p) ≥ α and (f_\ast p + F)(p) < α

We now consider each case separately.

**Case A:** (f_\ast p + F)(p) ≥ α and I(p) ≥ α. In this case, p ∈ Iα.
Now (f_\ast p + F)(p) ≥ α means
\[ \forall \text{ } r \in Q \left[ f_\ast (p, r) + F(r) \right] \geq \alpha \text{ so that } f_\ast (p, r) + F(r) \geq 2\alpha \text{ for some } r \in Q \]
This leads to the following three cases:

**Case A1:** f_\ast (p, r) ≥ α and F(r) ≥ α
**Case A2:** f_\ast (p, r) ≥ α and F(r) < α
**Case A3:** f_\ast (p, r) < α and F(r) ≥ α

**Case A1:** f_\ast (p, r) ≥ α and F(r) ≥ α
Thus \( r \in d_\alpha (p, s) \) and \( r \in F_\alpha \) so that \( d_\alpha (p, s) \cap F_\alpha = \emptyset \) where \( p \in I_\alpha \). This proves that \( s \in L(D_\alpha (M)) = L\alpha (M) \). \hspace{1cm} (1)

**Case A_2:** \( f_* (p, r) \geq \alpha \) and \( F(r) < \alpha \)

Let \( F(r) = \beta \) \( \beta \leq \alpha \) means \( r \in F_\beta \). \( I(p) \geq \alpha > \beta \) means \( p \in I_\beta \)

\[ f_* (p, r) \geq \alpha > \beta \text{ means } f^*(p, s, r) > \beta \]

Thus \( r \in d_\beta (p, s) \) and \( d_\beta (p, s) \cap F_\beta = \emptyset \) proving that

\[ s \in L(D_\beta (M)) = L\beta (M) \]. \hspace{1cm} (2)

**Case A_3:** \( f_* (p, r) < \alpha \) and \( F(r) \geq \alpha \)

Let \( f_* (p, r) = \gamma < \alpha \). Then \( f^*(p, s, r) = \gamma \) so that \( r \in d_\gamma (p, s) \)

\[ F(r) \geq \alpha > \gamma \text{ means } r \in F_\gamma \). \( I(p) \geq \alpha > \gamma \text{ means } p \in I_\gamma \)

Thus \( r \in d_\gamma (p, s) \) and \( d_\gamma (p, s) \cap F_\gamma = \emptyset \) proving that

\[ s \in L(D_\gamma (M)) = L\gamma (M) \]. \hspace{1cm} (3)

**Case B:** \( (f_* \circ F)(p) \geq \alpha \) and \( I(p) < \alpha \)

\[ \forall r \in Q \left[ f_* (p, r) + F(r) \right] \geq \alpha \text{ means } f_* (p, r) + F(r) \geq 2\alpha \] for some \( r \in Q \)\n
This leads to the following three cases.

**Case B_1:** \( f_* (p, r) \geq \alpha \) and \( F(r) \geq \alpha \)

**Case B_2:** \( f_* (p, r) \geq \alpha \) and \( F(r) < \alpha \)

**Case B_3:** \( f_* (p, r) < \alpha \) and \( F(r) \geq \alpha \)

**Case B_1:** \( f_* (p, r) \geq \alpha \) and \( F(r) \geq \alpha \). We already have \( I(p) < \alpha \).

Let \( I(p) = \lambda < \alpha \). Then \( p \in I_\lambda \). \( F(r) \geq \alpha > \lambda \text{ means } r \in F_\lambda \)

\[ f_* (p, r) \geq \alpha > \lambda \text{ implies } f^*(p, s, r) > \lambda \]

Thus \( r \in d_\lambda (p, s) \) and \( d_\lambda (p, s) \cap F_\lambda = \emptyset \) proving that

\[ s \in L(D_\lambda (M)) = L\lambda (M) \]. \hspace{1cm} (4)
.Case B2: $f^*_*(p, r) \geq \alpha$ and $F(r) < \alpha$. We already have $I(p) < \alpha$.

Let $I(p) = \rho < \alpha$. Then $p \in I_\rho$. Let $F(r) = \varphi < \alpha$. Then $r \in F_\varphi$.

If $\rho > \varphi$, then $I_\rho \subseteq I_\varphi$ so that $p \in I_\varphi$.

If $\rho < \varphi$, then $r \in F_\varphi \subseteq F_\rho$ and $f^*(p, s, r) \geq \alpha > \rho \varphi$ implies $r \in d_\varphi (p, s)$ and $d_\varphi (p, s) \cap F_\varphi \neq \emptyset$ proving that $s \in L(D_\varphi (M)) = L_\varphi (M)$. \hfill (5)

If $\rho > \varphi$, then $r \in d_\varphi (p, s)$ so that $r \in d_\rho (p, s)$.

Hence $d_\varphi (p, s) \cap F_\rho \neq \emptyset$ where $p \in I_\rho$ proving that

$s \in L(D_\rho (M)) = L_\rho (M)$. \hfill (6)

Case B3: $f^*_*(p, r) < \alpha$ and $F(r) \geq \alpha$. We already have $I(p) < \alpha$.

Let $I(p) = \pi < \alpha$. Then $p \in I_\pi$.

Now $F(r) \geq \alpha > \pi \Rightarrow r \in F_\pi$.

Let $f^*_*(p, r) = \mu < \alpha$.

If $\mu \leq \pi$, then $F(r) \geq \alpha > \mu \Rightarrow r \in F_\mu$. $f^*(p, s, r) = \mu$ means $r \in d_\mu (p, s)$. Hence $d_\mu (p, s) \cap F_\mu \neq \emptyset$. Also $l(p) = \pi > \mu \Rightarrow p \in I_\mu$.

This proves that $s \in L(D_\mu (M)) = L_\mu (M)$. \hfill (7)

If however $\mu > \pi$, then $f^*(p, s, r) = \mu > \pi$ means $r \in d_\pi (p, s)$. Also $r \in F_\pi$ so that $d_\pi (p, s) \cap F_\pi \neq \emptyset$ where $p \in I_\pi$. This proves that

$s \in L(D_\pi (M)) = L_\pi (M)$. \hfill (8)

Case C: $(f^*_* \circ F) (p) < \alpha$ and $I(p) \geq \alpha$.

$\forall \frac{1}{2} \left[ f^*_*(p, r) + F(r) \right] < \alpha \Rightarrow f^*_*(p, r) < F(r) < 2\alpha \text{ for some } r \in Q$

$r \in Q$

This leads to the following three cases.

Case C1: $f^*_*(p, r) < \alpha$ and $F(r) < \alpha$

Case C2: $f^*_*(p, r) < \alpha$ and $F(r) \geq \alpha$

Case C3: $f^*_*(p, r) \geq \alpha$ and $F(r) < \alpha$

Case C1: $f^*_*(p, r) < \alpha$ and $F(r) < \alpha$. We already have $I(p) \geq \alpha$.

Let $f^*_*(p, r) = v < \alpha$ and $F(r) = \Omega < \alpha$.

First assume that $v > \Omega$. Now $F(r) = \Omega$ means $r \in F_\Omega$. Also
Studies in Fuzzy Automata

\( f^*(p, s, r) = v \geq \Omega \) means \( r \in d_\Omega (p, s) \). Hence \( d_\Omega (p, s) \cap F_\Omega \neq \phi \). Also \( I(p) \geq \alpha > v > \Omega \) means \( p \in I_\Omega \). Hence

\[ s \in L(D_\Omega(M)) = L_\Omega(M). \tag{9} \]

Suppose \( v < \Omega \). Now \( F(r) = \Omega > v \) means \( r \in F_v \). \( f^*(p, s, r) = v \) means \( r \in d_\alpha (p, s) \) so that \( d_\alpha (p, s) \cap F_v \neq \phi \). Also \( I(p) \geq \alpha > v \) means \( p \in I_v \).

Hence \( s \in L(D_v(M)) = L_v(M). \tag{10} \)

**Case C2:** \( f^*(p, r) < \alpha \) and \( F(r) \geq \alpha \). We already have \( I(p) \geq \alpha \).

Let \( f^*(p, r) = \sigma < \alpha \). Then \( f^*(p, s, r) = \sigma \) so that \( r \in d_\sigma (p, s) \).

\( I(p) \geq \alpha > \sigma \) means \( p \in I_\sigma \). Also \( F(r) \geq \alpha > \sigma \) means \( r \in F_\sigma \). Hence

\[ d_\sigma (p, s) \cap F_\sigma \neq \phi \] where \( p \in I_\sigma \) proving that

\[ s \in L(D_\sigma(M)) = L_\sigma(M). \tag{11} \]

**Case C3:** \( f^*(p, r) \geq \alpha \) and \( F(r) < \alpha \). We already have \( I(p) \geq \alpha \).

Let \( F(r) = \eta < \alpha \). Then \( r \in F_\eta \). \( I(p) \geq \alpha > \eta \) means \( p \in I_\eta \).

\( f^*(p, r) \geq \alpha > \eta \) so that \( f^*(p, s, r) > \eta \). Thus \( r \in d_\eta (p, s) \) so that

\[ d_\eta (p, s) \cap F_\eta \neq \phi \] where \( p \in I_\eta \). This proves that

\[ s \in L(D_\eta(M)) = L_\eta(M). \tag{12} \]

From (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11) and (12), it follows that

\[ L_\alpha \subseteq L_\alpha (M) \cup L_\beta (M) \cup L_\gamma (M) \cup L_\lambda (M) \cup L_\varphi (M) \cup L_\rho (M) \cup L_\mu (M) \cup L_\pi (M) \cup L_\Omega (M) \cup L_v (M) \cup L_\sigma (M) \cup L_\eta (M) \]

where \( \alpha, \beta, \gamma, \lambda, \varphi, \rho, \pi, \nu, \Omega, \mu, \sigma, \eta \in [0, 1] \).

Let \( Q = \{q_0, q_1, q_2, \ldots, q_n\} \). As in max-product composition, there can be only finitely many values of \( L(s) \) \( (s \in S) \). Denote these fixed values (after arranging them in non-decreasing order) by \( \delta_1, \delta_2, \ldots, \delta_i \). Then \( \delta_1, \delta_2, \ldots, \delta_i \in [0, 1] \) and for each \( i \) \((1 \leq i \leq t)\),

\[ L_{\delta_i} \subseteq (L_{\alpha_i}(M) \cup L_{\beta_i}(M) \cup L_{\gamma_i}(M) \cup L_{\lambda_i}(M) \cup L_{\varphi_i}(M) \cup L_{\rho_i}(M) \cup L_{\mu_i}(M) \cup L_{\pi_i}(M) \cup L_{\Omega_i}(M) \cup L_{v_i}(M) \cup L_{\sigma_i}(M) \cup L_{\eta_i}(M)). \]

Since \( L_{\delta_i}(M) \) is the language accepted by a finite automaton, by Myhill Nerode theorem for finite automata, it follows that there exists a right invariant equivalence relation \( R_i \) of finite index. Let \( R_i^\prime \) denote its restriction on \( L_{\delta_i} \). Similarly, we obtain other restrictions like \( N_i^\prime, O_i^\prime, P_i^\prime, Q_i^\prime, S_i^\prime, T_i^\prime, U_i^\prime, V_i^\prime, W_i^\prime, X_i^\prime, \) and \( Y_i^\prime \) from
Studies in Fuzzy Automata

$L_{\delta i} (M), L_{\tau i} (M), L_{\lambda i} (M), L_{\phi i} (M), L_{\pi i} (M), L_{\ni i} (M), L_{\eta i} (M), L_{\phi i} (M), L_{\eta i} (M), L_{\tau i} (M), L_{\eta i} (M)$ respectively. Note that $N_i', O_i', P_i', Q_i', S_i', T_i', U_i', V_i', W_i', X_i', Y_i'$ are all right invariant equivalence relations of finite index and hence $Z_i' = R_i' \cap N_i' \cap O_i' \cap P_i' \cap Q_i' \cap S_i' \cap T_i' \cap U_i' \cap V_i' \cap W_i' \cap X_i' \cap Y_i'$ is a right invariant equivalence relation of finite index in $L_{\delta i}$. Let $[s]_{\delta i}$ denote the equivalence class of $s$ under this equivalence relation. Since the equivalence classes partition $L_{\delta i}$, it follows that $L_{\delta i} = \cup [s]_{\delta i}$. The fact that $L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L$ follows exactly as in max-min composition (Theorem 3.2.1 of chapter 3).

The proof of (ii) $\rightarrow$ (iii) is the same as in max-min composition (Theorem 3.2.1 of chapter 3).

Proof of (iii) $\rightarrow$ (i)

We have to define a fuzzy automaton $M$ such that $L = L (M)$. For every element $s \in S$, let $[s]$ denote the equivalence class of $s$ under the equivalence relation $R_L$.

Let $Q = \{ [s] / s \in S \}$. Since $R_L$ is of finite index, it follows that $Q$ is a finite set.

Define

$I: Q \rightarrow [0, 1], f^*: Q \times S \times Q \rightarrow [0, 1]$ and $F: Q \rightarrow [0, 1]$ as follows.

$I ([s]) = 1$ if $[s] = [e]$ 
$= 0$ otherwise.

$f^* ([v], s, [u]) = 2L (vs) - 1$ if $L (vs) \geq 0.5$ and $[u] = [vs]$ 
$= 0$ otherwise.

$F ([s]) = 2L (s) - 1$ if $L (s) \geq 0.5$
$= 0$ otherwise.

As in max-min composition ((iii) $\rightarrow$ (i) of theorem 3.2.1 in chapter 3), we can prove that $[s] = [v]$, then $L (s) = L (v)$. This proves that $F$ is well defined. To prove $f^*$ is well defined, assume that $w \in [v]$. Then $v R_L w$ and since $R_L$ is right invariant, $vs R_L ws$.

Hence $[vs] = [ws]$ so that $L (vs) = L (ws)$. If $L (vs) \geq 0.5$ and $[u] = [vs]$, then $L (ws) \geq 0.5$ and $[u] = [ws]$. Thus $f^* ([v], s, [u]) = 2L (ws) - 1$. 


Now

\[ L(M)(s) = \frac{1}{2} \vee \{I([p]) + (f^* \circ F)([p])\} \]

\[ [p] \]

\[ (f^* \circ F)([p]) = \frac{1}{2} \vee \{f^* ([p], [r]) + F([r])\} \]

\[ [r] \]

\[ = \frac{1}{2} \vee \{f^* ([p], [r]) + F([r])\} \]

\[ [u] \]

\[ = \frac{1}{2} \{ 2 L(ps) - 1 + 2 L(ps) - 1 \} \text{ where } [r] = [ps] \]

\[ = 2 L(ps) - 1 \]

\[ L(M)(s) = \frac{1}{2} \{ 1 + 2 L(ps) - 1 \} \text{ where } [p] = [e] \]

\[ = L(ps) \]

\[ = L(es) \]

\[ = L(s) \]

(Note that \([p] = [e]\) means \(p \text{ RL } e\) and since \(R_L\) is right invariant, \(ps \text{ RL } es\) so that \([ps] = [es]\) and hence \(L(ps) = L(es)\).)