Chapter 1

Prerequisites

1.1 Introduction

Distributions are introduced as functional on very good function spaces, called test function spaces.

If $\phi$ is a function defined on an open subset $\Omega$ of $\mathbb{R}$, the closure of the set $\{x \in \Omega : \phi(x) \neq 0\}$ is called the support of the function $\phi$ and denoted supp $\phi$.

The set of all complex-valued infinitely differentiable functions $\phi$ defined on $\Omega$ an open subset of $\mathbb{R}$ and having compact support is denoted by $\mathcal{D}(\Omega)$ (or $C^\infty_c(\Omega)$). It is a linear space. Its elements are called Schwartz test functions or simply test functions. The topology on $\mathcal{D}(\Omega)$ is generated by the sequence of seminorms $\{\gamma_\nu\}_{\nu=0}^\infty$ defined by

$$\gamma_\nu(\phi) = \sup_{x \in \mathbb{R}} |D^\nu \phi(x)| < \infty, \quad \forall \, \nu \in \mathbb{N}_0. \quad (1.1.1)$$

A continuous linear functional on $\mathcal{D}(\Omega)$ is called a distribution. The space of all such distributions is denoted by $\mathcal{D}'$. $\mathcal{D}'$ is called the dual space of $\mathcal{D}$.

1.2 The $L^p(\mathbb{R})$ space

Let $L^p(\mathbb{R})$ denote the class of measurable functions $f$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|^p dx < \infty$ for $1 \leq p < \infty$. 


Also let $L^\infty(\mathbb{R})$ be the collection of almost everywhere bounded function. Then we write

$$
\|f\|_p = \left\{ \begin{array}{ll}
\left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty \\
\text{ess sup}_{x \in \mathbb{R}} |f(x)| & \text{for } p = \infty.
\end{array} \right.
$$

Each $L^p(\mathbb{R})$, $1 \leq p < \infty$ is a Banach space.

1.3 Dirac delta function and its properties

The Dirac delta function, or $\delta$ function, is (informally) a generalized function depending on a real parameter such that it is zero for all values of the parameter except when the parameter is zero, and its integral over the parameter from $-\infty$ to $\infty$ is equal to one. It is a continuous analog of the Kronecker delta function which is usually defined on a finite domain, and takes values 0 and 1. From a purely mathematical viewpoint, the Dirac delta is not strictly a function, because any real function that is equal to zero everywhere but a single point must have total integral zero.

As a distribution, the Dirac delta is a linear functional on the space of test functions $\mathcal{D}$ and is defined by

$$
< \delta, \phi > = \phi(0)
$$

for every test function $\phi \in \mathcal{D}(\mathbb{R})$.

The following properties of Dirac delta function are given below:

(i) The Dirac delta function satisfies scaling property for a non-zero scalar $k$

$$
\int_{\mathbb{R}} \delta(kx) \, dx = \int_{\mathbb{R}} \delta(u) \frac{du}{|k|} = \frac{1}{|k|},
$$

and so

$$
\delta(kx) = \frac{\delta(x)}{|k|}.
$$

(ii) $\delta$ has (many) derivatives: and then $\int_{\mathbb{R}} \phi(x) \delta'(x-a) \, dx = -\phi'(a)$ and in general

$$
\phi^{(n)}(a) = (-1)^n \int_{\mathbb{R}} \delta^{(n)}(x-a) \phi(x) \, dx,
$$

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and

\[ x\delta(x) = 0, \ x\delta'(x) = -\delta(x). \]  \hspace{1cm} (1.3.5)

(iii) Dirac delta function can be defined as the following limit:

\[ \delta(x-a) = \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-\frac{(x-a)^2}{4\varepsilon}}. \]  \hspace{1cm} (1.3.6)

### 1.4 The Schwartz space $\mathcal{S}(\mathbb{R})$

**Definition 1.4.1** The space $\mathcal{S}$, the so-called space of smooth functions of rapid descent, is defined as follows: $\phi$ is member of $\mathcal{S}$ iff it is a complex valued $C^\infty$-function on $\mathbb{R}$ and for every choice of $\beta$ and $\nu$ of non-negative integers, it satisfies

\[ \gamma_{\beta,\nu}(\phi) = \sup_{x \in \mathbb{R}} \left| x^\beta D^\nu \phi(x) \right| < \infty. \]  \hspace{1cm} (1.4.1)

If $f$ is a locally integrable function on $\mathbb{R}$ (i.e. $f \in L^1(\mathbb{R})$) then $f$ generates a distribution in $\mathcal{S}'$ as follows:

\[ <f, \phi> = \int_{\mathbb{R}} f(x)\phi(x)dx, \quad \phi \in \mathcal{S}. \]

From [21, p. 58], we have a function $\phi \in C^\infty(\mathbb{R})$ that satisfies (1.4.1) if and only if

\[ \tau_{m,\beta}(\phi) = \sup_{x \in \mathbb{R}} \left| (1 + |x|^2)^{m/2} D^\beta \phi(x) \right| < \infty, \quad \forall \ m, \beta \in \mathbb{N}_0. \]  \hspace{1cm} (1.4.2)

A linear functional $T$ on $\mathcal{S}$ is called a tempered distribution if for any sequence $\{\phi_j\}$ of functions in $\mathcal{S}$ converging to zero in $\mathcal{S}$ we have

\[ <T, \phi_j> \to 0 \quad \text{as} \quad j \to \infty. \]  \hspace{1cm} (1.4.3)

The space of all tempered distribution is denoted by $\mathcal{S}'$, the dual of $\mathcal{S}$.

### 1.5 The Fourier transform

The classical Fourier transform of a function $\phi \in L^1(\mathbb{R})$ in one dimension, is defined by

\[ \hat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(x)dx, \quad \xi \in \mathbb{R}. \]  \hspace{1cm} (1.5.1)
If \( \hat{\phi} \in L^1(\mathbb{R}) \), then the inverse Fourier transform is given by

\[
\phi(x) = (\mathcal{F}^{-1}\hat{\phi})(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \hat{\phi}(\xi) d\xi, \quad x \in \mathbb{R}.
\]

(1.5.2)

The Fourier transform (1.5.1) was extended to distributions of slow growth using the method of adjoint operators by Schwartz [32]. The generalized Fourier transform of \( \phi \in \mathcal{S}' \) is defined by

\[
\langle \mathcal{F}'\phi, f \rangle = \langle \phi, \mathcal{F}f \rangle, \quad \forall f \in \mathcal{S}(\mathbb{R}).
\]

(1.5.3)

The classical Fourier transform is an automorphism on \( \mathcal{S} \) and the generalized Fourier transform is an automorphism on \( \mathcal{S}' \).

### 1.6 The fractional Fourier transform

The fractional Fourier transform \((FrFT)\) is a rotation operation on the time-frequency distribution and it can transform a function (either in the time domain or frequency domain) into the domain between time and frequency. The fractional Fourier transform is a generalization of the ordinary Fourier transform with a parameter \( \theta \), has many applications in several areas, including signal and image processing, optics and quantum physics, etc.

The one dimensional fractional Fourier transform [1, 4, 16, 28, 29, 38] with parameter \( \theta \) of \( f(x) \) denoted by \((\mathcal{F}\theta f(x))(\xi) = \hat{f}^\theta(\xi)\) is given by

\[
\hat{f}^\theta(\xi) = (\mathcal{F}\theta f(x))(\xi) = \int_{\mathbb{R}} K^\theta(x, \xi) f(x) dx,
\]

(1.6.1)

where the kernel

\[
K^\theta(x, \xi) = \begin{cases} 
  e^{i\theta} e^{-\frac{(ix^2 + \xi^2)\cot\theta}{2}} \csc\theta & \text{if } \theta \neq n\pi, \quad \forall n \in \mathbb{Z}, \\
  \frac{1}{\sqrt{2\pi}} e^{ix\xi} & \text{if } \theta = \frac{\pi}{2}
\end{cases}
\]

and

\[
c^\theta = \frac{(2\pi i\sin\theta)^{\frac{1}{2}}}{2\pi \sin\theta} e^{-\frac{i\theta}{2}} = \sqrt{\frac{1 + i\cot\theta}{2\pi}}
\]

and its inversion formula is

\[
f(x) = \int_{\mathbb{R}} K^\theta(x, \xi) \hat{f}^\theta(\xi) d\xi,
\]

(1.6.2)
where
\[
\mathcal{K}^\theta (x, \xi) = (2\pi i \sin \theta)^{-\frac{1}{2}} e^{i\theta x^2 \csc \theta} e^{-\frac{i}{2} e^{i\theta} (x^2 + \xi^2)} e^{-ix \xi \cot \theta} = K^{-\theta} (x, \xi),
\]
\[
c^\theta = (2\pi \sin \theta)^{-\frac{1}{2}} e^{i\theta x^2} = \sqrt{\frac{1 - \cot \theta}{2\pi}} = c^{-\theta}.
\]
From this, we conclude that the inverse of a FrFT with parameter \(\theta\) is the FrFT with parameter \(-\theta\).

### 1.7 Relationship between fractional Fourier transform and Fourier transform

The fractional Fourier transform
\[
(\mathcal{F}^\theta f(x))(\xi) = c^\theta \int e^{-i(x^2 + \xi^2)\cot \theta} + i\xi \csc \theta f(x) dx
\]
\[
= \sqrt{2\pi} c^\theta e^{-\frac{i\xi^2}{2}} \mathcal{F} (e^{-\frac{i\xi^2}{2}} f(x))(\xi \csc \theta).
\]
(1.7.1)

Replacing \(f(x) = e^{i\frac{x^2 \cot \theta}{2}} \phi(x)\) in equation (1.7.1), we obtain
\[
\mathcal{F}^\theta \left( e^{i\frac{x^2 \cot \theta}{2}} \phi(x) \right)(\xi) = \sqrt{2\pi} c^\theta e^{-\frac{i\xi^2}{2}} \mathcal{F} (\phi(x))(\xi \csc \theta).
\]
(1.7.2)

Substituting \(\xi = \tau \sin \theta\) in equation (1.7.2), we have
\[
\mathcal{F}^\theta \left( e^{i\frac{x^2 \cot \theta}{2}} \phi(x) \right)(\tau \sin \theta) = \sqrt{2\pi} c^\theta e^{-\frac{i\xi^2}{4}} \mathcal{F} \left( e^{-i\xi^2/4} \phi(x) \right)(\tau).
\]
(1.7.3)

### 1.8 The Hankel transform

The Hankel transform \(h_\mu\) of a conventional function \(\phi \in L^1(\mathbb{R}_+), \mathbb{R}_+ = (0, \infty)\) is defined by
\[
\hat{\phi}(x) = (h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \phi(y) dy, \quad \mu \geq -1/2
\]
(1.8.1)
and its inverse is given by
\[ \phi(y) = \left(h_\mu^{-1}\hat{\phi}(x)\right)(y) = \int_0^\infty (xy)^{1/2}J_\mu(xy)\hat{\phi}(x)dx, \quad (1.8.2) \]
where \( J_\mu \) is the Bessel function of the first kind of order \( \mu \). The precise conditions of validity of reciprocal relations (1.8.1) and (1.8.2) are given by:

**Theorem 1.8.1** Let \( \phi(y) \in L^1(\mathbb{R}_+) \) be of bounded variation in a neighborhood of the point \( y = y_0 > 0 \), \( \mu \geq -1/2 \) and \( \hat{\phi}(x) \) be defined by (1.8.1). Then we have
\[ \frac{1}{2} [\phi(y_0 + 0) + \phi(y_0 - 0)] = \left(h_\mu^{-1}\phi\right)(y_0) = \int_0^\infty \hat{\phi}(x)(y_0x)^{1/2}J_\mu(y_0x)dx. \quad (1.8.3) \]

The proof of the following Parseval theorem can be found from Zemanian [39].

For \( \mu \geq -1/2 \), let \( \hat{\phi}(x) = (h_\mu\phi)(x) \) and \( \hat{\psi}(x) = (h_\mu\psi)(x) \). Assume further that \( \phi(y) \) and \( \psi(y) \) belong to \( L^1(\mathbb{R}_+) \). Then
\[ \int_0^\infty \phi(y)\psi(y)dy = \int_0^\infty \hat{\phi}(x)\hat{\psi}(x)dx. \quad (1.8.4) \]

If the boundary conditions are impulsive in nature then (1.8.1) is not applicable to the problem and we need a generalization of (1.8.1), which is called the distributional Hankel transform.

Zemanian [39] introduced the function space \( H_\mu \) consisting of all complex valued infinitely differentiable functions \( \phi \) defined on \( \mathbb{R}_+ = (0, \infty) \) satisfying
\[ \gamma_{m,k}^\mu(\phi) = \sup_{x \in \mathbb{R}_+} \left|x^m(x^{-1}d/dx)^kx^{-\mu-1/2}\phi(x)\right| < \infty, \quad \forall \ m, k \in \mathbb{N}_0. \quad (1.8.5) \]

The space \( L^p_\mu(\mu \geq -1/2) \) is the set of all measurable functions \( \phi \) on \( \mathbb{R}_+ = (0, \infty) \) such that
\[ \|\phi\|_p^\mu = \int_0^\infty |\phi(x)|^p x^{\mu+1/2}dx < \infty. \quad (1.8.6) \]

The Hankel translation of \( \phi \in L^1_\mu(\mathbb{R}_+) \) is defined by
\[ \phi(w,z) = (\tau_z\phi)(w) = \int_0^\infty \phi(y)D_\mu(y,w,z)dy, \quad \forall \ w, z \in \mathbb{R}_+ \quad (1.8.7) \]
where
\[ D_\mu(y, w, z) = \int_0^\infty t^{-\mu - 1/2} j_\mu(yt) j_\mu(wt) j_\mu(zt) \, dt, \] (1.8.8)
and
\[ j_\mu(wt) = (wt)^{1/2} J_\mu(wt). \] (1.8.9)

The Hankel convolution transform of two functions \( \phi, \psi \in L^1_\mu(\mathbb{R}^+) \) is defined by
\[ (\phi \# \psi)(z) = \int_0^\infty \phi(w)(\tau_z \psi)(w) \, dw, \quad a.e., \quad z \in \mathbb{R}^+ \] (1.8.10)
we shall also make use of the following results [3, p. 285]
\[ h_\mu(\tau_z \phi)(u) = u^{-\mu - 1/2} j_\mu(uz)(h_\mu \phi)(u), \quad \forall u, z \in \mathbb{R}^+ \] (1.8.11)
and
\[ h_\mu(\phi \# \psi)(u) = u^{-\mu - 1/2}(h_\mu \phi)(u)(h_\mu \psi)(u), \quad \forall u \in \mathbb{R}^+. \] (1.8.12)

The following properties of the Bessel function will be needed in the sequel.
\[ J_\mu(x) = \sum_{k=0}^{\infty} (-1)^k (x/2)^{\mu + 2k} k! \Gamma(\mu + k + 1), \] (1.8.13)
\[ J_\mu(x) = \begin{cases} \frac{O(x^\mu)}{x} & x \to 0^+ \\ (2/\pi x)^{1/2} \cos(x - \frac{\mu \pi}{2} - \frac{x}{4}) + O(x^{-1}) & x \to \infty, \end{cases} \] (1.8.14)
\[ (x^{-1} d/dx)^k (x^\mu J_\mu(x)) = (-1)^k x^{-\mu - k} J_{\mu + k}(x), \] (1.8.15)
\[ (x^{-1} d/dx)^k (x^\mu J_\mu(x)) = x^{\mu - k} J_{\mu - k}(x). \] (1.8.16)

From the integral representation
\[ J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin x - \nu x) \, dx - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-(2 \sinh \gamma + \nu \gamma)} \, d\gamma, \] (1.8.17)
given in [10, Equation (9), p. 17] it follows that there exists a constant \( P > 0 \) independent of \( \nu \in \mathbb{R} \), such that
\[ \sup_{z \in \mathbb{R}^+} |J_\nu(z)| \leq P. \] (1.8.18)
1.9 Properties of the Hankel transform

The differential operators $N_\mu, M_\mu$ and $S_\mu$ are defined by

\[ N_\mu = N_\mu x = x^{\mu+1/2} (d/dx) x^{-\mu-1/2}, \]
\[ (1.9.1) \]

\[ M_\mu = M_\mu x = x^{-\mu-1/2} (d/dx) x^\mu + 1/2 \]
\[ (1.9.2) \]

\[ S_\mu = S_\mu x = M_\mu N_\mu = \frac{d^2}{dx^2} + \frac{(1 - 4\mu^2)}{4x^2}. \]
\[ (1.9.3) \]

We have the following relations for any $\phi \in H_\mu$:

\[ h_{\mu+1}(-x\phi) = N_\mu h_\mu \phi, \]
\[ (1.9.4) \]

\[ h_{\mu+1}(N_\mu \phi) = -y h_\mu \phi, \]
\[ (1.9.5) \]

\[ h_\mu (S_\mu \phi) = -y^2 h_\mu \phi \]
\[ (1.9.6) \]

and

\[ S'_\mu \phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2} (x^{-1} d/dx)^{r+j} x^{-\mu-1/2} \phi(x), \]
\[ (1.9.7) \]

where the $b_j$ are constants depending on $\mu$.

The following formula is given in [39, pp. 129, 134, 141] and [20, pp. 240, 242, 295].

\[ (x^{-1} d/dx)^k (x^{-\mu-1/2} \psi \phi) = \sum_{v=0}^{k} \binom{k}{v} (x^{-1} d/dx)^v (x^{-\mu-1/2} \phi)(x^{-1} d/dx)^{k-v} \psi, \]
\[ (1.9.8) \]

and

\[ (-1)^k x^k \left( x^{-1} \frac{d}{dx} \right)^m x^{-\mu-1/2} (h_\mu \phi)(x) = \int_0^\infty y^{2\mu+k+2m+1} \left( y^{-1} \frac{d}{dy} \right)^k y^{-\mu-1/2} \phi(y) \]
\[ \times (xy)^{-(\mu+m)} J_{\mu+k+m}(xy) \ dy, \ \forall \ k, \ m \in \mathbb{N}_0. \]
\[ (1.9.9) \]
1.10 The spaces of type $S$

Gel’fand and Shilov [11] introduced spaces of type $S$, viz. $S_\alpha$, $S_\beta$ and $S^\beta_\alpha$ as follows:

**Definition 1.10.1** The space $S_\alpha$, $\alpha \geq 0$, consists of all infinitely differentiable functions $\phi(x)$, $-\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} \left| x^k \left( \frac{d}{dx} \right)^q \phi(x) \right| \leq C_q A_k^{k \alpha}, \quad k, q \in \mathbb{N}_0,$$

where the constants $A$ and $C_q$ depend on the function $\phi$. For $k = 0$, the expression $k^{k \alpha}$ is considered equal to 1.

**Definition 1.10.2** The space $S_\beta$, $\beta \geq 0$, consists of all infinitely differentiable functions $\phi(x)$, $-\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} \left| x^k \left( \frac{d}{dx} \right)^q \phi(x) \right| \leq C_k B^q q^{\alpha \beta}, \quad k, q \in \mathbb{N}_0,$$

where the constants $B$ and $C_k$ depend on the function $\phi$.

**Definition 1.10.3** The space $S^\beta_\alpha$, $\alpha \geq 0$, $\beta \geq 0$, consists of all infinitely differentiable functions $\phi(x)$, $-\infty < x < \infty$, satisfying the inequalities

$$\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} \left| x^k \left( \frac{d}{dx} \right)^q \phi(x) \right| \leq C A_k^k B^q q^{\alpha \beta}, \quad k, q \in \mathbb{N}_0,$$

where the constants $A$, $B$ and $C$ depend on the function $\phi$.

Type-S spaces are closely interrelated by means of the Fourier transformation; namely, the formulae

$$\mathcal{S}_\alpha = S_\alpha, \quad \mathcal{S}_\beta = S_\beta \quad \text{and} \quad \mathcal{S}^\beta_\alpha = S^\beta_\alpha$$

hold.

We shall make use of the following inequalities in our investigation:

$$\frac{q!}{(q-k)!} = k! \binom{q}{k} \leq k! \sum_{k=0}^{q} \binom{q}{k} = k! 2^q,$$
and

\[(k + q)^{(k+q)\alpha} \leq k^{k\alpha} q^{q\alpha} e^{k\alpha} e^{q\alpha}, \forall k, q \in \mathbb{N}_0. \quad (1.10.6)\]

Pathak and Singh [26] defined the following similar test function spaces, on the upper half-space \( H = \mathbb{R} \times \mathbb{R}_+ \), called spaces of type \( \tilde{S} \) as follows:

**Definition 1.10.4** The space \( \tilde{S}_{\alpha}(\mathbb{R} \times \mathbb{R}_+) \), \( \tilde{\alpha} = (\alpha_1, \alpha_2) \), \( \alpha_1, \alpha_2 \geq 0 \), is defined as the space of all functions \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \) such that, for all \( l, s, k, t \in \mathbb{N}_0 \),

\[
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} \sup_{l+s \leq k+t} \left| a^l b^s \left( \frac{\partial}{\partial a} \right)^k \left( \frac{\partial}{\partial b} \right)^t \phi(b,a) \right| \\
\leq C_{k,t} A_1^l A_2^s l^\alpha_1 s^\alpha_2 t^\alpha_2, \quad (1.10.7)
\]

where the constants \( A_1, A_2 \) and \( C_{k,t} \) depend on the testing function \( \phi \).

**Definition 1.10.5** The space \( \tilde{S}_{\beta}(\mathbb{R} \times \mathbb{R}_+) \), \( \tilde{\beta} = (\beta_1, \beta_2) \), \( \beta_1, \beta_2 \geq 0 \), is defined as the space of all functions \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \) such that, for all \( l, s, k, t \in \mathbb{N}_0 \),

\[
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} \sup_{l+s \leq k+t} \left| a^l b^s \left( \frac{\partial}{\partial a} \right)^k \left( \frac{\partial}{\partial b} \right)^t \phi(b,a) \right| \\
\leq C_{l,s} B_1^k B_2^s k^\beta_1 t^\beta_2, \quad (1.10.8)
\]

where the constants \( B_1, B_2 \) and \( C_{l,s} \) depend on the testing function \( \phi \).

**Definition 1.10.6** The space \( \tilde{S}_{\alpha}(\mathbb{R} \times \mathbb{R}_+) \), \( \tilde{\alpha} = (\alpha_1, \alpha_2) \), \( \tilde{\beta} = (\beta_1, \beta_2) \), \( \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0 \), is defined to be the space of all functions \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \) such that

\[
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} \sup_{l+s \leq k+t} \left| a^l b^s \left( \frac{\partial}{\partial a} \right)^k \left( \frac{\partial}{\partial b} \right)^t \phi(b,a) \right| \\
\leq C A_1^l A_2^s B_1^k B_2^s l^\alpha_1 s^\alpha_2 k^{\beta_1} t^{\beta_2}, \forall l, s, k, t \in \mathbb{N}_0, \quad (1.10.9)
\]

where the constants \( A_1, A_2, B_1, B_2 \) and \( C \) depend on \( \phi \) defined as above.
1.11 The spaces of type $W$

Gel’fand and Shilov [12] introduced spaces of type $W$, viz. $W_{M,\alpha}$, $W_{\Omega,\beta}$ and $W_{M,\alpha}^{\Omega,\beta}$.

Let $\mu(\xi), (0 \leq \xi < \infty)$ and $\omega(\eta), (0 \leq \eta < \infty)$ be continuous increasing functions such that $\mu(0) = 0$, $\mu(\xi) \to \infty$ for $\xi \to \infty$ and $\omega(0) = 0$, $\omega(\eta) \to \infty$ for $\eta \to \infty$. For $x \geq 0$, $y \geq 0$, we define

$$M(x) = \int_0^x \mu(\xi) d\xi, \quad M(x) = M(-x) \quad for \quad x < 0, \quad (1.11.1)$$

and

$$\Omega(y) = \int_0^y \omega(\eta) d\eta, \quad \Omega(y) = \Omega(-y) \quad for \quad y < 0. \quad (1.11.2)$$

The functions $M(x)$ and $\Omega(y)$ are continuous, increasing and convex with $M(0) = 0$, $M(x) \to \infty$ for $x \to \infty$ and $\Omega(0) = 0$, $\Omega(y) \to \infty$ for $y \to \infty$. We have the following fundamental convex inequalities:

$$M(x_1) + M(x_2) \leq M(x_1 + x_2), \quad \Omega(y_1) + \Omega(y_2) \leq \Omega(y_1 + y_2). \quad (1.11.3)$$

If the functions $\mu(\xi)$ and $\omega(\eta)$ are mutually inverse, that is, $\mu(\omega(\eta)) = \eta$, $\omega(\mu(\xi)) = \xi$ then the corresponding functions $M(x)$ and $\Omega(y)$ will be said to be dual in the sense of Young.

In this case, the following Young’s inequality

$$xy \leq M(x) + \Omega(y), \quad (1.11.4)$$

holds for $x \geq 0$, $y \geq 0$.

**Definition 1.11.1** The space $W_{M,\alpha}$ consists of all complex valued infinitely differentiable functions $\phi(x)$, $(-\infty < x < \infty)$ which for any $\delta > 0$ satisfy the inequality

$$\left| \left( \frac{d}{dx} \right)^q \phi(x) \right| \leq C_{q,\delta} \exp[-M(\alpha - \delta)x], \quad q = 0, 1, 2, \ldots \quad (1.11.5)$$

where $C_{q,\delta} > 0$ and $\alpha > 0$ are constants which may depend on the function $\phi$. 

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Definition 1.11.2 The space $W_{\Omega, \beta}$ consists of all entire analytic functions $\psi(z)$, ($z = x + iy \in \mathbb{C}$) which for any $\rho > 0$ satisfy the inequality

$$|z^k \psi(z)| \leq C_{k, \rho} \exp[\Omega(\beta + \rho)y], \quad k = 0, 1, 2, \ldots$$  \hspace{1cm} (1.11.6)

where the constants $C_{k, \rho} > 0$ and $\beta > 0$ depend on the function $\psi$.

Definition 1.11.3 The space $W_{M, \alpha}$ consists of all entire analytic functions $\phi(z)$, ($z = x + iy \in \mathbb{C}$) which for any $\delta, \rho > 0$ satisfy the inequality

$$|\phi(z)| \leq C_{\delta, \rho} \exp[-M(\alpha - \delta)x + \Omega(\beta + \rho)y];$$  \hspace{1cm} (1.11.7)

where the constants $C_{\delta, \rho} > 0$ and $\alpha, \beta > 0$ depend on the function $\phi$.

Pathak and Pandey [27] defined the following similar test function spaces, called spaces of type $\tilde{W}$ which will be used in the study of the continuous fractional wavelet transform.

Definition 1.11.4 The space $\tilde{W}_{M, \alpha}$, where $\alpha > 0$, is defined to be the set of all complex valued infinitely differentiable functions $\phi_a(b) = \phi(b, a) \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ which for any $\delta > 0$ satisfy the inequality

$$\left| \left( \frac{\partial}{\partial b} \right)^k \left( \frac{\partial}{\partial a} \right)^l \phi(b, a) \right| \leq C_{k, l, \delta} e^{-M\left[ \frac{\delta}{1+\delta} (\alpha - \delta) \right]}; \quad k, l = 0, 1, 2, \ldots$$  \hspace{1cm} (1.11.8)

where $C_{k, l, \delta} > 0$ is constants depend on the function $\phi$.

Definition 1.11.5 The space $\tilde{W}^*_{\Omega, \alpha, \alpha, \alpha}$, where $\alpha > 0$, is defined to be the set of all functions $\phi_a(s) = \phi(s, a) \in C^\infty(\mathbb{C} \times \mathbb{R}_+)$ entire analytic with respect to $s = b + i\lambda$ which for any $\rho, \rho' > 0$ satisfy the inequality

$$\left| \left( \frac{1}{1+|a|} \right)^t \left( \frac{\partial}{\partial a} \right)^l \phi(s, a) \right| \leq C_{t, \rho} \exp[\Omega(\lambda + \rho)] + \Omega[\lambda(\alpha + \rho')]; \quad t = 0, 1, 2, \ldots$$  \hspace{1cm} (1.11.9)

where $C_{t, \rho} > 0$ is constants depend on the function $\phi$. 

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1.12 The wavelet transform

The word wavelet stands for small wave. Wavelets are relatively a new development in mathematics and have already made a significant impact in many areas of pure mathematics and applied sciences ranging from harmonic analysis and approximation theory to signal analysis and image compression. The concept of wavelets originated from the study of time-frequency signal analysis, wave propagation, and sampling theory. One of the main reasons for the discovery of wavelets and wavelet transforms is that the Fourier transform analysis does not contain the local information of signals. So the Fourier transform cannot be used for analyzing signals in a joint time and frequency domain. Wavelets were introduced by J. Morlet et al. [18] as a tool for signal analysis. Their theory was developed by Chui [5], Daubechies [6], Debnath [7], Grossmann [13], Pathak [22] and others.

Wavelets constitute a family of functions indexed by two levels, one for position and other for dilation. More explicitly, we define

\[ \psi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \psi \left( \frac{x-b}{a} \right), \quad a \neq 0, \ b \in \mathbb{R}. \]  

(1.12.1)

The parameter ‘b’ in \( \psi_{b,a} \) gives the position of the wavelet, while the dilation parameter ‘a’ governs its oscillation.

The single function \( \psi(x) \) is called the mother wavelet. The mother wavelet appears as a local oscillation or wave, in which most of the energy of the oscillation is located in a narrow region in the physical space. This localization in the physical space limits the localization in the frequency or wave number domain due to the uncertainty principle. The dilation parameter ‘a’ controls the width and rate of this local oscillation and intuitively can be thought of controlling the frequency of \( \psi_{b,a}(x) \). The translation parameter ‘b’ simply moves the wavelet throughout the domain.

The fractional mother wavelet defined by Shi et al. [33] as:

\[ \psi_{b,a,\theta}(x) = \frac{1}{\sqrt{a}} \psi \left( \frac{x-b}{a} \right) e^{i(x^2-b^2)\cot \theta}, \]  

(1.12.2)

for all \( a, \ b \) and \( \theta \) as above.
The continuous wavelet transform of $\phi \in L^2(\mathbb{R})$ is defined by

$$(W_\psi \phi)(b,a) = \langle \phi, \psi_{b,a} \rangle = \int_{\mathbb{R}} \phi(x) \overline{\psi_{b,a}(x)} dx = \frac{1}{|a|} \int_{\mathbb{R}} \phi(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx, \quad a \neq 0, \quad (1.12.3)$$

where $\overline{\psi}$ denotes the complex conjugate of $\psi$. Using the Parseval relation of the Fourier transform it can be expressed in the form

$$(W_\psi \phi)(b,a) = \int_{\mathbb{R}} e^{-ib\omega} \hat{\phi}(\omega) \overline{\hat{\psi}(a\omega)} d\omega, \quad (1.12.4)$$

where $\hat{\phi}$ denotes the Fourier transform of function $\phi$.

Notice that the introduction of parameter ‘$a$’ allows $\psi_{b,a}$ to compress or expand depending on the choice of ‘$a$’. The normalizing factor $|a|^{-1/2}$ automatically takes care of the relative frequency band. For example if $a > 1$, $\psi_{b,a}$ is stretched by a factor ‘$a$’ in the horizontal direction, whereas if $0 < a < 1$ then it is compressed in the same horizontal direction. When $\psi_{b,a}$ is dilated in horizontal direction, the factor $|a|^{-1/2}$ automatically reduces its size in the vertical direction and for small ‘$a$’, with the compression of $\psi_{b,a}$ in the horizontal direction, $\psi$ is enlarged in the vertical direction. In fact the total energy of $\psi_{b,a}$ remains independent of ‘$b$’ and ‘$a$’, that is, $\| \psi_{b,a} \|^2 = \| \psi \|^2$.

### 1.13 The discrete wavelet transform

The continuous wavelet transform (1.12.3) is a two-parameter representation of a function. In many applications, especially in signal processing, data are represented by a finite number of values, so it is important and often useful to consider discrete versions of the continuous wavelet transform (1.12.3). From a mathematical point of view, a continuous representation of a function of two continuous parameters ‘$a$’ and ‘$b$’ in (1.12.3) can be converted into a discrete one by assuming that ‘$a$’ and ‘$b$’ take only integral values. It turns out that it is better to discretize it in a different way.

First, we fix two positive constants $a_0$ and $b_0$ and define

$$\psi_{m,n}(x) = a_0^{-m/2} \psi\left(a_0^{-m}x - nb_0\right), \quad (1.13.1)$$
where both $m$ and $n \in \mathbb{Z}$. Then, for $\phi \in L^2(\mathbb{R})$, the discrete wavelet coefficients $(\phi, \psi_{m,n})$ are computed.

The discrete wavelet transform is defined by

$$ (W_\psi \phi)(m,n) = (\phi, \psi_{m,n}) = \int_{\mathbb{R}} \phi(x) \overline{\psi_{m,n}(x)} dx $$

$$ = a_0^{-m/2} \int_{\mathbb{R}} \phi(x) \overline{\psi(a_0^{-m}x - nb_0)} dx, \quad (1.13.2) $$

where both $\phi$ and $\psi$ are continuous and $\psi_{0,0}(x) = \psi(x)$.

### 1.14 Bessel wavelet transform

In this section, the Bessel wavelet transform introduced recently by Pathak and Dixit [25] and Upadhyay et al. [37], is given.

Pathak and Dixit [25] construct the Bessel wavelet transform as follows:

Let $\psi \in L_{p,\sigma}(0,\infty), 1 \leq p < \infty$ be given. For $b \geq 0$ and $a > 0$ define the Bessel wavelet

$$ \psi_{b,a}(t) = D_a \tau_t \psi(t) = D_a \psi(b,t) = a^{-2\gamma-1} \psi \left( \frac{b}{a} \frac{t}{a} \right) $$

$$ = a^{-2\gamma-1} \int_0^\infty D \left( \frac{b}{a} \frac{t}{a}, z \right) \psi(z) d\sigma(z), $$

where $\gamma$ is a positive real number and the integral being convergent.

Using the wavelet $\psi_{b,a}$ we now define the Bessel wavelet transform (BWT) which will send each $L^q$-function defined on the positive half line to a function $B(b,a)$ on the first quadrant.

$$ B(b,a) = (B_\psi f)(b,a) = \langle f(t), \psi_{b,a}(t) \rangle $$

$$ = \int_0^\infty f(t) \overline{\psi_{b,a}(t)} d\sigma(t) $$

$$ = a^{-2\gamma-1} \int_0^\infty \int_0^\infty f(t) \overline{\psi(z)} D \left( \frac{b}{a} \frac{t}{a}, z \right) d\sigma(z) d\sigma(t) $$

provided the integral is convergent.

The Bessel wavelet transform introduced recently by Upadhyay et al.[37] as follows:
A Bessel wavelet is a function $\psi \in L^2(0, \infty)$ which satisfies the condition

$$C_{\mu, \psi} = \int_0^\infty x^{-2\mu - 2} \left| (h_\mu \psi)(x) \right|^2 dx < \infty, \quad \mu \geq -1/2, \quad (1.14.1)$$

where $(h_\mu \psi)(x)$ is the Hankel transform of $\psi(t)$ and $C_{\mu, \psi}$ is called the admissibility condition of the Bessel wavelet.

Let $\psi \in L^2(0, \infty)$. For $b \geq 0$ and $a > 0$, then the Bessel wavelet transform of a function $f \in L^2(0, \infty)$ by

$$(B_\psi f)(b, a) = a^{-1/2} \int_0^\infty f(t) \psi \left( \frac{t}{a}, \frac{b}{a} \right) dt, \quad (1.14.2)$$

where $\psi$ is a wavelet which satisfies (1.14.1) and $\psi(x, y)$ has the same meaning as given by (1.8.7).

The continuous Bessel wavelet transform of a function $f \in L^2(0, \infty)$ defined by Upadhyay et al. [37] as:

$$(B_\psi f)(b, a) = a^{-\mu} \int_0^\infty (bx)^{1/2} J_\mu(bx)(h_\mu f)(x) x^{-\mu - 1/2} \left( \overline{(h_\mu \psi)(ax)} \right) dx. \quad (1.14.3)$$

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