CHAPTER-VI

TWO DIMENSIONAL SOLUTE DISPERSION ALONG AND AGAINST TRANSIENT GROUNDWATER FLOW IN A HOMOGENEOUS SEMI-INFINITE AQUIFER: INITIALLY NOT SOLUTE FREE WITH PULSE TYPE BOUNDARY CONDITIONS

6.1 Introduction

Mathematical models, based on certain simplifying assumptions, have been used to predict groundwater flow and solute transport. The transport of solutes in saturated media is governed by a partial differential equation (PDE) known as advection-dispersion equation (ADE). Analytical solutions of one-dimensional solute transport problems, subject to different initial and boundary conditions, in finite, semi-finite domain have been reported in the literature (e.g., van Genuchten and Alves, 1982; Lindstorm and Boersma, 1989; Fry et al., 1993; Kumar and Kumar, 1998; Singh et al., 2008, 2009; Chen and Liu, 2011a). With growing importance of groundwater resources, attention to prevent, reduce and eliminate groundwater pollution has been growing in various disciplines. Multidimensional solute transport problems in saturated media have been attracting the attention of many researchers. Two-dimensional solute transport problems involve both longitudinal as well as transverse dispersion along with porous media flow in addition to advection. Solutions of two-, three-dimensional deterministic ADE’s have been investigated in numerous studies and are still being actively studied (e.g., Latinopoulos et al., 1988; Yates, 1988; Batu, 1989, 1993; Leij et al., 1991, 1993; Serrano, 1995). A Laplace transform solution for tracer tests have been developed by Chen et al. (1996) with the assumption that advection and longitudinal dispersion are the transport mechanisms in a radially converging flow field. Also, Aral and Liao (1996) developed instantaneous and continuous point-source solution for a two-dimensional ADE model with constant, linear, asymptotic, and exponentially varying dispersion coefficients. An analytical solution of a spatially variable coefficient advection-diffusion equation has been solved by Zoppou and Knight (1999) in up to three dimensions with the assumption that the velocity component is proportional to the distance and that the diffusion coefficient is proportional to the square of the corresponding velocity.
component. Park and Zhan (2001) discussed analytical solutions of contaminant transport from finite one-, two-, and three-dimensional sources in a finite thickness aquifer using Green’s function method. Chen et al. (2003) discussed a Laplace-transformed power series (LTPS) technique to solve a two-dimensional ADE in cylindrical co-ordinate with non-axisymmetry solute transport in a radially convergent flow field and the solution was compared with a numerical solution to evaluate its robustness and accuracy. Kumar et al. (2006) discussed the time-dependent dispersivity behaviour of non reactive solutes in a system of parallel structures. Smedt (2006) has presented an analytical solution for transport of decaying solutes in rivers with transient storage. Chen (2007) has also presented a two-dimensional power series solution for non-axisymmetrical transport in a radially convergent tracer test with scale dependent dispersion. The power series technique used was the combination of the Laplace and finite Fourier cosine transformation which yielded the analytical solution to the two-dimensional, scale-dependent ADE in cylindrical coordinates with variable-dependent coefficients. Also James and Jawitz (2007) have discussed a two-dimensional reactive transport ADE model using a splitting technique where advective, dispersive and reactive parts of the equation were solved separately. An explicit finite-volume Godunov method was used to approximate the advective part while a hybridized mixed finite element method was used to solve for the dispersive step and the backward Euler method for the reactive component.

Recently many investigations have been carried out for two-, three-dimensional solute transport in saturated or unsaturated porous media using different mathematical approaches. A two-dimensional contaminant transport through saturated porous media using a mesh free method called the radial point interpolation method (RPIM) with polynomial reproduction, has been derived by Kumar and Dodagourdar (2008). They also developed a computer program for implementation of the RPIM procedure. Zhan et al. (2009) presented an analytical solution of two-dimensional solute transport in an aquifer-aquitard system by maintaining rigorous mass conservation at the aquifer-aquitard interface with the first-type and the third type boundary conditions. Li and Cleall (2010) presented an analytical solution for contamination diffusion in double layered porous media. Chen et al. (2011a) presented an analytical solution presented for two-dimensional ADE in
cylindrical geometry, finite-length medium, subject to the third-type inlet boundary condition. They used the second kind finite HTT and the generalized integral transform technique (GIT) to solve the model and the solutions were compared with the solutions for semi-infinite domain available in the literature to illustrate the impacts of inlet and outlet boundary conditions. Chen et al. (2011b) have also presented an exact analytical solution to the two-dimensional ADE in cylindrical coordinates in a finite domain with first- and third-type inlet boundary conditions. The second kind finite HTT and GIT have been adopted to solve the model. The developed analytical solutions were compared with the solutions in a semi-infinite domain, subject to the first- and third- type inlet boundary condition available in the literature to illustrate the exit boundary conditions. Jaiswal et al. (2011) discussed analytical solutions obtained for two-dimensional ADE describing the dispersion of pulse-type point source along temporally and spatially dependent flow domain. In order to obtain an analytical solution, they used the relations that the dispersion parameter was proportional to the square of the velocity.

The above literature review shows that the previous investigations were mainly related to determining the behavior of solute transport along the flow only. In the present work we derived an analytical solution in both the domains, i.e., the dispersion along and against the flow. The main objective of our present work therefore is to predict the distribution of two-dimensional solute transport in a semi-infinite aquifer along and against the flow. For this purpose, we consider that the aquifer is homogeneous and semi-infinite in nature. Initially the aquifer is supposed to be not solute free. A pulse type input source concentration is considered at the top layer of the outer boundary at the far end from the origin, i.e., at the intermediate portion of the aquifer. The accidental spill frequently referred to is a pulse type problem, and represents another potential contamination problem discussed by Domenico and Schwartz (1997). This situation is realistic in the sense that generally industries or waste sites release pollution in a finite period, either because industrial firms have finite life or controlled after a certain time with the awareness of the pollution or government regulation. For this reason two cases are arises involving, i) concentration distribution of solute dispersion along the flow and ii) concentration distribution of solute dispersion against the flow. Analytical solutions are obtained by the
Laplace Transform Technique (LTT). For numerical discussions two type of temporally dependent velocities, such as i) sinusoidal form and ii) exponentially decreasing form of velocities, are considered.

6.2 Mathematical Formulation of the Problem

Consider an isotropic and homogeneous aquifer which is initially not solute free. Let the aquifer be subjected to a pulse type source contamination at the uppermost groundwater level. This point defines the origin for the coordinates of measurement. The longitudinal and lateral directions at the origin are taken as \(x\) [L] and \(y\) [L] axes, respectively. Let \(c\) [ML\(^{-3}\)] denote the solute concentration in the aquifer at any time \(t\) [T]; \(u\) [LT\(^{-1}\)] and \(v\) [LT\(^{-1}\)] denote the groundwater velocity components along the \(x\) and \(y\) directions, respectively; and \(D_x\) [L\(^2\)T\(^{-1}\)] and, \(D_y\) [L\(^2\)T\(^{-1}\)] denote the dispersion coefficients along \(x\) and \(y\) axes, respectively. The source of solute is introduced as the point source on the surface; due to infiltration, it reaches the water table at the intermediate point \(x = x_0, y = y_0\) which is at the far end from the origin whereas the direction of flow is from the origin towards the other end of the aquifer.

For describing the two-dimensional hydrodynamic dispersion in a homogenous, anisotropic aquifer, the PDE can be expressed in general as (Batu, 2006):

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_{xx} \frac{\partial c}{\partial x} + D_{xy} \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial y}\left(D_{yx} \frac{\partial c}{\partial x} + D_{yy} \frac{\partial c}{\partial y}\right) - u(t) \frac{\partial c}{\partial x} - v(t) \frac{\partial c}{\partial y} \tag{6.1}
\]

where \(D_{xx} = a \frac{u^2}{\sqrt{u^2 + v^2}} + b \frac{v^2}{\sqrt{u^2 + v^2}}\), \(D_{yy} = a \frac{v^2}{\sqrt{u^2 + v^2}} + b \frac{u^2}{\sqrt{u^2 + v^2}}\), and \(D_{xy} = (a - b) \frac{uv}{\sqrt{u^2 + v^2}} = D_{yx}\)

Here, \(a\) [L] and \(b\) [L] are the longitudinal and transverse dispersivity respectively.

We assume that one of the axes say \(x\)-axis coincides with the direction of average velocity and therefore, the off diagonal elements say \(D_{xy} = D_{yx} = 0\) and \(D_{xx} \equiv D_x = au,\ D_{yy} \equiv D_y = bu\).

Hence, for homogeneous, isotropic aquifer the PDE given in Eq. (6.1) can be written as
\[
\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u(t) \frac{\partial c}{\partial x} - v(t) \frac{\partial c}{\partial y}
\]  
(6.2)

Let \(u\) and \(v\) be expressed as
\[
u = u_0 f(t) \quad \text{and} \quad v = v_0 f(t)
\]  
(6.3)

where \(u_0 [LT^{-1}]\) and \(v_0 [LT^{-1}]\) are the initial values of \(u\) and \(v\), respectively. Here \(V(t)\) is considered non-dimensional expression such as
\[
f(t) = 1 - \sin mt
\]  
(6.4a)
\[
f(t) = \exp(-mt), \ mt < 1
\]  
(6.4b)

where \(m [T^{-1}]\) is a flow resistance coefficient.

Here, \(mt = 3k + 2\) is chosen where \(k\) represents the whole number, i.e., \(k = 0, 1, 2, 3\) and so on. Now for particular values \(m = 0.0165 (/\text{day})\); yields, \(t = 182k + 121\) approximately, and accordingly Eq. (6.4a) represents minimum and maximum values alternatively. We choose the particular values of \(m\) only because it represents the specific behavior of time at the interval of 182 days which is approximately equivalent to six months. For example, for \(k = 0, t = 121\) days assumed say 1st year June which represents minimum values of groundwater level and velocity during that period and for next values of \(k = 1, t = 303\) days assumed 1st year December which represents maximum values of groundwater level and velocity during that period in a year. This cycle will continue for the subsequent year. Now, the time in the month of June (just before the rainy season) and December (just after the rainy season) in a year is equivalent to six months represented by a mathematical expression as stated above. For these values of \(mt\), velocity \(u\), is alternatively minimum and maximum.

The dispersion is proportional to a power of the seepage velocity which ranges between 1 and 2 (Freeze and Cherry, 1979). This has also been experimentally observed in India that the dispersion is directly proportional to the seepage velocity with a power ranging from 1 to 1.2 (Ghosh and Sharma, 2006). The dispersion is directly proportional to the seepage velocity is employed in the present discussion. This is applicable for different types of porous media. The relationship established between dispersion and seepage
velocity, for steady flow is also approximately valid for unsteady flow (Ebach and White, 1958; Rumer, 1962; Kumar, 1983). Therefore, let

\[ D_x = au_0 f(t) \quad \text{and} \quad D_y = bu_0 f(t) \]  

(6.5)

Using Eq. (6.3), Eq.(6.5) can be written as

\[ D_x = D_{x_0} f(t) \quad \text{and} \quad D_y = D_{y_0} f(t) \]  

(6.6)

where \( D_{x_0} = au_0 \) and \( D_{y_0} = bu_0 \) are the initial values of \( D_x \) and \( D_y \), respectively.

As stated earlier, the initial solute concentration is not zero at all points in the flow domain at \( t = 0 \), i.e., some concentration \( c_i \) [ML\(^{-3}\)] is already present initially. At the top layer of the outer boundary at the far end from the origin, i.e., at the intermediate point \( x = x_0, y = y_0 \), the input concentration is considered pulse type. Suppose \( c_0 \) [ML\(^{-3}\)] is the solute concentration at the intermediate point \( x = x_0, y = y_0 \) of the aquifer till \( t = t_0 \) and beyond that it become zero. For this reason two cases arises entailing: i) the concentration distribution of the solute in the region \( x \geq x_0, y \geq y_0 \) which is along the flow and ii) the concentration distribution of the solute in the region \( 0 \leq x \leq x_0, 0 \leq y \leq y_0 \) which is treated as against the flow. At the other end of the aquifer, the concentration gradient is supposed to be zero.

### 6.3 Analytical Solution

There are various approaches available in mathematics, such as analytical approach, numerical approach, statistical approach, finite element approach, etc., to deal with a problem of solute transport modeling in homogeneous porous formations. Though, the most powerful approaches among them are analytical because, they provide close form solutions which can be applied to benchmark numerical solutions. The complexity of the solute transport problem is not an easier task so the solution obtained in this fashion may give an idea to understand the feasibility of the practical application in solute transport modeling up to a certain extent. The significance of analytical methods for one-, two-, and three-dimensional advection-dispersion equations (ADEs) is also to estimate the pattern of solute concentration and various transport parameters from field data (Bear and Verruijt, 1987; Batu, 2006; Ghosh and Sharma, 2006). In the present work, a two-dimensional ADE
is formulated with dispersion along the flow and dispersion against the flow in the presence and absence of pollution source and solved, subject to suitable initial and boundary conditions.

6.3.1 Analytical solution for concentration distribution of solute in the region $x \geq x_0, y \geq y_0$:

The boundary conditions describe the nature of interaction of flow and dispersion with the surroundings. Hence, the initial and boundary conditions are expressed as

$$c(x, y, t) = c_i; \quad x \geq x_0, y \geq y_0, \quad t = 0$$

$$c(x, y, t) = \begin{cases} c_0; & 0 < t \leq t_0 \\ 0; & t > t_0 \end{cases}; x = x_0, y = y_0$$

$$\frac{\partial c}{\partial x} = 0 \text{ and } \frac{\partial c}{\partial y} = 0; \quad x \to \infty, y \to \infty, \quad t \geq 0$$

The source of contamination in the horizontal plane is the input point source and is uniform at the far end from the origin, i.e., at $x = x_0, y = y_0$, which helps predict the pattern of solute concentration distribution in comparison to at the origin.

Using the transformation $x^* = x - x_0, y^* = y - y_0$ and Eqs. (6.3) and (6.6), Eq. (6.2) can be written as

$$\frac{1}{f(t)} \frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u_0 \frac{\partial c}{\partial x} - v_0 \frac{\partial c}{\partial y^*}$$

Introducing the new time variable $T^*$ by the following transformation (Crank, 1975)

$$T^* = \int_0^t f(t) dt$$

Eq. (6.9) can be written as

$$\frac{\partial c}{\partial T^*} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u_0 \frac{\partial c}{\partial x} - v_0 \frac{\partial c}{\partial y^*}$$

Let a new space variable be introduced as

$$z = x^* + y^* \sqrt{\frac{D_y}{D_x}} \quad \text{or} \quad x^* + y^* \sqrt{\frac{D_y}{u_0}}$$
With the aid of Eq. (6.12), Eq. (6.11) can be written as

$$\frac{\partial c}{\partial T^*} = D \frac{\partial^2 c}{\partial z^2} - U \frac{\partial c}{\partial z}$$  \hspace{1cm} (6.13)

where

$$D = D_{wo} \left(1 + \frac{D_z}{D_{wo}}\right) \text{ and } U = \left(u_0 + v_0 \sqrt{\frac{v_0}{u_0}}\right)$$

Also, Eqs (6.7) and (6.8) can be written as

$$c\left(z, T^*\right) = c_i; \quad z \geq 0, \quad T^* = 0$$  \hspace{1cm} (6.14)

$$c\left(z, T^*\right) = \begin{cases} c_0; & 0 < T^* \leq T_0^* \\ 0; & T^* > T_0^* \end{cases}; \quad z = 0$$  \hspace{1cm} (6.15a)

$$\frac{\partial c}{\partial z} = 0; \quad z \to \infty, \quad T^* \geq 0$$  \hspace{1cm} (6.15b)

The transformation

$$c\left(z, T^*\right) = K\left(z, T^*\right) \exp\left(\frac{U_z}{2D} - \frac{U^2 T^*}{4D}\right)$$  \hspace{1cm} (6.16)

is used to remove the convective term $U \frac{\partial c}{\partial z}$ from Eq. (6.13) so that transformed partial differential equation can be solved easily using the Laplace Transformation Technique.

The terms of Eq. (6.13) can be expressed as

$$\frac{\partial c}{\partial z} = \exp\left(\frac{U_z}{2D} - \frac{U^2 T^*}{4D}\right) \left(\frac{\partial K}{\partial z} + \frac{U K}{2D}\right)$$  \hspace{1cm} (6.17a)

$$\frac{\partial^2 c}{\partial z^2} = \exp\left(\frac{U_z}{2D} - \frac{U^2 T^*}{4D}\right) \left(\frac{\partial^2 K}{\partial z^2} + \frac{U \partial K}{D \partial z} + \frac{U^2 K}{4D^2}\right)$$  \hspace{1cm} (6.17b)

$$\frac{\partial c}{\partial T^*} = \exp\left(\frac{U_z}{2D} - \frac{U^2 T^*}{4D}\right) \left(\frac{\partial K}{\partial T^*} - \frac{U^2 K}{4D}\right)$$  \hspace{1cm} (6.17c)

Substituting Eqs. (6.17a)-(6.17c) in Eq. (6.13), one gets

$$\frac{1}{D} \frac{\partial K}{\partial T^*} = \frac{\partial^2 K}{\partial z^2}$$  \hspace{1cm} (6.18)

Now the initial condition becomes
\begin{align*}
K(z,0) &= c_i \exp\left(-\frac{Uz}{2D}\right) \quad (6.19)
\end{align*}

and the boundary conditions become
\begin{align*}
K(0,T^*) &= c_0 \exp\left(\frac{U^2T^*}{4D}\right); \quad 0 < T^* \leq T_0^* \\
0; \quad T^* > T_0^* \quad (6.20a)
\end{align*}
\begin{align*}
\frac{\partial K}{\partial z} + U K &= 0; \quad z \to \infty, \quad T^* \geq 0 \quad (6.20b)
\end{align*}

Applying the Laplace transform (Sneddon, 1974) to Eq. (6.18) and using Eq. (6.19) one gets
\begin{align*}
\tilde{K}(z,p) &= c_i \exp\left(\frac{p}{\sqrt{Dz}}\right) + c_2 \exp\left(-\frac{p}{\sqrt{Dz}}\right) + \frac{c_i}{U^2} \exp\left(-\frac{Uz}{2D}\right) \quad (6.21)
\end{align*}

where \(c_1\) and \(c_2\) are constants.

Taking the Laplace transform of Eqs. (6.20a) and (6.20b), one gets
\begin{align*}
\tilde{K}(0,p) &= \frac{c_0 \left[1 - \exp\left(-\frac{p - U^2}{4D}T_0^*\right)\right]}{p - \frac{U^2}{4D}} \quad (6.22a)
\end{align*}
\begin{align*}
\frac{d\tilde{K}}{dz} + \frac{UK}{2} &= 0; \quad z \to \infty \quad (6.22b)
\end{align*}

Using Eqs. (6.22a) and (6.22b) in the solution given by Eq. (6.21), constants \(c_1\) and \(c_2\) can be obtained as
\begin{align*}
c_1 = 0 \quad \text{and} \quad c_2 = \frac{c_0 \left[1 - \exp\left(-\frac{p - U^2}{4D}T_0^*\right)\right]}{p - \frac{U^2}{4D}} - \frac{c_i}{p - \frac{U^2}{4D}} \quad (6.23)
\end{align*}

Thus, the solution given by Eq. (6.21) can be written as
\[
\bar{K}(z, p) = \frac{c_0}{\left(\frac{p - U^2}{4D}\right)} \left[ 1 - \exp\left\{ -\left(\frac{p - U^2}{4D}\right)T_0^* \right\} \right] \exp\left( -\sqrt{\frac{p}{D}} z \right)
\]
\[
- \frac{c_i}{\left(\frac{p - U^2}{4D}\right)} \exp\left( -\sqrt{\frac{p}{D}} z \right) + \frac{c_i}{\left(\frac{p - U^2}{4D}\right)} \exp\left( -\frac{Uz}{2D} \right)
\]

(6.24)

\[
\bar{K}(z, p) = c_0 \bar{K}_1(z, p) - c_i \bar{K}_2(z, p) + \frac{c_i}{\left(\frac{p - U^2}{4D}\right)} \exp\left( -\frac{Uz}{2D} \right)
\]

(6.25)

where

\[
\bar{K}_1(z, p) = \frac{1}{\left(\frac{p - U^2}{4D}\right)} \left[ 1 - \exp\left\{ -\left(\frac{p - U^2}{4D}\right)T_0^* \right\} \right] \exp\left( -\sqrt{\frac{p}{D}} z \right)
\]

(6.26a)

\[
\bar{K}_2(z, p) = \frac{1}{\left(\frac{p - U^2}{4D}\right)} \exp\left( -\sqrt{\frac{p}{D}} z \right)
\]

(6.26b)

Now taking the inverse Laplace transform of Eq. (6.25), \( K(z, T^* \) can be obtained as follows (Bateman, 1954)

\[
K(z, T^*) = c_0 K_1(z, T^*) - c_i K_2(z, T^*) + c_i \exp\left( \frac{U^2 T^*}{4D} - \frac{Uz}{2D} \right)
\]

(6.27)

where

\[
K_1(z, T^*) = \begin{cases} 
F(z, T^*); & 0 < T^* \leq T_0^* \\
F(z, T^*) - F(z, T^* - T_0^*); & T^* > T_0^*
\end{cases}
\]

(6.28a)

\[
K_2(z, T^*) = F(z, T^*)
\]

(6.28b)

Therefore,
\[ K(z,T^*) = \begin{cases} 
  c_0 F(z,T^*) - c_i F(z,T^*) + c_i \exp\left( \frac{U^2 T^* - Uz}{4D} \right); & 0 < T^* \leq T_0^* \\
  c_0 \left[ F(z,T^*) - F(z,T^* - T_0^*) \right] - c_i F(z,T^*) + c_i \exp\left( \frac{U^2 T^* - Uz}{4D} \right); & T > T_0^* 
\end{cases} \] (6.29)

and hence by using Eq. (6.16), the final solution can be written as

\[ c(z,T^*) = \begin{cases} 
  \exp\left( \frac{Uz - U^2 T^*}{4D} \right) \left[ c_0 F(z,T^*) - c_i F(z,T^*) \right] + c_i; & 0 < T^* \leq T_0^* \\
  \exp\left( \frac{Uz - U^2 T^*}{4D} \right) \left[ c_0 \left[ F(z,T^*) - F(z,T^* - T_0^*) \right] - c_i F(z,T^*) \right] + c_i; & T > T_0^* 
\end{cases} \] (6.30)

where

\[ F(z,T^*) = \frac{1}{2} \left[ \exp\left( \frac{U^2 T^*}{4D} - \frac{Uz}{2D} \right) \text{erfc}\left( \frac{z}{\sqrt{2}D T^*} - \frac{UT^*}{\sqrt{2}D T^*} \right) \right. \\
\left. + \exp\left( \frac{U^2 T^*}{4D} + \frac{Uz}{2D} \right) \text{erfc}\left( \frac{z}{\sqrt{2}D T^*} + \frac{UT^*}{\sqrt{2}D T^*} \right) \right] \] (6.31)

One can express this solution in terms of original space and time variable, i.e., \( x, y \) and \( t \) by using Eq. (6.12) followed by the transformations given in Eq. (6.10) and \( x = x^* + x_0, \quad y = y^* + y_0 \).

6.3.2 **Analytical solution for concentration distribution of solute in the region** \( 0 \leq x \leq x_0, 0 \leq y \leq y_0 \):

In this domain, the initial and boundary conditions of the problem can be written as

\[ c(x, y, t) = c_i; \quad x > 0, \ y > 0, \ t = 0 \] (6.32)

\[ \frac{\partial c}{\partial x} = 0 \text{ and } \frac{\partial c}{\partial y} = 0; \quad x = 0, \ y = 0, \ t \geq 0 \] (6.33a)

\[ c(x, y, t) = \begin{cases} 
  c_0; & 0 < t \leq t_0 \\
  0; & t > t_0 \end{cases} \quad x = x_0, \ y = y_0 \] (6.33b)
Using the relation given in Eqs. (6.3) and (6.6), the PDE given in Eq. (6.2) can be written as

\[ \frac{1}{f(t)} \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + D \frac{\partial^2 c}{\partial y^2} - u_0 \frac{\partial c}{\partial x} - v_0 \frac{\partial c}{\partial y} \]  
\hfill (6.34)

The new time variable given in Eq. (6.10) transforms the PDE of Eq. (6.34) as

\[ \frac{\partial c}{\partial T^*} = D \frac{\partial^2 c}{\partial \beta^2} - U \frac{\partial c}{\partial \beta} \]  
\hfill (6.35)

Let a new space variable is introduced as

\[ \beta = x + y \sqrt{\frac{D_{xx}}{D_{yy}}} \text{ or } x + y \sqrt{\frac{v_0}{u_0}} \]  
\hfill (6.36)

With the help of this transformation, Eq. (6.35) can be written as

\[ \frac{\partial c}{\partial T^*} = D \frac{\partial^2 c}{\partial \beta^2} - U \frac{\partial c}{\partial \beta} \]  
\hfill (6.37)

where

\[ D = D_{xx} \left(1 + \frac{D_{xy}^2}{D_{yy}}\right) \text{ and } U = \left(u_0 + v_0 \sqrt{\frac{v_0}{u_0}}\right) \]

Using transformation given by Eqs. (6.10) and (6.36), the initial and boundary conditions given by Eqs. (6.32) - (6.33) become

\[ c(\beta, T^*) = c_i; \quad \beta \geq 0 \text{, } T^* = 0 \]  
\hfill (6.38)

\[ \frac{\partial c}{\partial \beta} = 0; \quad \beta = 0 \text{, } T^* \geq 0 \]  
\hfill (6.39a)

\[ c(\beta, T^*) = \begin{cases} c_0; & 0 < T^* \leq T_0^* \\ 0; & T^* > T_0^* \end{cases}; \beta = \beta_0 \]  
\hfill (6.39b)

Using the following transformation

\[ c(\beta, T^*) = K(\beta, T^*) \exp\left(\frac{U\beta}{2D} - \frac{U^2 T^*}{4D}\right) \]  
\hfill (6.40)

Eqs. (6.37) – (6.39) can be transformed as follows:
\[
\frac{1}{D} \frac{\partial K}{\partial T^*} = \frac{\partial^2 K}{\partial \beta^2} \quad (6.41)
\]

\[
K(\beta, T^*) = c_i \exp \left( -\frac{U \beta}{2D} \right); \quad \beta \geq 0, \quad T^* = 0 \quad (6.42)
\]

\[
\frac{\partial K}{\partial \beta} + \frac{UK}{2D} = 0; \quad \beta = 0, \quad T^* \geq 0 \quad (6.43a)
\]

\[
K(\beta, T^*) = c_0 \exp \left( \frac{U^2 T^*}{4D} - \frac{U \beta_0}{2D} \right); \quad 0 < T^* \leq T^*_0 \quad \beta = \beta_0 \quad (6.43b)
\]

\[
T^* > T^*_0
\]

Applying the Laplace transformation to Eqs. (6.41) - (6.43), one can get the solution of the obtained boundary value problem as

\[
\bar{K}(\beta, p) = \exp \left( -\frac{U \beta_0}{2D} \right) \left[ c_i \left( \bar{K}_1(\beta, p) + \bar{K}_2(\beta, p) - \bar{K}_3(\beta, p) - \bar{K}_4(\beta, p) \right) \right]
\]

\[
+ c_i \left( \bar{K}_5(\beta, p) + \bar{K}_6(\beta, p) - \bar{K}_7(\beta, p) - \bar{K}_8(\beta, p) \right)
\]

\[
\left( p - \frac{U^2}{4D} \right)
\]

where

\[
\bar{K}_1(\beta, p) = \frac{1}{p - \frac{U^2}{4D}} \left[ 1 - \exp \left( -\left( p - \frac{U^2}{4D} \right) T^*_0 \right) \right] \exp \left( -\beta_0 (\beta - \beta_0) \sqrt{\frac{p}{D}} \right) \quad (6.45a)
\]

\[
\bar{K}_2(\beta, p) = \frac{1}{p - \frac{U^2}{4D}} \left[ 1 - \exp \left( -\left( p - \frac{U^2}{4D} \right) T^*_0 \right) \right] \left( 1 + \frac{U}{\sqrt{\frac{p}{D} - \frac{U}{2D}} \right) \right)
\]

\[
\exp \left( -\beta_0 (\beta + \beta_0) \sqrt{\frac{p}{D}} \right) \quad (6.45b)
\]
\[ \tilde{K}_3(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \left[ 1 - \exp \left\{ \left( p - \frac{U^2}{4D} \right) T_0^* \right\} \right]\left[ 1 + \frac{U}{D} \sqrt{\frac{p}{D} - \frac{U}{2D}} \right] \exp \left\{ -(3\beta_0 - \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45c)

\[ \tilde{K}_4(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \left[ 1 - \exp \left\{ \left( p - \frac{U^2}{4D} \right) T_0^* \right\} \right]\left[ 1 + \frac{U}{D} \sqrt{\frac{p}{D} - \frac{U}{2D}} \right]^2 \exp \left\{ -(3\beta_0 + \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45d)

\[ \tilde{K}_5(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \exp \left\{ -(\beta_0 - \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45e)

\[ \tilde{K}_6(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \left[ 1 + \frac{U}{D} \sqrt{\frac{p}{D} - \frac{U}{2D}} \right] \exp \left\{ -(\beta_0 + \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45f)

\[ \tilde{K}_7(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \left[ 1 + \frac{U}{D} \sqrt{\frac{p}{D} - \frac{U}{2D}} \right] \exp \left\{ -(3\beta_0 - \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45g)

\[ \tilde{K}_8(\beta, p) = \frac{1}{(p - \frac{U^2}{4D})} \left[ 1 + \frac{U}{D} \sqrt{\frac{p}{D} - \frac{U}{2D}} \right]^2 \exp \left\{ -(3\beta_0 + \beta) \sqrt{\frac{p}{D}} \right\} \] (6.45h)

Taking inverse Laplace transform of Eq. (6.44), the following is obtained (Bateman, 1954)
\[ K(\beta, T^*) = \exp\left(- \frac{U \beta_0}{2D}\right) \left[ c_0 \left\{ K_1(\beta, T^*) + K_2(\beta, T^*) - K_3(\beta, T^*) - K_4(\beta, T^*) \right\} \right. \\
+ \left. c_1 \exp\left( \frac{U^2 T^*}{4D} - \frac{U \beta}{2D} \right) \right] \]

(6.46)

where

\[ K_1(\beta, T^*) = \begin{cases} 
F_1(\beta_0 - \beta, T^*); & 0 < T^* \leq T_0^* \\
F_1(\beta_0 - \beta, T^*) - F_1(\beta_0 - \beta, T^* - T_0^*); & T > T_0^* 
\end{cases} \]

(6.47a)

\[ K_2(\beta, T^*) = \begin{cases} 
F_1(\beta_0 + \beta, T^*) + F_2(\beta_0 + \beta, T^*); & 0 < T^* \leq T_0^* \\
\left[ F_1(\beta_0 + \beta, T^*) - F_1(\beta_0 + \beta, T^* - T_0^*) \right] + \left[ F_2(\beta_0 + \beta, T^*) - F_2(\beta_0 + \beta, T^* - T_0^*) \right]; & T > T_0^* 
\end{cases} \]

(6.47b)

\[ K_3(\beta, T^*) = \begin{cases} 
F_1(3\beta_0 - \beta, T^*) + F_2(3\beta_0 - \beta, T^*); & 0 < T^* \leq T_0^* \\
\left[ F_1(3\beta_0 - \beta, T^*) - F_1(3\beta_0 - \beta, T^* - T_0^*) \right] + \left[ F_2(3\beta_0 - \beta, T^*) - F_2(3\beta_0 - \beta, T^* - T_0^*) \right]; & T > T_0^* 
\end{cases} \]

(6.47c)

\[ K_4(\beta, T^*) = \begin{cases} 
F_1(3\beta_0 + \beta, T^*) + 2F_2(3\beta_0 + \beta, T^*) + F_3(3\beta_0 + \beta, T^*); & 0 < T^* \leq T_0^* \\
\left[ F_1(3\beta_0 + \beta, T^*) - F_1(3\beta_0 + \beta, T^* - T_0^*) \right] + \left[ F_2(3\beta_0 + \beta, T^*) - F_2(3\beta_0 + \beta, T^* - T_0^*) \right] + \left[ F_3(3\beta_0 + \beta, T^*) - F_3(3\beta_0 + \beta, T^* - T_0^*) \right]; & T > T_0^* 
\end{cases} \]

(6.47d)

\[ K_5(\beta, T^*) = F_1(\beta_0 - \beta, T^*) \]

(6.47e)

\[ K_6(\beta, T^*) = F_1(\beta_0 + \beta, T^*) + F_2(\beta_0 + \beta, T^*) \]

(6.47f)

\[ K_7(\beta, T^*) = F_1(3\beta_0 - \beta, T^*) + F_2(3\beta_0 - \beta, T^*) \]

(6.47g)

\[ K_8(\beta, T^*) = F_1(3\beta_0 + \beta, T^*) + 2F_2(3\beta_0 + \beta, T^*) + F_3(3\beta_0 + \beta, T^*) \]

(6.47h)

Finally, \( K(\beta, T^*) \) can be written as follows
and hence by using Eq. (6.39), the final solution can be written as follows

\[ c(\beta,T^*) = \begin{cases} 
\exp\left(-\frac{U\beta_0}{2D}\right)\left(c_0-c_1\right) \left[F_1(\beta_0-\beta,T^*) + F_1(\beta_0+\beta,T^*) + F_2(\beta_0+\beta,T^*) - F_1(3\beta_0-\beta,T^*) \right] 
- F_2(3\beta_0-\beta,T^*) - F_1(3\beta_0+\beta,T^*) - 2F_2(3\beta_0+\beta,T^*) - F_1(3\beta_0+\beta,T^*) \right] 
+ c_1 \exp\left(\frac{U^2T^*}{4D} - \frac{U\beta}{2D}\right): 0 < T^* \leq T_0^* 
\end{cases} \]  

(6.49)

where

\[ F_1(\beta,T^*) = \frac{1}{2} \exp\left(\frac{U^2T^*}{4D} - \frac{U\beta}{2D}\right) \text{erfc}\left(\frac{\beta}{2\sqrt{DT^*}} - \frac{UT^*}{2\sqrt{DT^*}}\right) \]  

+ \frac{1}{2} \exp\left(\frac{U^2T^*}{4D} + \frac{U\beta}{2D}\right) \text{erfc}\left(\frac{\beta}{2\sqrt{DT^*}} + \frac{UT^*}{2\sqrt{DT^*}}\right) \]  

(6.50a)
\[ F_2(\beta, T^*) = \frac{U}{\sqrt{D}} \sqrt{\frac{T^*}{\pi}} \exp \left( -\frac{\beta^2}{4DT^*} \right) \]
\[ - \frac{1}{2} \exp \left( \frac{U^2T^*}{4D} + \frac{U\beta}{2D} \right) \text{erfc} \left( \frac{\beta}{2\sqrt{DT^*}} + \frac{UT^*}{2\sqrt{DT^*}} \right) \]
\[ + \frac{1}{2} \left( 1 - \frac{U\beta}{D} + \frac{U^2T^*}{D} \right) \exp \left( \frac{U^2T^*}{4D} - \frac{U\beta}{2D} \right) \text{erfc} \left( \frac{\beta}{2\sqrt{DT^*}} - \frac{UT^*}{2\sqrt{DT^*}} \right) \]  
(6.50b)

\[ F_3(\beta, T^*) = \frac{1}{2} \exp \left( \frac{U^2T^*}{4D} + \frac{U\beta}{2D} \right) \text{erfc} \left( \frac{\beta}{2\sqrt{DT^*}} + \frac{UT^*}{2\sqrt{DT^*}} \right) \]
\[ - \frac{U}{\sqrt{D}} \sqrt{\frac{T^*}{\pi}} \left( 1 - \frac{U\beta}{2D} + \frac{U^2T^*}{2D} \right) \exp \left( -\frac{\beta^2}{4DT^*} \right) \]
\[ + \frac{1}{2} \left\{ -1 - \frac{U\beta}{D} + 2\frac{U^2T^*}{D} + \frac{U^2}{2D} \left( \frac{\beta}{\sqrt{D}} - \frac{UT^*}{\sqrt{D}} \right) \right\} \]
\[ \exp \left( \frac{U^2T^*}{4D} - \frac{U\beta}{2D} \right) \text{erfc} \left( \frac{\beta}{2\sqrt{DT^*}} - \frac{UT^*}{2\sqrt{DT^*}} \right) \]  
(6.50c)

One can express the above solution in terms of original space and time variable, i.e., \( x, y \) and \( t \) by using Eq. (6.36), followed by the transformations given in Eq. (6.10).

### 6.4 Numerical Example and Discussion

The analytical solution given by Eq.(6.30) is computed for the values of \( c_0 = 1.0 \), \( c_i = 0.05 \), \( x = 100 \text{ km} \), \( y = 100 \text{ km} \), \( x_0 = 30 \text{ km} \), \( y_0 = 30 \text{ km} \), \( a = 6 \text{ km} \), \( b = 0.06 \text{ km} \), \( u_0 = 0.025 \text{ km/day} \), \( v_0 = 0.0025 \text{ km/day} \), \( m = 0.0165 \text{ (/day)} \); whereas the solution given by Eq. (6.49) is computed for the same values except for \( c_0 = 0.0 \), \( u_0 = 0.0025 \text{ km/day} \), \( v_0 = 0.00025 \text{ km/day} \), \( a = 22.8 \text{ km} \) and \( b = 2.28 \text{ km} \). The concentrated values in the presence of uniform input source pollution till \( t = t_0 (1500 \text{ days}) \) are depicted graphically at \( mt = 3k + 2 \), \( 4 \leq k \leq 7 \). Fig. 6.1(a) illustrates the concentration distribution for sinusoidally form of velocity along and against the flow during the month of June and December in the 3\textsuperscript{rd} and 4\textsuperscript{th} year, respectively. It is observed from Fig. 6.1(a) that the solute concentration decreases with distance of travel and increases with time at each position in both the domains. The solute concentrations are slowly and gradually getting minimum or harmless.
concentration at the far end of the aquifer in both the domains. In comparison to along the flow case, the values of solute concentrations are closer during against the flow case. This representation is made in the presence of source in the time domain $0 < t \leq t_0$. Now, when the source is eliminated in the time domain $t > t_0$ where $t = t_0 (1500 \text{ days})$ is the time of elimination of source, the solutions given in Eqs. (6.30) and (6.49) are further computed at $mt = 3k + 2$, $8 \leq k \leq 11$ and the solute concentration distribution patterns are depicted graphically in Fig. 6.1(b). This represents the solute concentration pattern during the month of June and December in the 5th and 6th year of prediction. It is observed that the solute concentration becomes zero in the absence of source. It is also observed that the solute concentration increases to a peak and then attains minimum and harmless concentration values in both the domains of the aquifer. The solute concentration decreases with time at each position, however it increases up to a certain distance and then starts decreasing just after achieving the peak. The trend of solute concentration is just the reverse in the absence of source pollution as depicted in Fig. 6.1(b) to that in the presence of source pollution depicted in Fig. 6.1(a) with respect to the sinusoidal form of velocity.

For the same set of inputs stated earlier except for $m = 0.0002 \text{ (day)}$, $mt < 1$ Eq. (6.30) and (6.49) are also computed for the exponentially decreasing form of velocity in the presence and absence of pollution source, respectively, in both the domains. It is observed that the solute concentration patterns follow almost the same trends in the presence and absence of pollution source, respectively, in both the domains. The tendency of solute concentration with time and distance traveled is depicted graphically in Fig. 6.2(a) for the time domain $0 < t \leq t_0$ in the presence of pollution source. However, the same is depicted in Fig. 6.2(b) in the absence of pollution source in time domain $t > t_0$. These representations are made for exponentially decreasing form of velocity. It is observed the rate of decreasing tendency of solute concentration for the sinusoidal form of velocity is slightly slower in both the domains than in the case of exponential form of velocity.

Two surface graphs are drawn for the same set of data taking the sinusoidal form of velocity under consideration $mt = 8$, i.e., in the presence of input source concentration and $mt = 38$, i.e., after the source pollutant is eliminated and depicted in Fig. 6.3(a) and Fig.
6.3(b), respectively. These figures illustrate the distribution of solute concentration along and against the flow in both longitudinal and lateral directions during month of June in the 2\textsuperscript{nd} year (i.e., the presence of input source concentration) and 7\textsuperscript{th} year (i.e., after the source of pollutant is eliminated), respectively. In both the time durations, it is observed that at the other end of the aquifer the solute concentration values attain minimum and harmless concentration values. This result may also be used as a preliminary predictive tool for groundwater resource management to estimate transport parameters.

\textbf{6.5 Conclusion}

Two-dimensional solute transport problem is discussed analytically using the Laplace Transform Technique (LTT) in the presence of uniform input source concentration. The input source concentration is considered as pulse type in the intermediate portion of the aquifer. The aquifer is assumed to be homogeneous and semi-infinite in nature. The nature of solute concentration is predicted along and against flow in both the domains \(x \geq x_0, y \geq y_0\) and \(0 \leq x \leq x_0, 0 \leq y \leq y_0\). Initially the aquifer is not supposed to solute free. Temporally dependent velocities, such as sinusoidally varying velocity and exponentially decreasing velocity, are considered for numerical discussion. The line graphs as well as surface graphs are drawn to illustrate the results. The conclusion drawn from this study as: for sinusoidally form of velocity along and against the flow, the solute concentration decreases with distance of travel and increases with time at each position. The solute concentration gradually becomes minimum or harmless concentration at the far end of the aquifer. In comparison to along the flow case, the values of solute concentrations are closer during against the flow case in the presence of pollution source in the time domain \(0 < t \leq t_0\). The solute concentration eventually becomes zero in the absence of pollution source. For sinusoidal form of velocity, the solute concentration increases to a peak and then attains minimum and harmless concentration values in both the domains of the aquifer. The solute concentration decreases with time at each position, however, it increases up to a certain distance and then starts decreasing just after achieving the peak. The trend of solute concentration is just the reverse in the absence of source pollution to that in the presence of source pollution. For the exponentially decreasing form of velocity,
that the solute concentration patterns follow almost the same trends in the presence and absence of pollution source in both the domains. The rate of decreasing tendency of solute concentration for the sinusoidal form of velocity is slightly slower in both the domains than in the case of the exponential decreasing form of velocity. The distribution of solute concentration along and against the flow in both longitudinal and lateral directions during month of June in the 2\textsuperscript{nd} year (i.e., the presence of input source concentration) and 7\textsuperscript{th} year (i.e., after the source of pollutant is eliminated) shows that in both time durations the solute concentration at the other end of the aquifer attains minimum and harmless concentration values.
Fig. 6.1(a) Concentration distribution patterns of solute in two-dimensional homogeneous aquifer during the time domain $0 < t \leq t_0$ for sinusoidal form of velocity.
<table>
<thead>
<tr>
<th>Curve No.</th>
<th>mt</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26</td>
<td>5th Year June</td>
</tr>
<tr>
<td>2</td>
<td>29</td>
<td>5th Year Dec.</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>6th Year June</td>
</tr>
<tr>
<td>4</td>
<td>35</td>
<td>6th Year Dec.</td>
</tr>
</tbody>
</table>

Fig. 6.1(b) Concentration distribution patterns of solute in two-dimensional homogeneous semi-infinite aquifer during the time domain $t > t_0$ for sinusoidal form of velocity.
Fig. 6.2(a) Concentration distribution patterns of solute in two-dimensional homogeneous aquifer during the time domain $0 < t \leq t_0$ for exponentially decreasing form of velocity.
Curve No. | Days  | Duration          
---       |------|------------------
 1        | 1576 | 5th Year June    
 2        | 1758 | 5th Year Dec.    
 3        | 1940 | 6th Year June    
 4        | 2122 | 6th Year Dec.    

Fig. 6.2(b) Concentration distribution patterns of solute in two-dimensional homogeneous aquifer during the time domain $t > t_0$ for exponentially decreasing form of velocity.
Fig. 6.3(a) Concentration distribution patterns of solute in a two-dimensional homogeneous aquifer during the time domain $0 < t \leq t_0$ for sinusoidal form of velocity: Surface graph at $mt = 8$.
Fig. 6.3(b) Concentration distribution patterns of solute in a two-dimensional homogeneous aquifer during the time domain $t > t_0$ for sinusoidal form of velocity: Surface graph at $mt = 38$. 
